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SOME ESTIMATES OF THE REMAINDER IN THE EXPRESSIONS FOR THE EIGENVALUE ASYMPTOTICS OF SOME SINGULAR INTEGRAL OPERATORS

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Abstract. Some estimates are given for the remainder in the expressions for eigenvalue asymptotics of operators of the type of Riesz' potential and logarithmic potential. As a corollary an estimate is obtained of the spectral asymptotics of an operator playing a role in the thin airplane wing theory. The method is elementary and it is based on the properties of the singular values of compact operators.

0. Introduction and notation

The general method of Birman and Solomyak [1] gives the first term of the convolution operator's singular values asymptotics if the operator kernels satisfy some conditions. In the general case there is not any known method to determine higher terms of the asymptotics, or at least an estimate of the remainder in the known asymptotic formulas. But in some special cases, it is still possible thanks to the special kernel structure (see [5]).

In this paper some estimates of the remainder in asymptotic formulas for convolution operator's eigenvalues, where the kernels of the operator are $|x-y|^{\alpha-1}$ and $\ln|x-y|$, are obtained using elementary methods.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$ and let A be a compact operator. The singular values of the operator A $(s_n(A))$ are the eigenvalues of the operator $(A^*A)^{1/2}$ (or $(AA^*)^{1/2}$).

Denote by $\lambda_n(A)$ eigenvalues of A in the order of decreasing absolute values, taking into account their multiplicity.

By $\int_a^b m(x,y)\cdot dy$ we denote the integral operator on $L^2(a,b)$ with the kernel $m(x,y)$.

The operator A is a Hilbert-Schmidt one if $\sum_{n\geqslant 1} s_n^2(A) < +\infty$.

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A Hilbert-Schmidt operator $A=\int_a^b m(x,y)\cdot dy$ satisfies the following equality $\sum_{n\geqslant 1} s_n^2(A)=\iint |m(x,y)|^2\,dx\,dy$.

1. Results

THEOREM 1. *Let* $\alpha_0=\dfrac{5-\sqrt{17}}{4}$.

a) *If* $0<\alpha\leqslant\alpha_0$, *then*

$$\lambda_n\left(\int_{-1}^{1}|x-y|^{\alpha-1}\cdot dy\right)=2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha}\cos\frac{\alpha\pi}{2}\,\left(1+O(n^{-\delta})\right),$$

where $\delta=\dfrac{1-2\alpha}{3}$.

b) *If* $\alpha_0\leqslant\alpha<1$, *then*

$$\lambda_n\left(\int_{-1}^{1}|x-y|^{\alpha-1}\cdot dy\right)=2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha}\cos\frac{\alpha\pi}{2}\,\left(1+O(n^{-\delta})\right),$$

where $\delta=\dfrac{3\alpha-2\alpha^2}{2+5\alpha-2\alpha^2}$.

THEOREM 2. $\lambda_n\left(\displaystyle\int_{-1}^{1}-\frac{1}{\pi}\ln|x-y|\cdot dy\right)=\dfrac{2}{n\pi}\left(1+O\left(\dfrac{\sqrt{\ln n}}{n^{1/5}}\right)\right)$.

Consider the operator

$$Hf(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{f'(t)}{x-t}\,dt,$$

whose domain consists of the functions f satisfying the following conditions: $f'\in L^1(-1,1)$, $Hf\in L^2(-1,1)$, $f(-1)=f(1)=0$ [3] (the integral is taken in the sense of principal value). This operator plays an important role in the theory of thin airplane wing.

In [4] M. Kac heuristically deduced the formula

$$\lambda_n(H)=\frac{\pi n}{2}(1+o(1)).$$

In [3] H. M. Hogan and L. A. Sahnovič gave a strong proof of this asymptotic formula. Applying Theorem 2 we give an estimate of the remainder in the asymptotic formula for $\lambda_n(H)$.

THEOREM 3. *The following asymptotic formula holds:*

$$\lambda_n(H)=\frac{\pi n}{2}\left(1+O(n^{-1/11})\right).$$

Remark 1. Since the function

$$\alpha \mapsto \begin{cases} \frac{1-2\alpha}{3}, & 0<\alpha\leqslant\alpha_0, \\ \frac{3\alpha-2\alpha^2}{2+5\alpha-2\alpha^2}, & \alpha_0\leqslant\alpha<1, \end{cases}$$

attains its minimum when $\alpha=\alpha_0$, Theorem 1 implies a more rough but unified estimate for $0<\alpha<1$, i.e.

$$\lambda_n\left(\int_{-1}^{1}|x-y|^{\alpha-1}\cdot\,dy\right)=2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha}\cos\frac{\alpha\pi}{2}\left(1+O(n^{\frac{2\alpha_0-1}{3}})\right).$$

Remark 2. Part b) in Theorem 1 holds in the case $0<\alpha<1$ but in the interval $(0,\alpha_0)$ the better estimate is given by a). That is the reason why these two cases are separated.

2. Proofs

To prove the previous Theorems we need some Lemmas.

Lemma 1. *If A and B are compact operators on a Hilbert space $\mathcal{H}$ such that*

$$\begin{aligned} &s_n(A)=a\cdot n^{-\alpha}+O(n^{-\beta}), && a>0,\ \alpha<\beta<\alpha+1, \\ &s_n(B)=O(n^{-\beta_1}), && \beta_1\geqslant\frac{\beta}{\alpha+1-\beta}, \end{aligned} \tag{1}$$

then $s_n(A+B)=a\cdot n^{-\alpha}+O(n^{-\beta})$.

Proof. Observe that from the assumptions of Lemma 1 it follows $\beta_1>\beta$.

For $n\in\mathbf{N}$ let $k=k(n)=\left[n^{1-\frac{\beta}{\beta_1}}\right]-1$, $m=m(n)=\left[n^{\frac{\beta}{\beta_1}}\right]$ ($[x]$ denotes the greatest integer not exceeding x). Then $n=(k+1)m+j$, where $j=j(n)$ is a nonnegative integer and $j(n)\leqslant n^{\frac{\beta}{\beta_1}}+n^{1-\frac{\beta}{\beta_1}}$. From the properties of singular values of the sum of two operators [2] it follows $s_{(k+1)m+j}(A+B)\leqslant s_{km+j}(A)+s_{m+1}(B)$ and hence

$$\begin{aligned} &n^{\beta}\left(s_n(A+B)-\frac{a}{n^{\alpha}}\right)\leqslant \\ &n^{\beta}\left(s_{km+j}(A)-\frac{a}{(km+j)^{\alpha}}\right)+n^{\beta}\left(\frac{a}{(km+j)^{\alpha}}-\frac{a}{((k+1)m+j)^{\alpha}}\right)+n^{\beta}s_{m+1}(B). \end{aligned} \tag{2}$$

The assumptions (1) and the properties of the sequences $k(n)$, $m(n)$ and $j(n)$ give

$$\begin{aligned} \overline{\lim}\, n^{\beta}\left(s_{km+j}(A)-\frac{a}{(km+j)^{\alpha}}\right)&<+\infty, \\ \overline{\lim}\, n^{\beta}s_{m+1}(B)&<+\infty, \end{aligned} \tag{3}$$

and since $\beta_1 \geqslant \frac{\beta}{\alpha+1-\beta}$, we obtain $\overline{\lim}\, n^\beta \left(\frac{1}{(km+j)^\alpha} - \frac{1}{((k+1)m+j)^\alpha}\right) < +\infty$. Combining (2) and (3) we get $\overline{\lim}\, n^\beta \left(s_n(A+B) - \dfrac{a}{n^\alpha}\right) < +\infty$. Since $s_{(k+1)m+j}(A) = s_{(k+1)m+j}(A+B-B) \leqslant s_{(k+1)m+j}(A+B) + s_{m+1}(B)$, i.e. $s_{km+j}(A+B) \geqslant s_{(k+1)m+j}(A) - s_{m+1}(B)$.

In the similar way we conclude $\underline{\lim}\, n^\beta \left(s_n(A+B) - \dfrac{a}{n^\alpha}\right) > -\infty$, and so $s_n(A+B) = an^{-\alpha} + O(n^{-\beta})$. ■

Remark 3. The proof of the following statement is carried out in the same way.

(a) If A and B are compact operators such that

$$s_n(A) = an^{-\alpha} + O(n^{-\beta}L(n)), \quad s_n(B) = O(n^{-\beta_1}L(n))$$

and if L is a slowly varying function, $L(x) \to +\infty$ when $x \to +\infty$, $\alpha < \beta < \alpha + 1$, $\beta_1 \geqslant \beta/(\alpha+1-\beta)$, then

$$s_n(A+B) = an^{-\alpha} + O(n^{-\beta}L(n)).$$

(b) From Lemma 1 it follows: If A and B are compact operators such that

$$s_n(A) = an^{-\alpha}, \quad a > 0; \qquad s_n(B) = O(n^{-\beta_1}), \quad \beta_1 > \alpha,$$

then $s_n(A+B) = an^{-\alpha} + O\big(n^{-\frac{\beta_1}{1+\beta_1}(1+\alpha)}\big)$.

Lemma 2. *Let $T = \int_0^1 (x+y)^{\alpha-1} \cdot dy$. If $\alpha > 0$, then*

$$s_n(T) = o(n^{-1/2}) \quad (= O(n^{-1/2})). \tag{4}$$

If $0 < \alpha < 1$, then

$$s_n(T) = O\big(n^{-\alpha-\alpha(\frac{3}{2}-\alpha)}\big). \tag{5}$$

(Observe that for small (large) values of α a better estimate is given by (4) ((5)), respectively.)

Proof. From $\int_0^1\int_0^1 (x+y)^{2\alpha-2}\,dx\,dy < +\infty$ $(\alpha > 0)$ it follows that T is a Hilbert-Schmidt operator and we have

$$s_n(T) = o(n^{-1/2}) = O(n^{-1/2}).$$

Let us now prove the asymptotic relation (5). Let $\delta > 0$ be a fixed number $(\delta < 1)$. First, we prove

$$s_n\left(\int_0^\delta (x+y)^{\alpha-1} \cdot dy\right) \leqslant C_1 \frac{\delta^\alpha}{n^\alpha} \qquad (0 < \alpha < 1), \tag{6}$$

where the constant C_1 does not depend on n and δ.

Let $J^\alpha\colon L^2(0,\delta) \to L^2(0,\delta)$ be the fractional integral operator defined by

$$J^\alpha f(x) = \frac{1}{\Gamma(x)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy$$

and let the operator $C\colon L^2(0,\delta) \to L^2(0,\delta)$ be defined by

$$Cf(x) = \int_0^x (x+y)^{\alpha-1} f(y)\, dy.$$

According to the formula for fractional partial integration [6] we get

$$Cf(x) = \int_0^x (J^\alpha f)(y) \left(-\frac{1}{\Gamma(1-\alpha)} \frac{d}{dy} \int_y^x (t-y)^{-\alpha} (x+t)^{\alpha-1}\, dt \right) dy,$$

i.e.

$$C = -\frac{1}{\Gamma(1-\alpha)} M J^\alpha, \tag{7}$$

where $M\colon L^2(0,\delta) \to L^2(0,\delta)$ is a linear operator defined by

$$Mf(x) = \int_0^x H(x,y) f(y)\, dy \quad \text{and} \quad H(x,y) = \frac{d}{dy} \int_y^x (x+t)^{\alpha-1} (t-y)^{-\alpha}\, dt.$$

The function H can be represented in the form (for $y > x$)

$$H(x,y) = -(2x)^{\alpha-1}(x-y)^{-\alpha} + (\alpha-1) y^{-1} \Phi\left(\frac{x}{y}\right),$$

where $\Phi(s) = \int_1^s (r-1)^{-\alpha} (r+s)^{\alpha-2}\, dr$, $s > 1$.

Now, we prove that the integral operator with the kernel $H(x,y)\theta(x-y)$ is bounded on $L^2(0,+\infty)$. ($\theta(x) = 0$ for $x < 0$, $\theta(x) = 1$ for $x \geqslant 0$). Since the function $H(x,y)\theta(x-y)$ is homogenous of order -1, according to the Hardy-Littlewood-Polya inequality [6] it is enough to prove that $k = \int_0^\infty x^{-1/2} |\theta(x-1) H(x,1)|\, dx < +\infty$, i.e.

$$k = \int_1^\infty x^{-1/2} |-(2x)^{\alpha-1}(x-1)^{-\alpha} + (\alpha-1)\Phi(x)|\, dx < +\infty.$$

Since $0 < \alpha < 1$, from asymptotic behavior of the function Φ when $x \to 1$ and $x \to +\infty$ it follows that the previous integral converges. Therefore, we have

$$\|M\| \leqslant k. \tag{8}$$

Having in mind that $s_n(J^\alpha) \leqslant \mathrm{const} \cdot \delta^\alpha / n^\alpha$ (const. does not depend on n and δ), from (7) and (8) and from the properties of singular values of the sum of two operators it follows $s_{2n}(C) \leqslant \mathrm{const} \cdot \delta^\alpha / n^\alpha$, i.e. $s_n(C) \leqslant \mathrm{const} \cdot \delta^\alpha / n^\alpha$ (const. does not depend on n and δ). From

$$s_{2n}\left(\int_0^\delta (x+y)^{\alpha-1} \cdot dy \right) = s_{2n}(C + C^*) \leqslant 2 s_n(C) \leqslant 2\, \mathrm{const}\, \frac{\delta^\alpha}{n^\alpha}$$

we obtain the estimate (6).

Now, we prove that if the operator $D_\delta\colon L^2(\delta,1)\to L^2(0,1)$ is defined by

$$D_\delta f(x)=\int_\delta^1 (x+y)^{\alpha-1} f(y)\,dy,$$

then the following estimate

$$s_n(D_\delta)\leqslant C_2\,\frac{\delta^{\alpha-1}}{n^{3/2}} \tag{9}$$

holds (C_2 does not depend on n and δ).

Let $\mathcal{L}_0=\{\,f\in L^2(\delta,1)\;:\;\int_\delta^1 f(t)\,dt=0\,\}$ and let $Q\colon L^2(\delta,1)\to\mathcal{L}_0$ be the orthoprojector. Consider the operator $A_0=D_\delta Q\colon L^2(\delta,1)\to L^2(0,1)$. Since the range of $D_\delta-A_0=D_\delta(I-Q)$ is one (I is the unit operator), we have [2]

$$s_{n-1}(A_0)\leqslant s_n(D_\delta)\leqslant s_{n+1}(A_0). \tag{10}$$

As in [2] we prove that $A_0=D'_\delta J'$, where $D'_\delta,\ J'\colon L^2(\delta,1)\to L^2(0,1)$ are the operators defined by

$$D'_\delta f(x)=(1-\alpha)\int_\delta^1 (x+y)^{\alpha-2} f(y)\,dy,\qquad J'f(x)=\int_\delta^x f(t)\,dt.$$

Since D'_δ is a Hilbert-Schmidt operator we have

$$ns_n^2(D'_\delta)\leqslant(1-\alpha)^2\int_0^1 dx\int_\delta^1 (x+y)^{2\alpha-4}\,dy,$$

and hence $s_n(D'_\delta)\leqslant \mathrm{const}\,\delta^{\alpha-1}/\sqrt{n}$ (const. does not depend on n and δ). Using (10) and

$$s_{2n}(A_0)\leqslant s_n(D'_\delta)\cdot s_n(J')\leqslant \mathrm{const}\,\frac{\delta^{\alpha-1}}{n^{3/2}}$$

(const. does not depend on n and δ) we obtain (9).

Let $P_\delta\colon L^2(0,1)\to L^2(0,1)$ be the orthoprojector defined by

$$P_\delta f(x)=\chi_{[0,\delta]}(x)f(x)$$

(by χ_S we denote the characteristic function of the set S). Then

$$T=T(I-P_\delta)+(I-P_\delta)TP_\delta+P_\delta TP_\delta$$

and we have $s_{3n}(T)\leqslant s_n(T(I-P_\delta))+s_n((I-P_\delta)TP_\delta)+s_n(P_\delta TP_\delta)$. Combining this with (6) and (9) we obtain

$$s_{3n}(T)\leqslant 2C_2\,\frac{\delta^{\alpha-1}}{n^{3/2}}+C_1\,\frac{\delta^\alpha}{n^\alpha}$$

(C_1, C_2 do not depend on n and δ). This inequality holds for each $\delta\in(0,1)$. Substituting $\delta=n^{\alpha-\frac{3}{2}}$ we get

$$s_{3n}(T)\leqslant(2C_2+C_1)n^{-\alpha-\alpha(\frac{3}{2}-\alpha)}$$

and therefore $s_n(T)=O\big(n^{-\alpha-\alpha(\frac{3}{2}-\alpha)}\big)$. ∎

LEMMA 3. $s_n\left(\int_0^1 \ln(x+y)\ \cdot\ dy\right) = O\left(\frac{\sqrt{\ln n}}{n^{3/2}}\right)$.

Proof. Let $V\colon L^2(0,1) \to L^2(0,1)$ denote the operator defined by

$$Vf(x) = \int_0^1 \ln(x+y) f(y)\, dy.$$

The operator V can be expressed in the form

$$V = V(I-P_\delta) + (I-P_\delta)VP_\delta + P_\delta V P_\delta. \tag{11}$$

In the similar way as for the estimates (6) and (9) we get

$$s_n(P_\delta V P_\delta) \leqslant C_3\, \frac{\delta}{n}, \qquad s_n(V(I-P_\delta)) \leqslant C_4\, \frac{\sqrt{-\ln\delta}}{n^{3/2}} \tag{12}$$

(C_3, C_4 do not depend on n and δ).

From (11) and (12) and the properties of singular values of the sum of two operators we obtain

$$s_{3n}(V) \leqslant 2C_4\, \frac{\sqrt{-\ln\delta}}{n^{3/2}} + C_3\, \frac{\delta}{n}.$$

This inequality holds for each $\delta \in (0,1)$. If we put $\delta = \sqrt{(\ln n)/n}$, then we have $s_{3n}(V) \leqslant C_5(\sqrt{\ln n})/n^{3/2}$ (where the constant C_5 does not depend on n) and therefore

$$s_n(V) = O\left(\frac{\sqrt{\ln n}}{n^{3/2}}\right). \quad \blacksquare$$

Proof of Theorem 1. Let $k(x) = |x|^{\alpha-1}$ $(0<\alpha<1)$ and let

$$\mathcal{A}(x,y) = \sum_{n=-\infty}^{\infty} (k(x-y+4n) - k(x+y+4n+2)). \tag{13}$$

By direct calculation it can be proved that

$$\int_{-1}^1 \mathcal{A}(x,y)\varphi_n(y)\, dy = \hat{k}\left(\frac{n\pi}{2}\right)\varphi_n(x); \quad \varphi_n(x) = \sin\frac{n\pi(1+x)}{2}, \quad n = 1,2,\ldots,$$

where $\hat{k}(\xi) = \int_{-\infty}^{\infty} e^{it\xi}|t|^{\alpha-1}\, dt = 2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\,|\xi|^{-\alpha}$ and $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal base of $L^2(-1,1)$.

Let A_1, A_2, A_3, A_4 be integral operators acting on $L^2(-1,1)$ with kernels

$$\mathcal{A}_{x,y}, \quad k(x-y-4) + \sum_{\substack{n\neq 0\\ n\neq -1}} (k(x-y+4n) - k(x+y+4n+2)),$$
$$-k(x+y+2) \quad \text{and} \quad -k(x+y-2),$$

respectively. Then we have

$$\int_{-1}^{1} |x-y|^{\alpha-1} \cdot dy = A_1 - A_2 - A_3 - A_4. \tag{14}$$

From (13) it follows

$$s_n(A_1) = 2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha} \cos\frac{\alpha\pi}{2}. \tag{15}$$

Let $0 < \alpha < 1/2$. According to Lemma 2 (relation (4)) we have $s_n(A_3) = O(n^{-1/2})$, $s_n(A_4) = O(n^{-1/2})$. Since the kernel of the operator A_2 is a smooth function on $[-1,1]^2$, using the properties of singular values of the sum of two operaors [2] we obtain $s_n(A_2 + A_3 + A_4) = O(n^{-1/2})$. From this equality, (14), (15) and from Remark 3(b) it follows

$$s_n\left(\int_{-1}^{1} |x-y|^{\alpha-1} \cdot dy\right) = 2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha} \cos\frac{\alpha\pi}{2}\left(1 + O(n^{\frac{2\alpha-1}{3}})\right).$$

Since the operator $\int_{-1}^{1} |x-y|^{\alpha-1} \cdot dy$ $(0 < \alpha < 1)$ is positive, for $0 < \alpha < 1/2$ we obtain

$$\lambda_n\left(\int_{-1}^{1} |x-y|^{\alpha-1} \cdot dy\right) = 2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha} \cos\frac{\alpha\pi}{2}\left(1 + O(n^{\frac{2\alpha-1}{3}})\right). \tag{16}$$

Suppose now $0 < \alpha < 1$. According to Lemma 2 (relation (5)) we have $s_n(A_3 + A_4) = O(n^{-\alpha-\alpha(\frac{3}{2}-\alpha)})$ and therefore, using the same line of arguments as before, we have

$$s_n(A_2 + A_3 + A_4) = O\left(n^{-\alpha-\alpha(\frac{3}{2}-\alpha)}\right).$$

From (14), (15) and Remark 3(b) it follows

$$\lambda_n\left(\int_{-1}^{1} |x-y|^{\alpha-1} \cdot dy\right) = 2\Gamma(\alpha)\left(\frac{2}{n\pi}\right)^{\alpha} \cos\frac{\alpha\pi}{2}\left(1 + O(n^{-\delta})\right), \tag{17}$$

where $\delta = \dfrac{3\alpha - 2\alpha^2}{2 + 5\alpha - 2\alpha^2}$.

Observe that for $0 < \alpha \leqslant \alpha_0$ $(\alpha_0 \leqslant \alpha < 1)$ the better estimate is given by (16) ((17)), respectively. ■

Proof of Theorem 2. If $G_1(t) = \frac{1}{\pi} K_0(|t|)$ (K_0 is Mc-Donald's function) then

$$\int_{-\infty}^{\infty} G_1(t) e^{i\xi t}\, dt = (1 + \xi^2)^{-1/2}.$$

It is well known that the function K_0 decreases exponentially and

$$K_0(x) = (-\ln x) A(x) + B(x), \tag{18}$$

where A and B are even entire functions and $A(0) = 1$.

Let

$$R(x,y)=\sum_{n=-\infty}^{\infty}(G_1(x-y+4n)-G_1(x+y+4n+2)).$$

By direct calculation we obtain

$$\int_{-1}^{1}R(x,y)\varphi_n(y)\,dy=\hat{G}_1\left(\frac{n\pi}{2}\right)\varphi_n(x)=\left(1+\frac{n^2\pi^2}{4}\right)^{-1/2}\varphi_n(x)$$

($\varphi_n(x)=\sin\frac{n\pi(1+x)}{2}$, $n\in\mathbf{N}$), which implies that $\int_{-1}^{1}R(x,y)\cdot dy$ is positive. Using (18) and $\left(1+\frac{n^2\pi^2}{4}\right)^{-1/2}=\frac{2}{n\pi}(1+O(\sqrt{\ln n}/n^{3/2}))$ and applying technique from Theorem 1, taking into account Remark 3(a) we obtain

$$\lambda_n\left(\int_{-1}^{1}-\frac{1}{\pi}\ln|x-y|\cdot dy\right)=s_n\left(\int_{-1}^{1}-\frac{1}{\pi}\ln|x-y|\cdot dy\right)=\frac{2}{n\pi}\left(1+O\left(\frac{\sqrt{\ln n}}{n^{1/5}}\right)\right).\ \blacksquare$$

Proof of Theorem 3. It is well known [3] that the operator $B=H^{-1}$ is positive and has the form $Bf(x)=\int_{-1}^{1}b(x,t)f(t)\,dt$, where

$$b(x,t)=-\frac{1}{\pi}\ln|x-t|+\frac{1}{\pi}\ln|1-tx+\sqrt{(1-t^2)(1-x^2)}|.$$

By the method from Lemma 2 and the paper [3] we get

$$s_n\left(\int_{-1}^{1}\frac{1}{\pi}\ln|1-tx+\sqrt{(1-t^2)(1-x^2)}|\cdot dt\right)=O(n^{-6/5}).$$

From Theorem 2 it follows

$$s_n\left(\int_{-1}^{1}-\frac{1}{\pi}\ln|x-t|\cdot dt\right)=\frac{2}{n\pi}\left(1+O(n^{-1/11})\right).$$

Applying Lemma 1 (Remark 3(a)) from the previous relation and (19) it follows $s_n(B)=\frac{2}{n\pi}\left(1+O(n^{-1/11})\right)$, i.e.

$$\lambda_n(H)=\frac{\pi n}{2}\left(1+O(n^{-1/11})\right).\quad\blacksquare$$

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