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# JULIA POINTS AND STRONG ANALYTIC NORMALITY

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**Abstract.** In this paper we prove that, if a function $f$ holomorphic on the open unit disk belongs to the class $JB^*$, then $f$ has no Julia points.

## 1. Introduction

In the Theory of cluster sets, many questions are in connection with the investigation of the behavior of some classes of functions in a neighborhood of a boundary point of a domain in $\mathbf{C}$. In this paper, we investigate a problem of this type concerning the Julia points.

Let $\mathbf{D}$ denote the open unit disk of the complex plane $\mathbf{C}$. Denote by $h(\alpha,\theta)$ the chord of the disk $\mathbf{D}$ which join a point $z \in \mathbf{D}$ with $\exp(i\theta)$, and which makes an angle $\alpha$, $-\pi/2 < \alpha < \pi/2$ with the diameter $\Lambda^\theta$. $h(\alpha,\theta)$ is called the Julia chord for a function $f$ holomorphic in $\mathbf{D}$, if in every angular domain of $\mathbf{D}$ whose bisector is the chord $h(\alpha,\theta)$, $f$ takes every value in $\mathbf{C}$, with at most one exception, infinitely often. The point $\exp(i\theta)$ is said to be the Julia point for a function $f$ if every chord of the form $h(\alpha,\theta)$, $-\pi/2 < \alpha < \pi/2$, is a Julia chord for $f$.

Collingwood and Piranian [1] studied functions meromorphic on $\mathbf{D}$, with bounded spherical length. Recall that a function $f$ meromorphic on $\mathbf{D}$ is a function with bounded spherical length if there holds

$$\sup_{0\leqslant r<1}\int_0^{2\pi}\frac{\left|f'\big(r\exp(i\theta)\big)\right|}{1+\left|f\big(r\exp(i\theta)\big)\right|^2}r\,d\theta<+\infty.$$

Collingwood and Piranian showed in [1] that there is a holomorphic function on $\mathbf{D}$ with bounded spherical length, and with finitely many Julia points. Further, W. Hayman proved in [2] that there exist functions holomorphic on $\mathbf{D}$ with bounded spherical length, and with infinitely many Julia points. Collingwood and Piranian proposed the conjecture that a normal holomorphic function with bounded spherical

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length contains no Julia points. This conjecture was proved by Hayman in [2]. In this paper we prove that a function belonging to the class $JB^*$ has no Julia points.

## 2. Preliminary notations, definitions and results

Let $\Gamma_{\mathbf{D}}$ denote the group of all conformal automorphisms of the disk $\mathbf{D}$. For any fixed $0 \leqslant \theta < \pi$, define the hyperbolic subgroup $T_H^\theta$ of $\Gamma_D$ as

$$T_H^\theta = \left\{ S_H^\theta : S_H^\theta(z) = \frac{z + a\exp(i\theta)}{1 + a\exp(-i\theta)}, a \in (-1, 1) \right\}.$$

The pseudohyperbolic disk $\Delta(\omega, r)$ is defined by

$$\Delta(\omega, r) = \left\{ z \in \mathbf{D} : \left| \frac{z - \omega}{1 - \bar{\omega} z} \right| < r \right\}, \omega \in \mathbf{D}.$$

A family $F$ of holomorphic functions in $\mathbf{D}$ is said to be strong analytic normal in $\mathbf{D}$ if every sequence $(f_n) \subset F$ contains a subsequence $(f_{n_k})$ such that $(f_{n_k}(z) - f_{n_k}(0))$ converges uniformly on each compact subset of $\mathbf{D}$ to a holomorphic function $f$. For a fixed $0 \leqslant \theta < \pi$, denote by $JB^*$ the class of holomorphic functions on $\mathbf{D}$ which are strongly analytic normal in $\mathbf{D}$ with respect to the hyperbolic subgroup $T_H^\theta$ of the group $\Gamma_{\mathbf{D}}$.

LEMMA. *Suppose $f \in JB_H^\theta$ for a fixed $0 \leqslant \theta < \pi$. Then $\exp(i\theta)$ and $\exp(-i\theta)$ are not Julia points for a function $f$.*

*Proof.* Suppose first that $\exp(i\theta)$ is a Julia point for $f$. Then there exists a sequence $\{z_n\} \subset \Delta_H^\theta(r) = \bigcup_{a \in (-1,1)} \Delta\big(a\exp(i\theta), r\big)$, such that $\lim_{n\to\infty} z_n = \exp(i\theta)$ and $\lim_{n\to+\infty} f\big(z_n\big) = \infty$. It follows that a function $f$ is not bounded on $\Delta_H^\theta(r)$. This contradicts the fact that $f \in JB_H^\theta$. Analogously, the assumption that $-\exp(i\theta)$ is a Julia point gives a contradiction, which proves our lemma. ∎

Put $JB^* = \bigcap_{0 \leqslant \theta < \pi} JB_H^\theta$. Then $JB^* \subset JB_H^\theta$ for each $0 \leqslant \theta < \pi$. The function $f(z) = \big(1 - z\exp(i\theta_1)\big)^{-1}$ with $\theta_1 \neq \theta$, shows that $JB^* \subset JB_H^\theta$ for each $0 \leqslant \theta < \pi$.

It is not difficult to see that $H^\infty \subset JB^*$, and this inclusion relation is proper. For the function $g_1(z) = \tan\left(z - \exp(i\theta_1) + \pi/2\right)$ with $0 \leqslant \theta < \pi/2$, it is easy to verify that $g_1 \in JB^*$, but $g_1 \notin H^\infty$. Note that there does not exist an inclusion relation between the class $JB^*$ and the class $B$ of Bloch's functions. Namely, $f(z) = (1 - z)^{-1} \in B$, but $f \notin JB^*$. On the other hand, the above function $g_1 \in JB^*$, but $g_1 \notin B$.

## 3. The main result

THEOREM. *A function $f$ belonging to the class $JB^*$ contains no Julia points.*

*Proof.* The assertion of the Theorem follows immediately from the above Lemma and the definition of the class $JB^*$. ∎

REMARKS. Observe that the above theorem gives a positive answer for the class $JB^*$ to a question proposed by Collingwood and Piranian [1]. This question is related with a characterization of classes of holomorphic functions which does not have Julia points on $\Gamma_{\mathbf{D}}$. We point out that the above theorem cannot be obtained from the results of Hayman [2], which prove the conjecture of Collingwood and Piranian [1]. This shows that the function $g_1$ which is holomorphic on $\mathbf{D}$ and belongs to the class $JB^*$ is not normal, and hence $g_1$ does not belong to the class investigated by Hayman [2]. Namely, the class investigated by Hayman, and for which the conjecture by Collingwood and Piranian is proposed, consists of normal holomorphic functions on $\mathbf{D}$ with bounded spherical length.

REFERENCES

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