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ON SEPARABLE SUBALGEBRAS OF AZUMAYA ALGEBRAS

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Abstract. Let A be an Azumaya C-algebra. Then the set of all commutative separable subalgebras of A and the set of separable subalgebras B such that $V_A(B) = V_B(B)$ are in a one-to-one correspondence, where $V_A(B)$ is the commutator subring of B in A, and the set of all separable subalgebras of A is a disjoint union of the Azumaya algebras in A over a commutative separable subalgebra of A. The results are used to compute splitting rings for an Azumaya skew group ring.

1. Introduction

The theory of central simple algebras has been generalized to Azumaya algebras. The separable subalgebras and splitting rings of an Azumaya algebra play an important role in Galois theory of rings ([3], Chapter IV). The purpose of the present paper is to investigate the set of separable subalgebras of an Azumaya algebra. Several relations between certain sets of separable subalgebras are found, and the results are applied to an Azumaya skew group ring S^*G to find a representation of the separable subalgebra arising from a subgroup of G, and to find splitting rings for S^*G. Let A be an Azumaya C-algebra, $V_A(B)$ the commutator subring of the subring B in A, $\mathcal{C}$ the set of all commutative separable subalgebras of A, $\mathcal{D}$ the set of separable subalgebras B such that $V_A(B) = V_B(B)$ (the center of B). Then $\mathcal{C}$ and $\mathcal{D}$ are in a one-to-one correspondence. Let P_D be the set of D-Azumaya algebras in A for a D in $\mathcal{C}$, and Q the set of all separable subalgebras of A, then Q is a disjoint union of P_D for all D in $\mathcal{C}$. The above results are applied to an Azumaya skew group ring S^*G where G is a finite automorphism group of S. We shall derive a repesentation for some separable subalgebra arising from some subgroup of G. Moreover, some separable splitting rings for S^*G are found from the above sets of separable subalgebras.

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2. Preliminaries

Throughout, we assume A is an Azumaya algebra over center C, S a ring with 1, G a finite automorphism group of S of order n for some integer n invertible in S, S^G the subring of the elements fixed under each element in G, S^*G the skew group ring of G over S. All notations in Section 1 are preserved. The definitions of separable algebras, Azumaya algebras and Galois extensions of rings can be found in References ([1], [2], [3]).

3. Azumaya algebras

In this section, we shall show that the set $\mathcal{C}$ and $\mathcal{D}$ are in one-to-one correspondence and that Q is a disjoint union of P_D for all D in $\mathcal{C}$. Also, a representation of the Azumaya D-algebra $V_A(D)$ is given for a D in $\mathcal{C}$.

THEOREM 3.1. *Let $\alpha\colon \mathcal{D} \to \mathcal{C}$ by $\alpha(B) = V_A(B)$ for any B in $\mathcal{D}$. Then α is bijective.*

Proof. By the definition of B, $V_A(B) = V_B(B)$ which is in $\mathcal{D}$, so α is well defined. Since $V_A(V_A(B)) = B$ by the commutator theorem for Azumaya algebras ([3], Theorem 4.3), α is injective. Also, for any D in $\mathcal{C}$, $V_A(V_A(D)) = D$ such that $V_A(D)$ is a separable algebra, so $V_A(D)$ is an Azumaya D-algebra. Hence α is surjective. ■

The above theorem shows that $V_A(D)$ is an Azumaya algebra over D. We denote the class of all Azumaya D-algebras in A by P_D.

THEOREM 3.2. *Let Q be the class of all separable subalgebras of A. Then (1) for each B in P_D, $V_A(B)$ is in P_D such that $V_A(D) \cong B \otimes V_A(B)$ where $\otimes$ is over D, and (2) Q is a disjoint union of $\{ P_D \mid$ for D in $\mathcal{C} \}$.*

Proof. Since B is a separable subalgebra with center D, $V_A(V_A(B)) = B$ by the commutator theorem for Azumaya algebras. Hence the center of $V_A(B) \subset B$. Thus $V_A(D)$ is an Azumaya D-algebra in P_D. Clearly, $V_A(B)$ is contained in $V_A(D)$ as an Azumaya D-algebra by Theorem 3.1, so $V_A(D) \cong B \otimes V_A(B)$ where $\otimes$ is over D by the commutator theorem for Azumaya algebras again. Therefore (1) holds. For any B in Q with center D, B is a separable C-algebra, so it is an Azumaya D-algebra and D is a separable C-algebra. Hence B is in P_D where D is in $\mathcal{C}$. Clearly, the union is disjoint, so (2) holds. ■

Theorem 3.2 implies that $V_A(D)$ is the maximal element in P_D. Next we give some relations between the subclasses of separable subalgebras of A, P_D, $\mathcal{C}$ and $\mathcal{D}$.

THEOREM 3.3. *For each D in $\mathcal{C}$, $P_D \cap \mathcal{D} = \{V_A(D)\}$ and $P_D \cap \mathcal{C} = \{D\}$.*

Proof. Let B be in $P_D \cap \mathcal{D}$. Then $V_A(B) = V_B(B) =$ the center of $B = D$ which is in $\mathcal{C}$ and $V_A(V_A(B)) = V_A(D) = B$. Thus $P_D \cap \mathcal{D} = \{V_A(D)\}$. Moreover, D is the only commutative Azumaya D-algebra, so $P_D \cap \mathcal{C} = \{D\}$. ■

Let S be a ring with 1, G a finite automorphism group of S of order n for some integer n invertible in S, S^G the subring of the fixed elements under each element in G, and S^*G the skew group ring of G over S. Then G induces an inner automorphism group G' of S^*G. When S^*G is an Azumaya algebra over its center Z, we shall derive a representation and correspondence relation for some separable subalgebras of S^*G.

COROLLARY 3.4. *Let H be a subgroup of G and H' the inner automorphism group of S^*G induced by H. Then (1) ZH is in P_D where D is the center of ZH, (2) ZH, $(S^*G)^{H'}$ and $V_{S^*G}(D)$ are in P_D, and (3) $V_{S^*G}(D) \cong ZH \otimes (S^*G)^{H'}$ as Azumaya D-algebras, where $\otimes$ is over D.*

Proof. Since the order of G is a unit in S, so is the order of H. Hence ZH is a separable subalgebra of S^*G. Let the center of ZH be D. Then (1) and (2) hold by Theorem 3.2. Also noting that $V_{S^*G}(ZH) = (S^*G)^{H'}$, we have $V_{S^*G}(D) \cong ZH \otimes (S^*G)^{H'}$ by Theorem 3.2 again. ■

COROLLARY 3.5. *Let Ω be the set of separable subalgebras $\{ZK\}$ generated by the elements of abelian subgroups K of G, and Φ the set of separable subalgebras $\{(S^*G)^{K'}\}$ of S^*G fixed under the elements of the inner automorphism subgroups K' induced by abelian subgroups K. Then Ω and Φ are in a one-to-one correspondence.*

Proof. Since Ω is contained in $\mathcal{C}$, Ω and $\{V_{S^*G}(ZK)\}$ are in a one-to-one correspondence by Theorem 3.1. Noting that $V_{S^*G}(ZK) = (S^*G)^{K'}$, the corollary is immediate. ■

For an interesting question in Galois theory, we give a necessary and sufficient condition for a separable subalgebra B of S^*G being $(S^*G)^{H'}$ for some subgroup H of G.

THEOREM 3.6. *Let B be a separable subalgebra of S^*G with center D and $H = \{\, g \text{ in } G \mid g(a) = a \text{ for each } a \text{ in } B\,\}$. Then, $B = (S^*G)^{H'}$ if and only if the ranks of B and $(S^*G)^{H'}$ are equal over D.*

Proof. Clearly, B is contained in $(S^*G)^{H'}$ and $V_{S^*G}(ZH) = (S^*G)^{H'}$, so ZH and $(S^*G)^{H'}$ are Azumaya algebras over the same center D by Theorem 3.2. Thus the theorem is immediate. ■

4. Splitting rings

Using the map $\alpha\colon \mathcal{D} \to \mathcal{C}$ as given by Theorem 3.1, we shall compute splitting rings for an Azumaya algebra A. We begin with a lemma.

LEMMA 4.1. *Let $\alpha\colon \mathcal{D} \to \mathcal{C}$ as given by Theorem 3.1. If α has a fixed point B, then B is a splitting ring for A.*

Proof. By definition, $\alpha(B) = V_A(B) = V_B(B) = B$, so B is a maximal commutative separable subalgebra of A. Thus B is a splitting ring for A ([3], Theorem 5.5). ■

Let C be the center of the ring S with a finite automorphism group G as given in Section 3 and S^*G an Azumaya skew group ring. In [1] and [2], the class of Azumaya G-Galois extensions was studied where S is called Azumaya G-Galois extension of S^G if S is a G-Galois extension of S^G which is an Azumaya C^G-algebra. It was shown that S is an Azumaya G-Galois extension of S^G if and only if S^*G is an Azumaya C^G-algebra ([2], Theorem 1). Also, if S is an Azumaya G-Galois extension, then $S^*G \cong S^G \otimes E$ where $\otimes$ is over C^G and E is a central Galois C^G-algebra with Galois group induced by and isomorphic with G ([1], Theorem 2). Now, for the Azumaya C^G-algebra S^G, if $\alpha(B) = B$ for some B in $\mathcal{D}$ where α is defined in Theorem 3.1, then B is a splitting ring for S^G. Since E is the identity of the Brauer group of C^G, B is also a splitting ring for S^*G ($\cong S^G \otimes E$). Thus we have the following theorem.

Theorem 4.2. *Let S^*G be an Azumaya C^G-algebra. If $\alpha(B) = B$ for some B in $\mathcal{D}$ where α is defined in Theorem 3.1, then B is a splitting ring for S^*G.*

Next we shall compute a splitting ring for an Azumaya skew group ring S^*G where S is an Azumaya algebra over C which is Galois with Galois group induced by and isomorphic with G (for more properties of such an S, see [4]). By the proof of Lemma 2 on p. 120 in [4], S^G is an Azumaya C^G-algebra and S is a G-Galois extension of S^G with the same G-Galois system for C, so S is an Azumaya G-Galois extension of S^G ([2], Theorem 1). Thus the following corollary holds by Theorem 4.2.

Corollary 4.3. *Let S be an Azumaya algebra over C which is a Galois algebra with Galois group induced by and isomorphic with G. If $\alpha(B) = B$ for some B in $\mathcal{D}$ where α is defined in Theorem 3.1 for the Azumaya algebra S^G over C^G, then B is a splitting ring for S^*G.*

For the Azumaya algebra S^*G as given in Theorem 4.2, we compute another splitting ring.

Theorem 4.4. *Let S be given in Theorem 4.2. If $\alpha(B) = B$ for some B in $\mathcal{D}$ where α is defined in Theorem 3.1 for the Azumaya algebra S over C, then B is a splitting ring for S^*G.*

Proof. By the proof of Lemma 2 on p. 120 in [4], S^G is an Azumaya C^G-algebra and S is a G-Galois extension of S^G with the same G-Galois system for C, so S is an Azumaya G-Galois extension of S^G ([2], Theorem 1). We claim that $\alpha(B) = B$ where α is defined in Theorem 3.1 for the Azumaya algebra S^*G. In fact, since B is a commutative separable subalgebra over C, it is also a commutative separable subalgebra over C^G (for C is separable over C^G). Let $G = \{g_1, g_2, \ldots, g_n\}$ and for any $\sum r_i g_i$ for some r_i in S such that $\sum r_i g_i$ is in $V_{S^*G}(B)$. Then $a(\sum r_i g_i) = (\sum r_i g_i)a$ for all a in C (for $C \subset B$). We have $\sum a r_i g_i = \sum r_i g_i(a) g_i$. Hence $(a - g_i(a)) r_i = 0$ for all a in C. But C is G-Galois over C^G with Galois group induced by and isomorphic with G, so $r_i = 0$ for each $g_i \neq g_1 = 1$ in G. Thus $\sum r_i g_i = r_1$. Moreover, $b r_1 = r_1 b$ for each b in B, so r_1 is in $V_S(B)$ which is B by hypothesis. This implies that B is a splitting ring for S^*G. ■

We conclude the paper with two examples of splitting rings for an Azumaya skew group rings as given in this section.

1. Let $Q[i,j,k]$ be the quaternion algebra over the rational field Q, $S = Q[i,j,k] \times Q[i,j,k]$, and $G = \{1, g\}$ where $g(a,b) = (b,a)$ for each (a,b) in S. Then (1) S is a G-Galois extension of $Q[i,j,k]$ with a G-Galois system $\{(1,0),(0,1);(1,0),(0,1)\}$, (2) $Q[i,j,k]$ is an Azumaya Q-algebra, so S is an Azumaya G-Galois extension of $Q[i,j,k]$, (3) $\alpha(B) = B$ for the Azumaya Q-algebra $Q[i,j,k]$, where $B = Q[i]$ as given in Theorem 3.1, so B is a splitting ring for S^*G by Theorem 4.2.

2. Let K be $Q \times Q$ where Q is the rational field, $G = \{1, g\}$ where $g(a,b) = (b,a)$ for all (a,b) in $Q \times Q$, and $S = (Q \times Q)[i,j,k]$ the quaternion algebra over $Q \times Q$. Then (1) S is an Azumaya algebra over $Q \times Q$, (2) $Q \times Q$ is a G-Galois algebra over Q with a G-Galois system $\{(1,0),(0,1);(1,0),(0,1)\}$, so S is a ring as given in Corollary 4.3 and Theorem 4.4 by extending the action of g to S such that $g(i) = i$, $g(j) = j$ and $g(k) = k$, (3) $\alpha(B) = B$ for the Azumaya algebra S where $B = (Q \times Q)[i]$, so B is a splitting ring for S^*G by Theorem 4.4.

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