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BEST POSSIBLE BOUNDS AND MONOTONICITY OF SEGMENTS OF HARMONIC SERIES (II)

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Abstract. We give an answer to a hypothesis formulated in one of our earlier papers, concerning boundaries and estimates of some segments of harmonic series.

In our previous article [4] we established some monotonicity criteria for the sequences $s(a_n, b_n)$ of the type

$$s(a_n, b_n) := \frac{1}{a_n} + \frac{1}{a_n + 1} + \frac{1}{a_n + 2} + \dots + \frac{1}{b_n},$$

where (a_n) and (b_n) are increasing sequences of positive integers and $a_n < b_n$, $n \in \mathbf{N}$.

Let $A_n := (a_n - 1/2)^2 + 1/12$, $B_n := (b_n + 1/2)^2 + 1/12$, $n \in \mathbf{N}$. Our results from [4] are contained in the following propositions:

Proposition A. *If the sequence (A_n/B_n) is nonincreasing for $n \in [n_1, n_2]$, then $s(a_n, b_n)$ is strictly decreasing for $n \in [n_1, n_2]$, $n_1, n_2 \in \mathbf{N}$.*

Proposition B. *If the sequence $(\frac{B_n - 1/12}{A_n - 1/12})$, i.e. $(\frac{b_n + 1/2}{a_n - 1/2})$ is nondecreasing, then $s(a_n, b_n)$ is strictly increasing for the same $n \in [n_1, n_2]$.*

Proposition C. *If*

$$\frac{B_{n+1}}{A_{n+1}} - \frac{B_n}{A_n} > \frac{B_{n+1} - B_n}{45a_n^2 A_n A_{n+1}}, \quad n \in [n_1, n_2),$$

then the sequence $s(a_n, b_n)$ is strictly increasing for $n \in [n_1, n_2]$.

There are numerous problems concerning boundaries and estimates of some segments of harmonic series (i.e. $s(a_n, b_n)$) for given integer sequences (a_n), (b_n); see for example [1], [2].

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Since the best possible bounds for $s(a_n, b_n)$ are represented by this sequence itself, we can define (as in [2]) the best lower bound of $s(a_n, b_n)$ by $s_* = \inf_n s(a_n, b_n)$, and the best possible upper bound by $s^* = \sup_n s(a_n, b_n)$; so these bounds do not depend on $n \in \mathbf{N}$.

Obviously, the question of the best possible bounds is in close connection with monotonicity of the sequence $s(a_n, b_n)$ and Propositions A–C, cited above, give an efficient tool for solving this problem for wide class of segments of harmonic series.

In [3] we considered the question of the best possible bounds for sequences (a_n), (b_n) being of the form of arithmetic progression, i.e.

$$s_n(p,q,A,B) := s(np+A, nq+B) = \frac{1}{np+A} + \frac{1}{np+A+1} + \cdots + \frac{1}{nq+B},$$

where p, q, A, B are fixed integers and $q > p > 0$, $B \geqslant A-1 \geqslant 0$. There we formulated a generalization of results from [2] and [3] in the following

Hypothesis 1. *Let $m := p(2B+1) - q(2A-1)$. Then the sequence $s_n(p,q,A,B)$ is:*

(a) strictly increasing if $m \leqslant 0$; hence the best possible bounds are

$$s_*(p,q,A,B) = \inf_n s_n(p,q,A,B) = s_1(p,q,A,B);$$
$$s^*(p,q,A,B) = \sup_n s_n(p,q,A,B) = \lim_n s_n(p,q,A,B) = \ln(q/p);$$

(b) strictly decreasing in the case that $m > 0$, so

$$s_*(p,q,A,B) = \ln(q/p); \qquad s^*(p,q,A,B) = s_1(p,q,A,B).$$

In the same article we proved the validity of the cited hypothesis for the cases $A = 1$, $B = 0$, $p, q \in \mathbf{N}$, and $A = 1$, $B = 1$, $p, q \in \mathbf{N}$. We also showed that $s_n(p,q,A,B)$ is monotonous for sufficiently large n under conditions on p, q, A, B, m as defined.

But closer examination shows that part (b) of the hypothesis is not valid, i.e. condition $m > 0$ is necessary but not sufficient for $s_n(p,q,A,B)$ to be strictly decreasing for each n, $n \in \mathbf{N}$. This can be illustrated by the following example.

Let $p = A = 1$, $q = 2B$; then $m = p(2B+1) - q(2A-1) = 1 > 0$, but

$$s_{n+1}(1,2B,1,B) - s_n(1,2B,1,B) = \sum_{s=n+2}^{B(2n+3)} \frac{1}{s} - \sum_{s=n+1}^{B(2n+1)} \frac{1}{s} = \sum_{s=B(2n+1)+1}^{B(2n+3)} \frac{1}{s} - \frac{1}{n+1}$$
$$> \int_{B(2n+1)+1}^{B(2n+3)+1} \frac{dt}{t} - \frac{1}{n+1} = \ln\left(1 + \frac{1}{n+\frac{1}{2}+\frac{1}{2B}}\right) - \frac{1}{n+1} > 0,$$

for B large enough and fixed n, since the inequality $\ln\left(1 + \dfrac{1}{n+\frac{1}{2}}\right) > \dfrac{1}{n+1}$ is satisfied for each n, $n \in \mathbf{N}$.

In this paper we will give definite solution of cited Hypothesis using Propositions A–C, i.e. we will show that part (a) is correct and, under conditions of part (b), we will determine $n_0 = n_0(m, p, q, A)$ such that for $n \leqslant n_0$ the sequence $s_n(p, q, A, B)$ is strictly increasing and for $n \geqslant n_0$ otherwise. From this, the best possible bounds follow immediately.

An interesting consequence of this proposition is the establishment of the best possible bounds for a generalized sequence $s_{f(n)}(p, q, A, B)$:

$$s_{f(n)}(p, q, A, B) := \frac{1}{pf(n) + A} + \frac{1}{pf(n) + A + 1} + \cdots + \frac{1}{qf(n) + B},$$

where $f\colon \mathbf{N} \to \mathbf{N}$ ia any increasing integer function.

An answer to cited Hypothesis 1 (which includes all cases considered in [3]) is given in the next

THEOREM D. *Let A, B, p, q be fixed integers satisfying $B \geqslant A - 1 \geqslant 0$, $q > p \geqslant 1$, and define*

$$m := p(2B + 1) - q(2A - 1); \quad c := \frac{q}{6mp} - \frac{2A - 1}{2p}; \quad C_1 := \left[c - \frac{1}{3}, c + \frac{1}{3}\right];$$
$$C_2 := \left(c + \frac{1}{3}, c + \frac{2}{3}\right); \quad r \in C_1 \cup C_2 \textit{ is an integer and } t := \max(1, r).$$

Considering the sequence $s_n(p, q, A, B)$ defined as above, we have:

PROPOSITION D1. *If $m \leqslant 0$, the sequence $s_n(p, q, A, B)$ is strictly increasing for each n, $n \in \mathbf{N}$; hence the best possible bounds are:*

$$s_*(p, q, A, B) := \inf_n s_n(p, q, A, B) = s_1(p, q, A, B) = \sum_{s=p+A}^{q+B} \frac{1}{s};$$
$$s^*(p, q, A, B) := \sup_n s_n(p, q, A, B) = \lim_n s_n(p, q, A, B) = \ln(q/p).$$

Proof. Checking validity of conditions of cited Proposition B (for $a_n = pn + A$; $b_n = qn + B$), we obtain that $m \leqslant 0$ is equivalent to

$$\frac{nq + B + 1/2}{np + A - 1/2} \geqslant \frac{(n + 1)q + B + 1/2}{(n + 1)p + A - 1/2},$$

i.e. $(\frac{b_n + 1/2}{a_n - 1/2})$ is nondecreasing. Therefore $s_n(p, q, A, B)$ is strictly increasing and conclusions of Proposition D1 follow. ■

PROPOSITION D2. *In the case $m > 0$, the sequence $s_n(p, q, A, B)$ has the maximal term with index n_0 in the sense that it strictly increases for $n \in [1, n_0]$ and strictly decreases for $n \geqslant n_0$. Index n_0 is determined by: (i) $n_0 = t$ if $r \in S_1$; (ii) $n_0 = t$ or $n_0 = t + 1$ if $r \in S_2$.*

Proof. Let $m>0$ and suppose $n_0\geqslant 2$. Considering case (i), for $n\in[1,n_0)$ we have $c-(n+1/2)\geqslant c-(n_0-1/2)\geqslant 1/6$, that is

$$6mp(c-(n+1/2))\geqslant mp>\frac{3m(2A-1)+3mp(2n+1)}{3{,}75(np+A)^2-1},$$

wherefrom $22{,}5(np+A)^2mp(c-(n+1/2))>q$, i.e.

$$45(np+A)^2mpq\left(c-\left(n+\frac{1}{2}\right)\right)\left(n+\frac{1}{2}+\frac{2B+1}{2q}\right)>q((2n+1)q+2B+1),$$

i.e.

$$45(np+A)^2mpq\left(\left(c-\left(n+\frac{1}{2}\right)\right)\left(n+\frac{1}{2}+\frac{2B+1}{2q}+\frac{p}{6mq}\right)+\frac{1}{4}-\frac{1}{36m^2}\right)$$
$$>q((2n+1)q+2B+1).$$

This is (with $a_n=pn+A$, $b_n=qn+B$) exactly

$$45a_n^2(B_{n+1}A_n-B_nA_{n+1})>B_{n+1}-B_n;$$

so we have proved the validity of the conditions of Proposition C. Hence, $s_n(p,q,A,B)$ is strictly increasing for $n\in[1,n_0]$.

Otherwise, for $m>0$ and $n\geqslant n_0$ we have

$$\begin{aligned}B_nA_{n+1}-B_{n+1}A_n&=\left(\left(nq+B+\frac{1}{2}\right)^2+\frac{1}{12}\right)\left(\left((n+1)p+A-\frac{1}{2}\right)^2+\frac{1}{12}\right)-\\&\quad-\left(\left((n+1)q+B+\frac{1}{2}\right)^2+\frac{1}{12}\right)\left(\left(np+A-\frac{1}{2}\right)^2+\frac{1}{12}\right)\\&=mpq\left(\left(n+\frac{1}{2}-c\right)\left(n+\frac{1}{2}+\frac{2B+1}{2q}+\frac{p}{6mq}\right)-\frac{1}{4}+\frac{1}{36m^2}\right)\\&\geqslant mpq\left(\left(n_0+\frac{1}{2}-c\right)\left(n_0+\frac{1}{2}+\frac{2B+1}{2q}+\frac{p}{6mq}\right)-\frac{1}{4}+\frac{1}{36m^2}\right)>0,\end{aligned}$$

since $n_0+\dfrac{1}{2}-c\begin{cases}>1/2; & c<1,\\ \geqslant 1/6; & c\geqslant 1.\end{cases}$

Hence, under the conditions of Proposition A, the sequence $s_n(p,q,A,B)$ is strictly decreasing for $n\geqslant n_0$. It follows that, in this case:

$$s^*(p,q,A,B)=s_{n_0}(p,q,A,B);\quad s_*(p,q,A,B)=\min(s_1(p,q,A,B),\ln(q/p)).$$

What is the exact value of this minimum stays ambiguous and depends how large is the ratio p/A (as could be seen using inequalities from [4]).

The proof of the case (ii) goes on similar lines, so we omit it here. ■

Now we expose a generalized form of Theorem D.

THEOREM E. *Let $f\colon \mathbf{N} \to \mathbf{N}$ be any strictly increasing integer function. Define:*

$$S_{f(n)} = S_{f(n)}(p,q,A,B) := \frac{1}{pf(n)+A} + \frac{1}{pf(n)+A+1} + \cdots + \frac{1}{qf(n)+B},$$

with p,q,A,B,m,c,r,t,n_0 same as in Theorem D. Let

$$S_* = S_*(p,q,A,B) := \inf_n S_{f(n)}, \quad S^* = S^*(p,q,A,B) := \sup_n S_{f(n)}.$$

PROPOSITION E1. *If $m < 0$ then the sequence $S_{f(n)}$ is strictly increasing for each n, $n \in \mathbf{N}$; hence, $S_* = S_{f(1)}$; $S^* = S_{f(\infty)} = \ln(q/p)$.*

PROPOSITION E2. *If $m > 0$, $(S_{f(n)})$ is strictly increasing for $n \in [1, k_0]$, where $k_0 := \max(1, \max_k(f(k) \leqslant n_0))$. So, in this case $S_* = S_{f(1)}$; $S^* = S_{f(k_0)}$.*

PROPOSITION E3. *If $m > 0$, $(S_{f(n)})$ is strictly decreasing for $n \geqslant k_1$, where $k_1 := \min_k(f(k) \geqslant n_0)$; hence: $S_* = S_{f(\infty)} = \ln(q/p)$; $S^* = S_{f(k_1)}$.*

Proof of Theorem E is obvious and immediately follows from Theorem D. For example, validity of Proposition E1 could be proved like this:

Let $f(i) = k_i$, $i \in \mathbf{N}$. Since $f(\cdot)$ is strictly increasing on $\mathbf{N}$, so is the sequence (k_i). According to Proposition D1, for $m \leqslant 0$, $s_n(p,q,A,B)$ is strictly increasing for each n, $n \in \mathbf{N}$. We conclude that $s_{k_1}(\cdot) < s_{k_2}(\cdot) < \cdots < s_{k_n}(\cdot) < \cdots$, i.e. $S_{f(1)} < S_{f(2)} < \cdots < S_{f(n)} < \cdots$.

Propositions E2,3 could be proved in a similar manner. ∎

For an illustration, we take sequences (a_n), (b_n) in the form of geometrical progression, i.e. $a_n = pa^n$, $b_n = qa^n$; $p < q$, $a > 1$; $p,q,a \in \mathbf{N}$.

PROPOSITION F. *The sequence*

$$s(pa^n, qa^n) := \frac{1}{pa^n} + \frac{1}{pa^n+1} + \frac{1}{pa^n+2} + \cdots + \frac{1}{qa^n}$$

is strictly decreasing for each n, $n \in \mathbf{N}$. Hence, $s_* = \lim_n s(\cdot) = \ln(q/p)$; $s^* = s(pa,qa)$.

Proof. Put in Theorem E: $A = p$, $B = q$, $f(n) = a^n - 1$; then: $m := p(2B+1) - q(2A-1) = p+q > 0$; $c < 0$; $n_0 = 1$; so, from Proposition E3 it follows that the considered sequence is strictly decreasing for each n, $n \in \mathbf{N}$, and conclusion follows. ∎

For the sake of completeness we note a possibility to prove (using Proposition A), that the sequence $(s(pa^n, qb^n))$; $a < b$, $pa \leqslant qb$; $a,b,p,q \in \mathbf{N}$; is strictly increasing for each n, $n \in \mathbf{N}$. Hence, in this case $s_* = s(pa,qb)$; $s^* = +\infty$.

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