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ON CERTAIN CLASS OF RATIONAL FUNCTIONS WHOSE DERIVATIVES HAVE POSITIVE REAL PARTS

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Abstract. For a certain class of rational functions we give a sufficient condition such that their derivatives have positive real parts.

1. Introduction and preliminaries

Let A denote the class of functions f analytic in the unit disc $U = \{\, z : |z| < 1 \,\}$ with $f(0) = f'(0) - 1 = 0$. Let R denote the subclass of A for which $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$. It is well known that the previous condition implies univalence of functions $f \in A$.

In their paper [3] Ozaki and Nunokawa proved the following criterion for univalence in U.

Theorem A. *If $f \in A$ and*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U, \tag{1}$$

then f is univalent in U.

But the condition (1) doesn't imply $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$. For example, for the function

$$f(z) = \frac{z}{(1 - z\sqrt{k})^2}, \qquad 0 < k \leqslant 1,$$

we have that

$$f'(z) = \frac{1 + z\sqrt{k}}{(1 - z\sqrt{k})^3}; \qquad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = |-kz^2| < k, \quad z \in U;$$

and $\operatorname{Re}\{f'(i)\} = \operatorname{Re}\{(1 - 6k + k^2)/(1 + k)^3\} < 0$, for $0.17157\ldots = 3 - 2\sqrt{2} < k \leqslant 1$.

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In that sense we have just showed that for $3-2\sqrt{2} < k \leqslant 1$ the condition

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < k, \quad z \in U \tag{2}$$

does not imply that $f \in R$.

In this paper we give some additional conditions to (2) such that they provide $f \in R$. We apply these results to a certain class of rational functions.

We also need the next lemma.

LEMMA 1. [1] *Let Ω be a subset of the complex plane $\mathbf{C}$ and suppose that the function $\psi\colon \mathbf{C}^2 \times U \to \mathbf{C}$ satisfies the conditions $\psi(ix, y; z) \notin \Omega$, for all real x, $y \leqslant -(1+x^2)/2$ and all $z \in U$. If the function p is analytic in U, $p(0) = 1$ and $\psi[p(z), zp'(z); z] \in \Omega$, then* $\operatorname{Re}\{p(z)\} > 0$.

2. Results and consequences

First, we prove

THEOREM 1. *Let $f \in A$ satisfy the condition (2) and the condition*

$$\left|\arg \frac{z}{f(z)}\right| \leqslant \frac{1}{2} \arctan \frac{\sqrt{1-k^2}}{k}, \qquad z \in U, \tag{3}$$

for some $0 < k \leqslant 1$. Then $f \in R$.

Proof. If we put $f'(z) = p(z)$ and $(z/f(z))^2 = g(z)$, then by the conditions (2) and (3) of the theorem we obtain

$$|g(z)p(z) - 1| < k \quad \text{and} \quad |\arg g(z)| = 2\left|\arg \frac{z}{f(z)}\right| \leqslant \arctan \frac{\sqrt{1-k^2}}{k}, \tag{4}$$

for some $0 < k \leqslant 1$. Also, if we put $g(z) = u + iv$, then by (4) we get $v^2 \leqslant \frac{1-k^2}{k^2} u^2$ and so

$$|g(z)(ix) - 1|^2 - k^2 = |(u+iv)(ix) - 1| - k^2 = (u^2+v^2)x^2 + 2vx + 1 - k^2 \geqslant 0.$$

By Lemma 1 we conclude that $\operatorname{Re}\{p(z)\} = \operatorname{Re}\{f'(z)\} > 0$, $z \in U$, i.e. $f \in R$. ∎

From Theorem 1 we easily obtain the following

COROLLARY 1. *Let $f \in A$ satisfy the condition (2) and let*

$$\left|\frac{z}{f(z)} - 1\right| \leqslant \sqrt{\frac{1-k}{2}}, \qquad z \in U,$$

for some $0 < k \leqslant 1$. Then $f \in R$.

In the next theorem we give an application to a certain class of rational functions.

THEOREM 2. *Let*

$$f(z) = z\left\{1 + \sum_{n=1}^{\infty} b_n z^n\right\}^{-1} \tag{5}$$

and let

$$\sum_{n=2}^{\infty}(n-1)|b_n| \leqslant k \quad \textit{and} \quad \sum_{n=1}^{\infty}|b_n| \leqslant \sqrt{\frac{1-k}{2}}, \tag{6}$$

for some $0 \leqslant k \leqslant 1$. *Then* $f \in R$.

Proof. First let us suppose that the function (5) satisfies (6) and let $0 < k < 1$ and $\sum_{n=2}^{\infty}|b_n| > 0$. Then we have

$$\begin{aligned}\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| &= \left|-z^2\left[\frac{1}{f(z)} - \frac{1}{z}\right]'\right| \leqslant \left|\left[\frac{1}{f(z)} - \frac{1}{z}\right]'\right| \\ &= \left|\sum_{n=2}^{\infty}(n-1)b_n z^{n-2}\right| < \sum_{n=2}^{\infty}(n-1)|b_n| \leqslant k, \quad z \in U.\end{aligned}$$

and

$$\left|\frac{z}{f(z)} - 1\right| = \left|\sum_{n=1}^{\infty} b_n z^{n-1}\right| \leqslant \sum_{n=1}^{\infty}|b_n| \leqslant \sqrt{\frac{1-k}{2}}, \quad z \in U.$$

Hence from Corollary 1 we conclude that $f \in R$.

The other cases are simple. Namely, for $k = 0$ or for $\sum_{n=2}^{\infty}|b_n| = 0$ we get $b_2 = b_3 = \cdots = 0$ with $|b_1| \leqslant \sqrt{2}/2$, and the function (5) becomes $f(z) = z/(1 + b_1 z)$. For this function we can obtain that $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$, for $|b_1| \leqslant \sqrt{2}/2$, i.e. that $f \in R$. For $k = 1$ we obtain the function $f(z) = z$, which also belongs to R. ■

Finally we have

THEOREM 3. *Let the function* f *be given by (5) and let*

$$\sum_{n=2}^{\infty}(n-1)|b_n| + 2\left(\sum_{n=1}^{\infty}|b_n|\right)^2 \leqslant 1. \tag{7}$$

Then $f \in R$.

Proof. Let us put $\sum_{n=2}^{\infty}(n-1)|b_n| = k$. Then from (7) we have that $0 \leqslant k \leqslant 1$ and $\sum_{n=1}^{\infty}|b_n| \leqslant \sqrt{(1-k)/2}$, which by Theorem 2 means that $f \in R$. ■

EXAMPLE. For the function $f(z) = z\left\{1 + \sum_{n=1}^{\infty}\frac{z^n}{2^{n+1}}\right\}^{-1}$ we have that $b_n = 2^{-(n+1)}$ and $\sum_{n=2}^{\infty}(n-1)|b_n| + 2\left(\sum_{n=1}^{\infty}|b_n|\right)^2 = 1$. By the previuous theorem we conclude that this function belongs to R, i.e. it is univalent with $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$.

We note that the method given here is the same as in [2].

REFERENCES

[**1**] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 167–171.

[**2**] M. Obradović, *Starlikeness and certain class of rational functions*, to appear

[**3**] S. Ozaki, M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (2) (1972), 392–394.

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