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ON CB-COMPACT, COUNTABLY CB-COMPACT AND CB-LINDELÖF SPACES

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Abstract. By a (countably) CB-compact space we call a topological space each cover (respectively each countable cover) of which by open sets with compact boundaries contains a finite subcover. By a CB-Lindelöf space we call a topological space each cover of which by open sets with compact boundaries contains a countable subcover. Basic properties of these spaces and relations of these spaces to some other classes of topological spaces are studied.

Introduction

Comapactnes is one of the most fundamental topological properties. Therefore it is quite natural that a very vast literature is devoted to the study of various generalizations of compactness. These generalizations proceed from different characterizations of compactness and are being developed into essentially different directions (cf. e.g. [1], [6], [3], [4]).

One of these directions relies on restricting open covers in the definition of compactness by allowing only special open sets to appear in these covers. The starting point of this direction was probably the paper [12] where Clp-compact spaces were introduced (i.e. spaces each clopen cover of which contains a finite subcover; note that [12] uses a different terminology). Later Clp-compact spaces were studied and used in a series of papers, see e.g. [5], [7], [8], [11]. Developing the idea of Clp-compactness, in [9] we introduced the property of FB-compactness (or finite-boundary compactness) by calling a space FB-compact whenever each its open cover, all sets in which have finite boundaries, has a finite subcover. The subject of this paper are CB-compact (or compact-boundary compact) spaces, i.e. spaces every cover of which consisting of open sets with compact boundaries has a finite subcover. The property of CB-compactness first appears in [11]. The main aim of our paper is to develop the origins of the theory of CB-compactness. Incidentally we introduce the concept of CB-Hausdorffness, which for CB-compactness can play a role similar to the role of Hausdorffness in theory of compactness. Besides, in the paper we consider CB-analogues of properties of countable compactness and Lindelöfness.

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1. General properties of CB-compact spaces

Definition 1.1. *A topological space is called CB-compact if every its cover, all elements of which are open sets with compact boundaries, contains a finite subcover.*

The next scheme shows the interdependence between some compactness type properties which are defined by open covers with sets satisfying certain properties of its boundaries:

Compactness $\Longrightarrow$ CB-compactness $\Longrightarrow$ FB-compactness $\Longrightarrow$ Clp-compactness.

Sometimes it can be useful to compare CB-compactness with the property of CCL-compactness where we call a space CCL-compact if every its cover by open sets with compact closures has a finite subcover:

CB-compactness $\Longrightarrow$ CCL-compactness.

That the converse in these conclusions related to CB-situation is not true is demonsatrated by the following examples.

Example 1.2. Let X be an uncountable set and let $\tau = \{ X, \varnothing, X \setminus A : A \subset X, |A| \leqslant \aleph_0 \}$. Obviuosly the space X is CB-compact, but fails to be compact.

Example 1.3. Hilbert space $X = \mathbf{R}^{\aleph_0}$ is CB-compact and it is not compact.

Let $U \subset \mathbf{R}^{\aleph_0}$ and the boundary ∂U is compact. Then we can prove that $\operatorname{int}(X \setminus \overline{U}) = \varnothing$, i.e. $\overline{U} = U \cup \partial U = \mathbf{R}^{\aleph_0}$. Therefore obviously the Hilbert space $X = \mathbf{R}^{\aleph_0}$ is CB-compact.

Example 1.4. The Euclidean space $\mathbf{R}^n$, $n \geqslant 2$, is FB-compct, but fails to be a CB-compact space.

Example 1.5. The rational numbers space $\mathbf{Q}$ is CCL-compact, but fails to be a CB-compact space.

The next two statements show when the properties of CB-compactness and compactness become equivalent.

Recall that a space X is called RIM-compact, if for each point x and each open neighbourhood U_x there exists an open neighbourhood $V_x \subset U_x$ whose boundary is compact.

Proposition 1.6. *An RIM-compact space is CB-compact iff it is compact.*

Corollary 1.7. *A zero-dimensional space is CB-compact iff it is compact.*

From the resemblance of the definitions of compactness and CB-compactness one can expect a certain analogy in the behaviour of these properties.

First we have to point out the difference between a CB-compact subspace and a CB-compact subset. A subset M of a space X is called a CB-compact *subset* if every cover of M in X by open sets with compact boundaries contains a finite subcover. On the other hand a subset M of a space X is called a CB-compact *subspace* if every cover of M by open sets with compact boundaries in M contains a finite subcover.

PROPOSITION 1.8. *If M is a CB-compact subspace of a space X, then M is also a CB-compact subset of X.*

Proof of this proposition easily follows if we use the relation $\partial_X A \supset \partial_M(A \cap M)$ whenever $M \subset X$ and $A \subset X$, see e.g. Lemma 1.11 from [9]. ■

LEMMA 1.9. *If $U \subset M \subset X$, then $\partial_X U \subset \partial_M U \cup \partial_X M$.*

Proof. Take an arbitrary point $x \in \partial_X U$, then for every its neighbourhood O_x: (1) $O_x \cap U \neq \varnothing$ and (2) $O_x \cap (X \setminus U) \neq \varnothing$. From (2) it follows that either $O_x \cap (M \setminus U) \neq \varnothing$ or $O_x \cap (X \setminus M) \neq \varnothing$. Now taking into account (1) we conclude that either $x \in \partial_M U$ or $x \in \partial_X M$. Therefore $x \in \partial_M U \cup \partial_X M$. ■

PROPOSITION 1.10. *If a topological space X is CB-compact and M is its closed subset with compact boundary, then M is a CB-compact subspace of X, too.*

Proof. Let $U = \{ U_i : i \in I \}$ be an open cover with compact boundaries $\partial_M U_i$ in the subspace M of the space X and let $\tilde{U}_i = U_i \cup (X \setminus M)$. Then each $\tilde{U}_i$ is an open subset in X. Further we observe that $\partial_X \tilde{U}_i \subset \partial_X U_i \cup \partial_X (X \setminus M) \subset \partial_M U_i \cup \partial_X M \cup \partial_X (X \setminus M)$ (the last inclusion follows from Lemma 1.9). Since the boundary of the set M coincides with the boundary of its complement $X \setminus M$, the set $\partial_X (X \setminus M)$ is compact. Therefore $\partial_X \tilde{U}_i$ is compact as a finite union of compact sets and hence the system $\{ \tilde{U}_i : i \in I \}$ is an open cover with compact boundaries of this space X. Now, the conclusion follows easily in the standard way. ■

PROPOSITION 1.11. *A topological space is CB-compact iff every system of its closed subsets with compact boundaries which has the finite intersection property has a non-empty intersection.*

Proof can be done in the standard way. ■

Since the boundary of the intersection of two sets is contained in the union of boundaries of these sets and the finite union of compact sets is compact, one can easily get the following corollary of the previous proposition.

COROLLARY 1.12. *A topological space is CB-compact iff every system of its non-empty closed subsets with compact boundaries which is invariant under finite intersections has a non-empty intersection.*

PROPOSITION 1.13. *If a space X is CB-compact and Y is the image of X under a continuous mapping $f \colon X \to Y$ such that $f^{-1}(K)$ is a compact subset in X whenever K is compact in Y, then Y is CB-compact, too.*

Proof. Since $\overline{f^{-1}(K)} \setminus f^{-1}(K) \subset f^{-1}(\overline{K}) \setminus f^{-1}(K) = f^{-1}(\overline{K} \setminus K)$ for every open subset K of Y, the preimage of each open set K with compact boundary is also an open set with compact boundary. Therefore the proof can be easily done in the standard way. ■

Turning now to the question of preserving CB-compactness by preimages, we first state the following, probably well-known lemma.

LEMMA 1.14. *If a continuous mapping $f\colon X \to Y$ is open and closed, then $\partial f(M) \subset f(\partial M)$—in other words, the boundary of the image of a set $M \subset X$ is contained in the image of its boundary.*

Proof. Since f is an open and closed mapping, it holds $\partial f(M) = \overline{f(M)} \setminus \operatorname{int} f(M) = f(\overline{M}) \setminus \operatorname{int} f(M) \subset f(\overline{M}) \setminus f(\operatorname{int} M) \subset f(\overline{M} \setminus \operatorname{int} M) = f(\partial M)$ for every subset M of X. ■

PROPOSITION 1.15. *If a space Y is CB-compact and a mapping $f\colon X \to Y$ has the following properties: f is an open and closed mapping and for every point y from Y the preimage $f^{-1}(y)$ is a CB-compact subset of X, then the space X is CB-compact, too.*

Proof. Let $U = \{\, U_i : i \in I \,\}$ be a system of closed subsets of X with compact boundaries which is invariant under finite intersections. We have to prove that $\bigcap U \neq \varnothing$.

Let $f(U) = \{\, f(U_i) : U_i \in U \,\}$. Since f is open and closed, from the previous lemma it follows that $\partial f(U_i) \subset f(\partial U_i)$ for every subset $U_i \in U$. Therefore $f(U)$ is a system of closed subsets of Y with compact boundaries and obviously $f(U)$ has the finite intersection property. Therefore $\bigcap f(U) \neq \varnothing$. Choose some point $y_0 \in \bigcap f(U)$ and let $K = f^{-1}(y_0)$. It is clear that for every $U_i \in U$, $U_i \cap K$ is a non-empty closed subset of K and $\partial(U_i \cap K) \subset \partial U_i \cap K$, and therefore also $\partial(U_i \cap K)$ is compact in K. Since U is invariant under finite intersections, it follows that $\{\, U_i \cap K : U_i \in U \,\}$ has the finite intersection property and since K is CB-compact, we conclude that $\bigcap U \supset (\bigcap U) \cap K \neq \varnothing$. ■

PROPOSITION 1.16. *A finite union of CB-compact subsets of a given space X is a CB-compact subset.*

PROPOSITION 1.17. *A direct sum $X = \bigoplus_{i \in I} X_i$ of non-empty spaces X_i is CB-compact iff all X_i, $i \in I$, are CB-compact and the set I is finite.*

Proofs are obvious and therefore omitted.

CONSTRUCTION 1.18. For every σ the quotient hedgehog J_σ is a CB-compact space. Generalizing this example, we can construct new CB-compact spaces from old ones. Let X_α, $\alpha \in A$ (A is a finite set) be CB-compact spaces and assume that in each space X_α, $\alpha \in A$ there exists a point $x^*_\alpha \in X_\alpha$ having a neighbourhood U_α with a compact boundary (in case there are several such points we fix one of them). In the direct sum $\bigoplus X_\alpha$ we define the equivalence relation by setting $x^*_\alpha \approx x^*_\beta$ for all $\alpha, \beta \in A$. The resulting quotient space $Z = \bigoplus X_\alpha / \approx$ is CB-compact.

Now we shall introduce the property of CB-Hausdorffness which in the context of CB-compact spaces plays a role similar to the role of Hausdorffness in the classic theory of compactness.

DEFINITION 1.19. A topological space X is called CB-Hausdorff if for any two different points x and y there exist disjoint open neighbourhoods A_x and B_y with compact boundaries.

Obviously,

$$\text{Clp-Hausdorff} \implies \text{FB-Hausdorff} \implies \text{CB-Hausdorff} \implies \text{Hausdorff}$$
$$\text{CCl-Hausdorff} \implies \text{CB-Hausdorff}$$

That the converse in these conclusions related to CB-Hausdorfness is not true is showed by the next examples.

EXAMPLE 1.20. The Hilbert space $\mathbf{R}^{\aleph_0}$ (see 1.3) is Hausdorff, but is not CB-Hausdorff, while the Euclidean space $\mathbf{R}^n$ for $n \geqslant 2$ is CB-Hausdorff, but is not FB-Hausdorff.

EXAMPLE 1.21. The space $\mathbf{Q}$ of rational numbers is CB-Hausdorff, but is not CCl-Hausdorff. (We call a space CCl-Hausdorff if for any two different points there exist disjoint open neighbourhoods with compact closures.)

The proof of the next proposition can be easily done in the standard way.

PROPOSITION 1.22. *A CB-compact subset of a CB-Hausdorff space is closed.*

The fundamental property of compactness is multiplicativity. Our next aim is to study whether CB-compactness is preserved by products. We do not know the answer in the general case. However, it is true in some special situations.

PROPOSITION 1.23. *The product of a CB-compact space and a compact space is CB-compact.*

Proof. Let X be a CB-compact and Y be a compact space. Since the projection $p_X \colon X \times Y \to X$ along the compact space Y is an open and closed mapping and for every point $x \in X$ the preimage $p_X^{-1}(x) = \{x\} \times Y \subset X \times Y$ is a CB-compact subset, the conclusion follows from Proposition 1.14. ∎

PROPOSITION 1.24. (see Proposition 2.9 in [11]) *Let W be an open subset of the product $X \times Y$ such that the boundary ∂W is compact. If the projection $p_X \colon X \times Y \to X$ is clopen, it is $p_X(\partial W) \supset \partial(p_X(W))$.*

LEMMA 1.25. (see Lemma 2.10 in [11]) *Let V be a regularly closed subset of the product $X \times Y$ and $p_X(\partial \operatorname{int} V) \supset \partial(p_X(\operatorname{int} V))$, then also $p_X(\partial V) \supset \partial(p_X(V))$.*

We can conclude now:

COROLLARY 1.26. *If V is a regularly closed subset of the product $X \times Y$ such that the boundary ∂V is compact and the projection $p_X \colon X \times Y \to X$ is clopen, then $p_X(\partial V) \supset \partial(p_X(V))$.*

The next fact was mentioned in [11]. Here we present the proof of this fact more completely.

Proposition 1.27. *If X and Y are CB-compact spaces and the projection $p_X : X \times Y \to X$ is clopen, then the product $X \times Y$ is CB-compact.*

Proof. Let $V = \{ V_i : i \in I \}$ be a family of closed subsets of $X \times Y$ with compact boundaries which is invariant under finite intersections. To show CB-compactness of the product it is sufficient to prove that $\bigcap V \neq \varnothing$ (Proposition 1.12).

For every $V_i \in V$ let $V'_i = \overline{\operatorname{int} V_i}$ and let $V' = \{ V'_i : V_i \in V \}$. Obviously $\partial V'_i = \partial \overline{\operatorname{int} V_i} \subset \partial \operatorname{int} V_i \subset \partial V_i$ and hence the boundaries of all $V'_i \in V'$ are also compact.

Consider the two possible cases:

1) V' is not closed under finite intersections. Since $V_i = \operatorname{int} V_i \cup \partial V_i = V'_i \cup \partial V_i$, this means that there exists $V_0 \in V$ such that ∂V_0 intersects every $V_i \in V$. Taking into account that ∂V_0 is compact, we conclude that $\bigcap V \supset (\bigcap V) \cap \partial V_0 \neq \varnothing$.

2) Let now V' be closed under finite intersections, and consider the family of sets $A = \{ \overline{p_X(V'_i)} : V'_i \in V' \}$. Then obviously A has the finite intersection property and sets from A are closed. The sets $V'_i \in V'$ for every $i \in I$ are regularly closed in $X \times Y$ and $\partial V'_i$ is compact. Then from Corollary 1.26 it follows that $p_X(\partial V'_i) \supset \partial(p_X(V'_i))$, but this means that also $\partial(p_X(V'_i))$ is compact as a closed subset of a compact subspace. Since $\partial(\overline{p_X(V'_i)}) \subset \partial(p_X(V'_i))$, the set $\partial(\overline{p_X(V'_i)})$ is also compact. Thus all sets of A are closed and have compact boundaries. Hence, by Proposition 1.12 it follows that $\bigcap A \neq \varnothing$. Then there exists a point $a \in \bigcap A$. For every $V'_i \in V'$ the intersection $(\{a\} \times Y) \cap V'_i \neq \varnothing$ because V' is closed under finite intersections and all sets V'_i are regularly closed and have compact boundaries. Thus $\bigcap V \supset \bigcap V' \supset (\{a\} \times Y) \cap V'_i \neq \varnothing$ and therefore $X \times Y$ is a CB-compact space. ∎

2. Countably CB-compact spaces

Definition 2.1. A topological space is called countably CB-compact if every its countable cover, all elements of which are open sets with compact boundaries, contains a finite subcover.

Obviously,

CB-compactness $\Longrightarrow$ countable CB-compactness

Countable compactness $\Longrightarrow$ countable CB-compactness

$\Longrightarrow$ countable FB-compactness $\Longrightarrow$ countable Clp-compactness.

Example 2.2. The Hilbert space $\mathbf{R}^{\aleph_0}$ (see 1.3) is a countably CB-compact space which is not countably compact.

Example 2.3. Since in the realm of zero-dimensional spaces CB-compactness (respectively, countable CB-compactness) is equivalent to compactness (respectively, to countable compactness), every zero-dimensional countably compact non-compact space presents also an example of a countably CB-compact space which fails to be CB-compact. One of such spaces is the space W_0 of all countable ordinals.

But for spaces of countable weight, the converse is also true.

PROPOSITION 2.4. *If* $w(X) \leqslant \aleph_0$, *then* X *is a CB-compact space iff* X *is countably CB-compact.*

Proof. Let $U = \{ U_i : i \in I \}$ be a cover of X all elements of which are open sets with compact boundaries. Since the space X is second countable, for every $i \in I$ and every $x \in U_i$ there exists a neighbourhood V_x from the countable base B_x such that $x \in V_x \subset U_i$. Clearly, the system $V = \{ V_x : V_x \in B_x \}$ is finite or countable, i.e. $V = \{V_{x_1}, \dots, V_{x_n}, \dots \}$ and taking the corresponding sets $U_{i_1} \supset V_{x_1}$, etc., we select at most a countable subcovering from the cover U. Since X is a countable CB-compact space, there exists a finite subcover of X. So X is a CB-compact space.

The converse conclusion is obvious. ∎

In a similar way as CB-compactness, obviously also countable CB-compactness has characterizations by systems of closed sets with finite intersection properties.

PROPOSITION 2.5. *A topological space is countably CB-compact iff every countable system of its closed subsets with compact boundaries which has the finite intersection property has a non-empty intersection.*

COROLLARY 2.6. *A topological space is countably CB-compact iff every countable system of its non-empty closed subsets with compact boundaries which is invariant under finite intersections has a non-empty intersection.*

PROPOSITION 2.7. *If a topological space* X *is countably CB-compact and* M *is its closed subset with compact boundary, then* M *is a countably CB-compact subspace of* X.

Proof is similar to that of Proposition 1.10. ∎

PROPOSITION 2.8. *Let* X *be a countably CB-compact space and* Y *be the image of* X *under a continuous mapping* $f\colon X \to Y$ *such that* $f^{-1}(K)$ *is a compact subset in* X *whenever* K *is compact in* Y. *Then* Y *is counatbly CB-compact, too.*

Proof is similar to that of Proposition 1.13. ∎

PROPOSITION 2.9. *Let* Y *be a countably CB-compact space and a mapping* $f\colon X \to Y$ *has the following properties:* f *is open and closed and for every point* y *from* Y *the preimage* $f^{-1}(y)$ *is a CB-compact subset of* X. *Then the space* X *is countably CB-compact, too.*

Proof is similar to that of Proposition 1.15. ∎

With respect to products the behaviour of countably CB-compact spaces has analogies with the CB-compactness. In particular, patterned after the proof of Proposition 1.23 one can easily prove the next

PROPOSITION 2.10. *The product of a countably CB-compact and a compact space is countably CB-compact.*

3. CB-Lindelöf spaces

DEFINITION 3.1. A topological space is called CB-Lindelöf if every its cover, all elements of which are open sets with compact boundaries, contains a countable subcover.

Obviously,

CB-compactness $\Longrightarrow$ CB-Lindelöfness

Lindelöfness $\Longrightarrow$ CB-Lindelöfness $\Longrightarrow$ FB-Lindelöfness $\Longrightarrow$ Clp-Lindelöfness.

Every countably CB-compact CB-Lindelöf space is CB-compact.

The space constructed in [2], 3.12.18 is not CB-Lindelöf (this space is not CB-compact and also is not countably CB-compact, but remains CB-Hausdorff.)

EXAMPLE 3.2. The Euclidean space $\mathbf{R}^n$ gives an example of a CB-Lindelöf space, which fails to be CB-compact.

EXAMPLE 3.3. Let X be an uncountable set, $p \in X$ and the topology $\tau = \{\varnothing, A : p \in A\}$. Obviously the space (X, τ) ic CB-Lindelöf, but fails to be a Lindelöf space.

PROPOSITION 3.4. *A RIM-compact space is CB-Lindelöf iff it is Lindelöf.*

COROLLARY 3.5. *A zero-dimensional space is CB-Lindelöf iff it is Lindelöf.*

From the resemblance of the definitions of Lindelöf and CB-Lindelöf spaces one can expect a certain analogy in the behaviour of these properties.

PROPOSITION 3.6. *If a topological space X is CB-Lindelöf and M is its closed subset with compact boundary, then M is a CB-Lindelöf subspace of X, too.*

PROPOSITION 3.7. *If a space X is CB-Lindelöf and Y is the image of X under a continuous mapping $f\colon X \to Y$ such that $f^{-1}(K)$ is a compact subset in X whenever K is compact in Y, then Y is CB-Lindelöf, too.*

PROPOSITION 3.8. *A direct sum $X = \bigoplus_{i \in I} X_i$ of non-empty spaces X_i is CB-Lindelöf iff all X_i, $i \in I$, are CB-Lindelöf and the set I is countable.*

PROPOSITION 3.9. *If a space Y is CB-Lindelöf and a mapping $f\colon X \to Y$ has the following properties: f is an open and closed mapping and for every point y from Y the preimage $f^{-1}(y)$ is a CB-compact subset of X, then the space X is CB-Lindelöf, too.*

Proofs of Propositions 3.6–3.9 are similar to proofs of 1.10, 1.13, 1.17, 1.15 respectively and therefore are omitted.

There is also a characterization of CB-Lindelöfness by countable intersection property.

PROPOSITION 3.10. *A topological space is CB-Lindelöf iff every system of its closed subsets with compact boundaries which has the countable intersection property has a non-empty intersection.*

Proof is straightforward.

EXAMPLE 3.11. The Sorgenfrey line $\mathbf{R}_b$ is CB-Lindelöf, but $\mathbf{R}_b \times \mathbf{R}_b$ is not Clp-Lindelöf and then also it is not CB-Lindelöf.

PROPOSITION 3.12. *The product of a CB-Lindelöf space and a compact space is CB-Lindelöf.*

Proof can be done in a similar manner as the proof of Proposition 1.23.

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