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ON HB-FLAT HYPERBOLIC KAEHLERIAN SPACES

Nevena Pušić

Abstract. We consider a hyperbolic Kaehlerian space with vanishing conformal invariant. We prove a theorem which is fully analogous to results for Riemannian and Kaehlerian spaces with vanishing conformal invariants. Also, we prove two theorems which are valid in some special cases.

1. Preliminaries

If an $n(=2m)$-dimensional pseudo-Riemannian space M_n with metric (g_{ij}) is equipped with a non-degenerate structure tensor (F^i_j) which satisfies the following conditions

$$\nabla_k F^i_j = 0 \tag{1.1}$$

$$F^i_j F^j_k = \delta^i_k \tag{1.2}$$

$$F_{jk} = F^i_j g_{ik} = -F_{kj} \tag{1.3}$$

then the space M_n is called a hyperbolic Kaehlerian space.

As we have mentioned in the paper [2], the nondegenerate structure has n (the dimension of the space is $n = 2m$) linearly independent eigenvectors in the tangent space. In the paper [2], we also proved

PROPOSITION 1. *(A) Every vector in the tangent space of a hyperbolic Kaehlerian space is transformed by the structure into an orthogonal vector.*

(B) The scalar square of a vector-original is opposite to the scalar square of the vector-image.

In accordance to the Proposition 1, eigenvectors of the structure are isotropic (null-vectors). As the structure has n linearly independent eigenvectors, there exists a basis of the tangent space of a hyperbolic Kaehlerian space where these isotropic

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vectors serve as basic vector fields. In such a basis, metric tensor is hybrid and the structure tensor is pure. Covariant structure tensor is also hybrid. Using this coordinate system, we can show that a hyperbolic Kaehlerian space admits isotropic vector fields which are not eigen for the structure. Moreover, such a coordinate system shows us that a hyperbolic Kaehlerian space is divided very naturally into two totally gedesic subspaces of equal dimension. Such a basis is called a separated besis. Later in this paper, we shall construct a separated basis effectively. Also, according to Propostion 1(B), there exist vectors of positive scalar square (space-like vectors) and vectors of negative scalar square (time-like vectors). Space-like vectors may serve as a domain for the involution (F^i_j) and its co-domain will be the set of time-like vectors. We may choose such a basis; then the metric tensor will be a pure tensor of signature (m, m) and (F^i_j) will be a hybrid tensor. Such a basis is called an adapted basis. Sometimes the adapted basis enables significant help in calculations.

In the article [2], we have investigated some properties of a hyperbolic Kaehlerian space. Among other properties, we investigated a conformal connection (as there cannot be a conformal transformation naturally introduced) and we found a tensor which is an invariant for all conformal connections on a hyperbolic Kaehlerian space:

$$\begin{aligned}HB^i_{jkl} = K^i_{jkl} - \frac{1}{n+4}[&\delta^i_l K_{kj} - \delta^i_k K_{lj} + g_{kj}K^i_l - g_{lj}K^i_k \\ &+ F^i_l S_{kj} - F^i_k S_{lj} + F_{kj}S^i_l - F_{lj}S^i_k + 2S^i_j F_{kl} + 2S_{kl}F^i_j \\ &- \frac{K}{n+2}(\delta^i_l g_{kj} - \delta^i_k g_{lj} + F_{lj}F^i_k - F^i_l F_{kj} - 2F^i_j F_{kl})]. \end{aligned} \tag{1.4}$$

By K^i_{jkl} we denote curvature tensor of Levi-Civita connection for the metric (g_{ij}), by K_{kj} – the corresponding Ricci tensor and by K the corresponding scalar curvature. Also, there holds

$$S_{lj} = K_{la}F^a_j. \tag{1.5}$$

In the paper [2] we proved that the tensor S_{lj} is skew-symmetric. The tensor HB is a curvature-like tensor and in [2] we proved the following algebraic properties of this tensor:

$$\begin{gathered}\text{(a) } HB_{ijkl} = -HB_{ijlk} \\ \text{(b) } HB_{ijkl} = -HB_{jikl} \\ \text{(c) } HB_{ijkl} = HB_{klij} \\ \text{(d) } HB_{ijkl} + HB_{iklj} + HB_{iljk} = 0 \\ \text{(e) } HB^t_{jkt} = 0 \\ \text{(f) } HB^i_{tkl}F^t_j - HB^t_{jkl}F^i_t = 0.\end{gathered} \tag{1.6}$$

We call the tensor HB the Bochner curvarure tensor of a hyperbolic Kaehlerian space, for the sake of two analogies: it looks much like the Bochner tensor (of the Kaehlerian space) and the Bochner tensor is, in some sense, an invariant tensor of conformal connections in a Kaehlerian space (K. Yano, [4]).

In this paper, we are going to investigate a hyperbolic Kaehlerian space with vanishing HB-tensor.

2. HB-flat hyperbolic Kaehlerian space with almost constant curvature

We shall give

DEFINITION 1. If the curvature tensor of a hyperbolic Kaehlerian space can be expressed in the following way

$$K_{ijkl} = \frac{K}{n(n+2)}(g_{li}g_{kj} - g_{ki}g_{lj} - F_{li}F_{kj} + F_{ki}F_{lj} - 2F_{kl}F_{ji}), \tag{2.1}$$

then such a hyperbolic Kaehlerian space is said to be a space of almost constant curvature.

Later, when we construct a separated coordinate system, we shall explain the fact of a hyperbolic Kaehlerian space being of almost constant curvature.

Now we shall prove

LEMMA 1. *If a hyperbolic Kaehlerian space with vanishing HB-curvature tensor is an Einstein space, then it is a space of almost constant curvature.*

Proof. Suppose that our hyperbolic Kaehlerian space is an Einstein space, what means

$$K_{ij} = \frac{K}{n}g_{ij}. \tag{2.2}$$

Then, in accordance to the definition of the tensor S_{ij},

$$S_{ij} = -\frac{K}{n}F_{ij}. \tag{2.3}$$

If we substitute (2.2) and (2.3) into the expression for HB-curvature tensor and if we take into accout that the space is HB-flat, then (2.1) holds.

As the space is an Einstein space, its scalar curvature is a global constant and, in accordance to the Definition 1, the space is a space of almost constant curvature. ■

In order to transfer our calculations into the separated coordinate system and to work in it, we are going to prove the following two lemmas:

LEMMA 2. *For two eigenvectors of the structure tensor in a hyperbolic Kaehlerian space there holds that either corresponding eigenvalues are mutually opposite or the eigenvectors are mutually orthogonal.*

Proof. Suppose that u and v are two eigenvectors for the structure, with corresponding eigenvalues λ and κ respectively. Then

$$u_a v^a = u_j v_k g^{jk} = \frac{1}{\kappa}u_j F^s_k v_s g^{jk} = \frac{1}{\kappa}u_j v_s F^{js} = -\frac{1}{\kappa}v_s u_j F^{sj} = -\frac{\lambda}{\kappa}u_s v^s,$$

whence

$$u_a v^a (1 + \frac{\lambda}{\kappa}) = 0, \tag{2.4}$$

what proves the statement. ■

LEMMA 3. *If the vector u is an eigenvector for the structure on a hyperbolic Kaehlerian space, then Fu is also an eigenvector of the structure. Then the only eigenvalues of the structure are* $+1$ *and* -1.

Proof.

$$v = Fu, v_i = F_i^a u_a = \lambda u_i,$$
$$F_j^i v_i = F_j^i F_i^a u_a = u_j = \lambda^2 u_j = \lambda v_j,$$

what proves the statement. ■

Now we can construct the separated coordinate system effectively. First $n = \frac{m}{2}$ places take those eigenvectors of the structure which correspondto the eigenvalue 1. Those are vectors $v_1, v_2, \dots, v_m$. The basic vectors $v_{m+1}, \dots, v_n$ are eigenvectors with eigenvalue -1. Every v_{m+i} we mark by $v_{\bar{i}}$. Subspaces of the tangent space of a hyperbolic Kaehlerian space V_m and $\bar{V}_m$ generated by first m or the last m of basic vectors are invariant subspaces of the structure. On each of them, by the sake of isotropy and by the Lemma 2, every basic vector is orthogonal to every basic vector (including itself) and integral manifolds of vector spaces V_m and $\bar{V}_m$ are totally geodesic.

In the separated coordinate system, the metric takes the form

$$\begin{bmatrix} 0 & g_{\alpha\bar{\beta}} \\ g_{\bar{\alpha}\beta} & 0 \end{bmatrix} \tag{2.5}$$

in blocks, where $\alpha, \beta = 1, \dots, m$ and $\bar{\gamma} = m + \gamma$. The structure looks this way

$$\begin{bmatrix} E_m & 0 \\ 0 & -E_m \end{bmatrix} \tag{2.6}$$

and the covariant structure has the form

$$\begin{bmatrix} 0 & g_{\alpha\bar{\beta}} \\ -g_{\bar{\alpha}\beta} & \end{bmatrix} \tag{2.7}$$

The metric tensor is, of course, symmetric.

By looks of metric tensor, only those components of Levi-Civita connection which are of the form $\Gamma^{\kappa}_{\lambda\mu}$, $\Gamma^{\bar{\kappa}}_{\bar{\lambda}\bar{\mu}}$ do not vanish identically.

The only components of the Riemann-Christoffel curvature tensor, by looks of metric tensor and Christoffel symbols will be of the form $K_{\alpha\bar{\beta}\gamma\bar{\delta}}$ in the separated coordinate system. Also, we have to take into account the all algebraic properties of the Riemann-Christoffel tensor.

We can see that only components og HB-tensor which do not vanish identically in the separated coordinate system are of the form $HB_{\alpha\bar{\beta}\gamma\bar{\delta}}$. Our proposal is that the space is HB-flat. There holds

$$\begin{aligned}0 = HB_{\alpha\bar{\beta}\gamma\bar{\delta}} = K_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{n+4}[&g_{\bar{\delta}\alpha}K_{\gamma\bar{\beta}} - g_{\gamma\alpha}K_{\bar{\delta}\bar{\beta}} + g_{\gamma\bar{\beta}}K_{\bar{\delta}\alpha} - g_{\bar{\delta}\bar{\beta}}K_{\gamma\alpha}\\ &+ F_{\bar{\delta}\alpha}S_{\gamma\bar{\beta}} - F_{\gamma\alpha}S_{\bar{\delta}\bar{\beta}} + F_{\gamma\bar{\beta}}S_{\bar{\delta}\alpha} - F_{\bar{\delta}\bar{\beta}}S_{\gamma\alpha} + 2S_{\bar{\beta}\alpha}F_{\gamma\bar{\delta}} + 2S_{\gamma\bar{\delta}}F_{\bar{\beta}\alpha}\\ &- \frac{K}{n+2}(g_{\bar{\delta}\alpha}g_{\gamma\bar{\beta}} - g_{\gamma\alpha}g_{\bar{\delta}\bar{\beta}} - F_{\bar{\delta}\alpha}F_{\gamma\bar{\beta}} + F_{\bar{\delta}\bar{\beta}}F_{\gamma\alpha} - 2F_{\bar{\beta}\alpha}F_{\gamma\bar{\delta}})].\end{aligned}$$

The all covariant tensors of second order appearing in the upper formula are hybrid and, besides, there hold

$$F_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}; F_{\bar{\beta}\alpha} = -g_{\bar{\beta}\alpha}; S_{\alpha\bar{\beta}} = -K_{\alpha\bar{\beta}}; S_{\bar{\beta}\alpha} = K_{\bar{\beta}\alpha}. \tag{2.8}$$

Substituting (2.8) in the upper formula, we obtain

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{2}{n+4}[g_{\alpha\bar{\delta}}K_{\gamma\bar{\beta}} + g_{\gamma\bar{\beta}}K_{\alpha\bar{\delta}} + g_{\gamma\bar{\delta}}K_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}K_{\gamma\bar{\delta}} - \frac{K}{n+2}(g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}})]. \tag{2.9}$$

If a hyperbolic Kaehlerian space has constant sectional curvature, then that sectional curvature vanishes and, moreover, the space is flat; also, any sectional curvature of a hyperbolic Kaehlerian space vanishes except of cases when (from the viewpoint of the separated coordinate system) the section directions of different totally geodesic subspaces or if one direction of the section is a vector which has nonvanishing components in tangent spaces of both totally geodesic submanifolds (again in separated coordnate system) and the second direction of the section is the image of this vector by the structure tensor. The first case of nonvanishing sectional curvature we call the *cross sectional curvature* and the second one is *totally holomorphic sectional curvature*, because of holomorphy of such a section. If the totally holomorphic sectional curvature of a hyperbolic Kaehlerian space is constant (or if it does not depend on choice of the section at a point), then the space is a space of almost constant curvature. This fact may be expressed in the form (2.1) in any general coordinate system.

In the same manner as in the article [1] it can be proved that if a hyperbolic Kaehlerian space is a space of almost constant curvature, then the cross sectional curvature equals to a quarter of totally holomorphic sectional curvature. Both these types of sectional curvature are in such a case, of course, constant.

Suppose, now, that the hyperbolic Kaehlerian space is a space of almost constant curvature. As the space is HB-flat, there holds formula (2.9) and we obtain

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}}u^{\gamma}v^{\bar{\delta}}u^{\alpha}v^{\bar{\beta}} = \frac{2}{n+4}[4\langle u,v\rangle K_{\gamma\bar{\delta}}u^{\gamma}v^{\bar{\delta}} - \frac{2K}{n+2}\langle u,v\rangle^2]$$

for vectors u, v which form a cross sectional curvature. By $\langle u,v\rangle$ we denote the scalar product of vectors u and v. The cross sectional curvature (which is constant) can be expressed in this way

$$K(u,v) = \frac{2}{n+4}\left(4\frac{K_{\gamma\bar{\beta}}u^{\gamma}v^{\bar{\beta}}}{g_{\gamma\bar{\beta}}u^{\gamma}v^{\bar{\beta}}} - \frac{2K}{n+2}\right). \tag{2.10}$$

If the space is HB-flat and Einstein, then, in accordance to the Lemma 1 its cross sectional curvature is constant, and, by (2.1) is equal to $\frac{-2K}{n(n+2)}$ where K denotes the scalar curvature of the hyperbolic Kaehlerian space.

We can notice that our HB-flat hyperbolic Kaehlerian space of almost constant curvature is an Einstein space, because of Ricci curvature, appearing in parentheses of the expression for constant cross sectional curvature.

Comparing the expressions (2.1) and (2.10) we obtain that $K = 0$, what, by (2.1) means that the space is flat.

We have proved that there holds

Theorem 1. *If a hyperbolic Kaehlerian space is HB-flat, then it is an Einstein space if and only if it is a space of almost constant curvature.*

We did not prove explicitely that a HB-flat space of almost constant curvature is an Einstein space (the converse of Lemma 1), but it is very easy to prove.

3. On the indicating quadratic form

In the present paragraph, we shall not introduce any additional proposals. Our hyperbolic Kaehlerian space will be just HB-flat.

We shall involve some abbreviations:

$$\Pi_{kj} = \frac{1}{n+4}[K_{kj} - \frac{K}{2(n+2)}g_{kj}],\ \Pi_{kj} = \Pi_{jk}, \tag{3.1}$$

$$T_{kj} = \frac{1}{n+4}[S_{kj} + \frac{K}{2(n+2)}F_{kj}],\ T_{kj} = -T_{jk}. \tag{3.2}$$

Also, there holds

$$T_{kj} = \Pi_{ka}F_j^a. \tag{3.3}$$

Now we can put

$$HB_{ijkl} = K_{ijkl} - D_{ijkl}, \tag{3.4}$$

where

$$\begin{aligned} D_{ijkl} = {} & g_{li}\Pi_{kj} - g_{ki}\Pi_{lj} + g_{kj}\Pi_{li} - g_{lj}\Pi_{ki} \\ & + F_{li}T_{kj} - F_{ki}T_{lj} + F_{kj}T_{li} - F_{lj}Tki + 2T_{ji}F_{kl} + 2T_{kl}F_{ji}. \end{aligned} \tag{3.5}$$

We can show that Π_{kj} and D_{ijkl} satisfy the following relations

$$\Pi = g^{ab}\Pi_{ab} = \frac{K}{2(n+2)}, \tag{3.6}$$

$$D_{ijkl} = -D_{jikl} = -D_{ijlk} = D_{klij}. \tag{3.7}$$

We are going to prove a most significant theorem, which shows full analogy between conformally flat Riemannian spaces, Bochner-flat Kaehlerian spaces and HB-flat hyperbolic Kaehlerian spaces:

THEOREM 2. *On a hyperbolic Kaehlerian space, HB-tensor vanishes if and only if there exists a quadratic form Q such that the sectional cutvature of a holomorphic section σ is equal to trace of the restriction of the form Q to σ, metric also being restricted to σ.*

Proof. We shall prove the theorem in both directions. Suppose, first, that HB-tensor vanishes. We consider a holomorphic section, in directions of vectors X and $F(X)$. The vector X is not an isotropic vector field. From the viewpoint of natural geomety of hyperbolic Kaehlerian space, the vector X can be expressed as a linear combination of vectors from V_m and $\bar{V}_m$. The sectional curvature of such a section looks this way:

$$k = \frac{-K_{ijkl}X^iFX^jX^kFX^l}{(g_{ik}g_{lj} - g_{kj}g_{li})X^iFX^jX^kFX^l} = \frac{-K_{ijkl}X^iFX^jX^kFX^l}{\langle X, X\rangle^2} \tag{3.8}$$

where we have taken into account the Proposition 1(B) from Preliminaries.

Now we take into account that $HB_{ijkl} = 0$ and, according to (3.4) $K_{ijkl} = D_{ijkl}$. If we pay more attention to local coordinates $X^i, FX^j = F^{ja}X_a$, the expression (3.8) can be rewritten as

$$k = \frac{8\Pi_{ab}X^aX^b}{\langle X, X\rangle}, \tag{3.9}$$

where Π_{ab} are components of the tensor (3.1). Suppose that the first direction of the section is the vector U, satisfying $\langle U, U\rangle = 1$; the second direction is the vector $F(U)$, with scalar square -1. Then, there holds for the sectional curvature

$$k(U, F(U)) = 4\Pi_{ab}U^aU^b - 4\Pi_{ab}F^a_cU^cF^b_dU^d.$$

Taking into account the formulae (3.1), (3.2) and (3.3), it can be easily seen that this holomorphic sectional curvature equals to restriction of the trace of the covariant tensor $4\Pi_{ab}$ of the second order to the holomorphic section σ.

Conversely, suppose that the sectional curvature $k(\sigma)$ of a hyperbolic Kaehlerian space in the direction of holomorphic section σ generated by nonisotropic vector U and its image $F(U)$ is given by $k(U, F(U)) = 8\Pi_{ab}U^aU^b$. U is a unit vector and coefficient of the quadratic form Π_{ij} satisfy the relation

$$\Pi_{qp}F^q_aF^p_b = -\Pi_{ab}. \tag{3.10}$$

We shall prove that in this case HB-tensor of this space vanishes. In accordance to this proposal about the holomorphic sectional curvature, there holds:

$$K_{sitk}X^kF^t_jX^jX^iF^s_hX^h = 8g_{kj}X^kX^j\Pi_{ih}X^iX^h. \tag{$*$}$$

If we take into account that the structure is parallel, applying the Ricci identity we obtain $F^t_jK_{sitk} = F^t_kK_{sitj}$. Further, there hold

$$F^t_jF^s_hK_{sitk} = F^t_kF^s_hK_{sitj} \tag{3.11}$$

$$F^t_hF^s_jK_{skti} = -F^s_kF^t_hK_{sjit} \tag{3.12}$$

$$F^t_jF^s_hK_{sitk} - F^t_hF^s_jK_{skti} = F^t_kF^s_hK_{sitj} + F^s_kF^t_hK_{sjit} = F^t_kF^s_h(K_{sitj} + K_{tjis}) = 0. \tag{3.13}$$

Now if we rewrite the formula marked by $(*)$ this way

$$K_{sitk}F_j^tF_h^sX^kX^jX^iX^h = 8g_{kj}\Pi_{ih}X^kX^jX^iX^h = 8g_{ih}\Pi_{kj}X^kX^jX^iX^h$$

and if this relation holds for components (X^s) of any nonisotropic vector, then there will hold

$$K_{sitk}F_j^tF_h^s = 4(g_{kj}\Pi_{ih} + g_{ih}\Pi_{kj}), \tag{3.14}$$

whence

$$K_{sitk}F_j^tF_h^s + K_{shtk}F_i^tF_j^s + K_{sjtk}F_h^tF_i^s \\ = 4[g_{kj}\Pi_{ih} + g_{ki}\Pi_{hj} + g_{hk}\Pi_{ji} + \Pi_{kj}g_{ih} + \Pi_{ki}g_{hj} + \Pi_{hk}g_{ji}]. \tag{3.15}$$

In the formula (3.15) we change index j for q and h for p and then contract by $F_j^qF_h^p$. We obtain

$$K_{hijk} + K_{jptk}F_i^tF_h^p + K_{ijkh} \\ = 4[F_{jk}T_{ih} - g_{ki}\Pi_{jk} + F_{hk}T_{ij} + T_{kj}F_{hi} - \Pi_{ki}g_{jh} + F_{ji}T_{kh}], \tag{3.16}$$

where the relation between T_{kj} and the form Π_{kj} is given by (3.2). Then we alternate the indices k and j in the last equality. We obtain

$$2K_{hijk} + (K_{jptk} - K_{kptj})F_i^tF_h^p + K_{ijkh} - K_{ikjh} \\ = 4[2F_{jk}T_{ih} - g_{ki}\Pi_{jh} + g_{ji}\Pi_{kh} + F_{hk}T_{ij} - F_{hj}T_{ik} + 2T_{kj}F_{li} \\ - \Pi_{ki}g_{jh} + \Pi_{ji}g_{kh} + F_{ji}T_{kh} - F_{ki}T_{jh}]. \tag{$**$}$$

There holds

$$(K_{jptk} - K_{kptj})F_i^tF_h^p = (K_{jptk} + K_{jtkp})F_i^tF_h^p = -K_{jkpt}F_i^tF_h^p \\ = -K_{ptjk}F_h^pF_i^t = -K_{tpkj}F_i^tF_h^p = K_{hijk}.$$

By this fact, we can rewrite the equality marked by $(**)$ in this way

$$2K_{hijk} + K_{hijk} + K_{ijkh} - K_{ikjh} = 3K_{hijk} - K_{ijhk} - K_{ikjh} = 3K_{hijk} + K_{ihkj} \\ = 4K_{hijk} = 4[2F_{jk}T_{ih} - g_{ki}\Pi_{jh} + g_{ji}\Pi_{kh} + F_{kh}T_{ij} - F_{hj}T_{ik} + 2T_{kj}F_{hi} \\ - \Pi_{ki}g_{jh} + \Pi_{ji}g_{kh} + F_{ji}T_{kh} - F_{ki}T_{jh}]. \tag{3.17}$$

After contraction by $\frac{1}{4}g^{kh}$, we obtain $K_{ij} = (n+4)\Pi_{ij} + \Pi g_{ij}$, and after contraction by g^{ij}, $\Pi = \frac{K}{2(n+2)}$ whence

$$\Pi_{ij} = \frac{1}{n+4}[K_{ij} - \frac{K}{2(n+2)}g_{ij}]$$

and then

$$T_{ij} = \frac{1}{n+4}[S_{ij} + \frac{K}{2(n+2)}F_{ij}] = \Pi_{ia}F_j^a,$$

just as it has been defined. As we showed what the symbols Π_{ij} and T_{ij} mean, from (3.17) we obtain $K_{ijkl} = D_{ijkl}$ and, consequently, $HB_{ijkl} = 0$. ∎

The analogous therem for Riemannian and Kaehlerian spaces has been proved by Yano and Chen in [5]. The technology of proof of the first part of our theorem is very similar to theirs.

4. *HB*-flat decomposable hyperbolic Kaehlerian spaces

We shall prove the following theorem:

THEOREM 3. *If a hyperbolic Kaehlerian space HK_n can be locally decomposed into two hyperbolic Kaehlerian spaces of dimensions p and $n-p$ (p and $n-p$ are even numbers) and if its HB-tensor vanishes, then each of these two spaces is a space of almost constant curvature and these almost constant curvatures are mutually opposite; conversely, the product of two hyperbolic Kaehlerian spaces with mutually opposite almost constant curvatures is a hyperbolic Kaehlerian space vith vanishing HB-tensor.*

The consideration of a hyperbolic Kaehlerian space which is itself a product space does not mean that we shall consider a space with two different structures.

Proof. We are going to prove this theorem using the separated coordinate system. The fact that the space is decomposable we can express in the following way

$$ds^2 = ds_1^2 + ds_2^2, \tag{4.1}$$

where

$$ds_1^2 = 2g_{a\bar{b}}dx^a dx^{\bar{b}} \text{ and } ds_2^2 = 2g_{r\bar{s}}dx^r dx^{\bar{s}} \tag{4.2}$$

are matrices of decomposition components HK_p and HK_{n-p}, which are themselves hyperbolic Kaehlerian spaces. The indices $a, b, c, \dots$ are associated to the space HK_p and the indices $r, s, t, \dots$ are associated to the space HK_{n-p}. Evidently, chosen coordinate systems in both spaces HK_p and HK_{n-p} are separated and they together make a separated coordinate system for the space HK_n.

For the space HK_p, there holds

$$K_{a\bar{b}c\bar{d}} = 0 \tag{4.3}$$

because of

$$K_{a\bar{b}c\bar{d}} = -K_{\bar{b}ar\bar{s}} = -g_{\bar{b}c}K^c_{ar\bar{s}} = -g_{\bar{b}c}\frac{\partial \Gamma^c_{ar}}{\partial x^{\bar{s}}} = 0,$$

where we use the fact about the form of only nonvanishing components of Levi-Civita connection in the separated coordinate system.

The space HK_n is HB-flat and there holds

$$HB_{a\bar{b}c\bar{d}} = -\frac{2}{n+4}\left(g_{r\bar{s}}K_{a\bar{b}} + g_{a\bar{b}}K_{r\bar{s}} - \frac{K}{n+2}g_{a\bar{b}}g_{r\bar{s}}\right).$$

After contraction by $g^{r\bar{s}}$, we obtain

$$K_{a\bar{b}} = \left(\frac{K}{n+2} - \frac{K_2}{n-p}\right) g_{a\bar{b}} \tag{4.4}$$

and after contraction by $g^{a\bar{b}}$,

$$K_{r\bar{s}} = \left(\frac{K}{n+2} - \frac{K_1}{p}\right) g_{r\bar{s}}. \tag{4.5}$$

That means that the spaces HK_p and HK_{n-p} are Einstein spaces.

Now we calculate $K_{a\bar{b}c\bar{d}}$, which are components of Riemann-Christoffel tensor of the space HK_p. We obtain

$$K_{a\bar{b}c\bar{d}} = \frac{2}{n+4}\left(\frac{K}{n+2} - \frac{K_2}{n-p}\right)\left(1 - \frac{K}{n+2}\right)(g_{a\bar{b}}g_{c\bar{d}} + g_{a\bar{d}}g_{c\bar{d}}),$$

using the fact that the space HK_n is HB-flat. The upper equality means, in accordance to Definition 1 (and taking into account that we use separated coordinate system), that the space HK_p is a space of almost constant curvature; in a similar way we can obtain that HK_{n-p} is a space of almost constant curvature. Taking into account the form of Ricci tensor of the space HK_p, there yields:

$$\frac{K_1}{p} + \frac{K_2}{n-p} = \frac{K}{n+2} \tag{4.6}$$

and by (4.1)

$$K_1 + K_2 = K. \tag{4.7}$$

Eliminating K from (4.6) and (4.7), we obtain

$$\frac{K_1}{p(p+2)} + \frac{K_2}{(n-p)(n-p+2)} = 0 \tag{4.8}$$

and this is the sum of almost constant curvatures of spaces HK_p and HK_{n-p}, what completes the proof in one direction.

Conversely, if HK_p and HK_{n-p} are spaces of almost constant curvatures and if these almost constant curvatures are mutually opposite, there yields that both of them are Einstein spaces; then so is their sum. The scalar curvarure of the sum is the sum of scalar curvatures of components etc. ■

The theorem which is analogous to the last one (for Kaehlerian spaces) has been proved in [3] by Tachibana and Liu.

REFERENCES

[**1**] M. Prvanoviċ, *Holomorphically projective transformations in a locally product space*, Mathematica Balkanica, **1** (1971), 193–213

[**2**] N. Pušiċ, *On invariant tensor of a conformal transformation of a hyperbolic Kaehlerian space*, Zbornik radova Filozofskog fakulteta u Nišu, Serija Matematika, **4** (1990), 55–64

[**3**] S. Tachibana and R. C. Liu, *Notes on Kaehlerian metrics with vanishing Bochner curvature tensor*, Kodai Math. Sem. Rep. **22** (1970), 313–321

[**4**] K. Yano, *On complex conformal connections*, Kodai Math. Sem. Rep. **26** (1975), 137–151

[**5**] K. Yano and B. Chen, *Manifolds with vanishing Weyl or Bochner curvature tensor*, J. Math. Soc. Japan, **27**, 1 (1975), 106–112

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Institute of Mathematics, University of Novi Sad, Trg Dositeja Obradoviča 3, 21000 Novi Sad, Yugoslavia

e-mail: nevena@unsim.im.ns.ac.yu