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ON GEODESIC MAPPINGS OF GENERAL AFFINE CONNEXION SPACES AND OF GENERALIZED RIEMANNIAN SPACES

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Abstract. In the present paper we define a geodesic mapping of two nonsymmetrical affine connexion spaces and obtain necessary and sufficient conditions that a mapping of two such spaces be geodesic (§1). Particularly we study a geodesic mapping of two generalized Riemannian spaces (§2). Finally, we generalize the notion of Thomas's projective parameters as an invariant of geodesic mappings (§3).

1. Geodesic mappings of general affine connexion spaces

Consider two N-dimensional spaces of nonsymmetrical affine connexion: GA_N and $G\overline{A}_N$. So, if connexion coefficients of these spaces are respectively L^i_{jk} and $\overline{L}^i_{jk}$, we suppose that in general the symmetry with respect to indices j, k is not valid.

One says that reciprocal one-valued mapping $f\colon GA_N \to G\overline{A}_N$ is *geodesic*, if geodesics of the space GA_N pass to geodesics of the space $G\overline{A}_N$. We can consider these spaces together with this mapping system of local coordinates, i.e. for $f\colon M \mapsto \overline{M}$ we have $M(x^1,\dots,x^N) \equiv M(x)$ and $\overline{M}(x^1,\dots,x^N) \equiv \overline{M}(x)$, where $M \in GA_N$, $\overline{M} \in G\overline{A}_N$. In the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{L}^i_{jk}(x) = L^i_{jk}(x) + P^i_{jk}(x), \quad (i,j,k = 1,\dots,N), \tag{1.1}$$

where $P^i_{jk}(x)$ is *the deformation tensor* of the connexion L of GA_N according to the mapping $f\colon GA_N \to G\overline{A}_N$.

The curve

$$l\colon x^i = x^i(t) \tag{1.2}$$

is *geodesic* of GA_N if and only if for $\lambda^i = dx^i/dt$ it is:

$$\frac{d\lambda^i}{dt} + L^i_{pq}\lambda^p\lambda^q = \rho(t)\lambda^i(t), \tag{1.3}$$

where $\rho(t)$ is an invariant.

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If $f: l \to \overline{l}$, then by the mapping f, coordinates $\overline{x}^i \equiv x^i$ and it is $\overline{\lambda}^i = d\overline{x}^i/dt = \lambda^i$, and $\overline{l}$ is geodesic in $G\overline{A}_N$, too, so we get

$$\frac{d\lambda^i}{dt} + \overline{L}^i_{pq}\lambda^p\lambda^q = \overline{\rho}(t)\lambda^i(t) \tag{1.$\overline{3}$}$$

Subtracting (1.$\overline{3}$) and (1.3), we obtain

$$(\overline{L}^i_{pq} - L^i_{pq})\lambda^p\lambda^q = (\overline{\rho}(t) - \rho(t))\lambda^i(t),$$

and, because of (1.1):

$$P^i_{pq}\lambda^p\lambda^q = 2\psi(t)\lambda^i(t). \tag{1.4}$$

Denoting by $P^i_{\underline{jk}}$, $P^i_{\underset{\vee}{jk}}$ the symmetric and antisymmetric part of P^i_{jk} respectively, we get

$$P^i_{jk} = P^i_{\underline{jk}} + P^i_{\underset{\vee}{jk}}, \tag{1.5}$$

and (1.4) reduces to

$$P^i_{\underline{pq}}\lambda^p\lambda^q = 2\psi(t)\lambda^i(t). \tag{1.4$'$}$$

As in the case of a symmetric connexion (see e.g. [4]) one concludes that $\psi(t) = \psi_p(x^1(t), \ldots, x^N(t))\lambda^p(t)$, and from (1.4$'$):

$$P^i_{\underline{pq}}\lambda^p\lambda^q = 2\psi_p\lambda^p\lambda^i = \psi_p\lambda^p\lambda^q\delta^i_q + \psi_q\lambda^q\lambda^p\delta^i_p = (\psi_p\delta^i_q + \psi_q\delta^i_p)\lambda^p\lambda^q,$$

wherefrom

$$P^i_{\underline{jk}} = \delta^i_j\psi_k + \delta^i_k\psi_j. \tag{1.6a}$$

Denoting also

$$P^i_{\underset{\vee}{jk}} = \xi^i_{jk} = -\xi^i_{kj}, \tag{1.6b}$$

substituting in (1.1) we obtain

$$\overline{L}^i_{jk} = L^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{1.7}$$

and the deformation tensor is

$$P^i_{jk}(x) = \delta^i_j\psi_k(x) + \delta^i_k\psi_j(x) + \xi^i_{jk}(x). \tag{1.8}$$

So, the condition (1.8) is necessary that the mapping f be geodesic. It is easy to prove that this condition is sufficient, too, and we have

Theorem 1.1. *A necessary and sufficient condition that the mapping f: $GA_N \to G\overline{A}_N$ be geodesic is that the deformation tensor P^i_{jk} from* (1.1) *according to the mapping f has the form* (1.8)*, where $\psi_j(x^1, \ldots, x^N)$ is a covariant vector, and $\xi^i_{jk}(x^1, \ldots, x^N)$ an antisymmetric tensor.*

For $k = i$, we obtain from (1.6a) that $P^i_{\underline{ji}} = \delta^i_j\psi_i + \delta^i_i\psi_j = \psi_j + N\psi_j$, wherefrom

$$\psi_i = \frac{1}{N+1}P^p_{\underline{ip}}, \tag{1.9}$$

which, by substitution in (1.7), gives

$$\overline{L}^i_{jk} = L^i_{jk} + \frac{1}{N+1}(\delta^i_j P^p_{\underline{kp}}(x) + \delta^i_k P^p_{\underline{jp}}(x)) + P^i_{\underset{\vee}{jk}}(x), \tag{1.7'}$$

where $P^i_{jk}(x)$ is the deformation tensor.

On the base of the facts given above, we get

THEOREM 1.2 *Let a space GA_N be given, i.e. on a differentiable manifold M_N let nonsymmetric connexion coefficients $L^i_{jk}(x)$ be defined. If on M_N a tensor $P^i_{jk}(x)$ is given, too and we determine $\overline{L}^i_{jk}(x)$ according to* (1.7′)*, then on M_N a space $G\overline{A}_N$ is defined, with connexion coefficients $\overline{L}^i_{jk}(x)$, so that GA_N and $G\overline{A}_N$ have common geodesics. We obtain the same result (on the base of* (1.7)*) choosing a vector $\psi_i(x)$ and antisymmetric tensor $\xi^i_{jk}(x) = P^i_{\underset{\vee}{jk}}(x)$.*

A question arises itself: Is there a geodesic mapping of a space GA_N with a nonsymmetric affine connexion onto a space $\overline{A}_N$ with a symmetric affine connexion? It is easy to see that the next theorem is valid.

THEOREM 1.3 *A necessary and sufficient condition that a mapping $f : GA_N \to \overline{A}_N$ of a nonsymmetric affine connexion space GA_N onto a symmetric affine connexion space $\overline{A}_N$ be geodesic, is that*

$$P^i_{\underset{\vee}{jk}}(x) = -L^i_{\underset{\vee}{jk}}(x), \tag{1.10}$$

where $P^i_{\underset{\vee}{jk}}(x)$, $L^i_{\underset{\vee}{jk}}(x)$, are antisymmetric parts of the deformation tensor and connexion coefficients of the GA_N respectively.

REMARK. It is easy to check that a set of geodesic mappings of a space GA_N forms a group.

2. Geodesic mappings of generalized Riemannian spaces

Generalized Riemannian space GR_N in the sense of Eisenhart's definition [1] is a differentiable N-dimensional manifold, equipped with nonsymmetric basic tensor g_{ij}. Connexion coefficients are generalized Cristoffel's symbols of the second kind Γ^i_{jk}, where

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad \Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}, \tag{2.1}$$

and $g^{\underline{ip}} g_{\underline{ip}} = \delta^i_j$.

Generally, it is $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. Based on this, everything that we exposed for GA_N, is valid for GR_N, too. We will expose some specifics.

In a space of nonsymmetric affine connexion (and in a generalized Riemannian space) one can define four kinds of covariant derivative [2,3]. For example, for a tensor a^i_j in GA_N we have

$$
\begin{aligned}
a^i_{j\underset{1}{|}m} &= a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p,\\
a^i_{j\underset{2}{|}m} &= a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p,\\
a^i_{j\underset{3}{|}m} &= a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p,\\
a^i_{j\underset{4}{|}m} &= a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p.
\end{aligned}
\tag{2.2}
$$

Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively. We define a geodesic mapping of GR_N in the similar manner as of GA_N (§1).

Suppose that a geodesic mapping $f\colon GR_N \to G\overline{R}_N$ is given. Then, with respect to (1.7), one obtains

$$
\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + P^i_{\underset{\vee}{jk}}. \tag{2.3}
$$

and

$$
\begin{aligned}
\overline{g}_{ij\underset{\overline{1}}{|}k} &= \overline{g}_{ij,k} - \overline{\Gamma}^p_{ik}\overline{g}_{pj} - \overline{\Gamma}^p_{jk}\overline{g}_{ip} \underset{(2.3)}{=}\\
&= \overline{g}_{ij,k} - (\Gamma^p_{ik} + \delta^p_i\psi_k + \delta^p_k\psi_i + P^p_{\underset{\vee}{ik}})\overline{g}_{pj} - (\Gamma^p_{jk} + \delta^p_j\psi_k + \delta^p_k\psi_j + P^p_{\underset{\vee}{jk}})\overline{g}_{ip}\\
&= (\overline{g}_{ij,k} - \Gamma^p_{ik}\overline{g}_{pj} - \Gamma^p_{jk}\overline{g}_{ip}) - \overline{g}_{ij}\psi_k - \overline{g}_{kj}\psi_i - \overline{g}_{ij}\psi_k - \overline{g}_{ik}\psi_j - P^p_{\underset{\vee}{ik}}\overline{g}_{pj} - P^p_{\underset{\vee}{jk}}\overline{g}_{ip}.
\end{aligned}
\tag{2.4}
$$

Because of $\overline{g}_{ij\underset{\overline{\theta}}{|}k} = \overline{g}_{\underline{ij}\underset{\overline{\theta}}{|}k} + \overline{g}_{\underset{\vee}{ij}\underset{\overline{\theta}}{|}k}$ and $\overline{g}_{\underline{ij}\underset{\overline{\theta}}{|}k} = 0 \quad (\theta = 1, 2)$, and in the parentheses on the right side of (2.4) we have $\overline{g}_{ij\underset{1}{|}k}$, this equation gives

$$
\overline{g}_{ij\underset{1}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{\overline{1}}{|}k} = 2\overline{g}_{ij}\psi_k + \psi_i\overline{g}_{kj} + \psi_j\overline{g}_{ik} + \overline{g}_{ip}P^p_{\underset{\vee}{jk}} + \overline{g}_{pj}P^p_{\underset{\vee}{ik}}. \tag{2.5a}
$$

Starting from $\overline{g}_{ij\underset{\overline{2}}{|}k}$, we obtain

$$
\overline{g}_{ij\underset{2}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{\overline{2}}{|}k} = 2\overline{g}_{ij}\psi_k + \psi_i\overline{g}_{kj} + \psi_j\overline{g}_{ik} + \overline{g}_{ip}P^p_{\underset{\vee}{kj}} + \overline{g}_{pj}P^p_{\underset{\vee}{ki}}, \tag{2.5b}
$$

With respect to the Theorem 1.1., the condition (2.3) is necessary and sufficient that the mapping $f\colon GR_N \to G\overline{R}_N$ be geodesic, so the conditions (2.5a,b) are necessary for this. Let us prove that these conditions are sufficient too. Start from (2.5b). Denoting the left and right side in (2.5b) by $\mathcal{L}$ and $\mathcal{R}$ respectively, we have

$$
\begin{aligned}
\mathcal{L} &\equiv \overline{g}_{ij\underset{2}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{\overline{2}}{|}k} = \overline{g}_{ij\underset{2}{|}k} - \overline{g}_{ij\underset{\overline{2}}{|}k}\\
&= \overline{g}_{ij,k} - \Gamma^p_{ki}\overline{g}_{pj} - \Gamma^p_{kj}\overline{g}_{ip} - \overline{g}_{ij,k} - \overline{\Gamma}^p_{ki}\overline{g}_{pj} - \overline{\Gamma}^p_{kj}\overline{g}_{ip}\\
&= (\overline{\Gamma}^p_{ki} - \Gamma^p_{ki})\overline{g}_{pj} + (\overline{\Gamma}^p_{kj} - \Gamma^p_{kj})\overline{g}_{ip},
\end{aligned}
$$

$$\mathcal{R} \equiv \psi_i \overline{g}_{kj} + \psi_k \overline{g}_{ij} + \overline{g}_{pj} P^p_{\underset{\vee}{ki}} + \psi_j \overline{g}_{ik} + \psi_k \overline{g}_{ij} + \overline{g}_{ip} P^p_{\underset{\vee}{kj}}$$
$$= (\psi_i \delta^p_k + \psi_k \delta^p_i + P^p_{\underset{\vee}{ki}}) \overline{g}_{pj} + (\psi_j \delta^p_k + \psi_k \delta^p_j + P^p_{\underset{\vee}{kj}}) \overline{g}_{ip}.$$

The equation (2.5b), i.e. $\mathcal{L} = \mathcal{R}$ is satisfied for

$$\overline{\Gamma}^p_{kj} - \Gamma^p_{kj} = \psi_j \delta^p_k + \psi_k \delta^p_j + P^p_{\underset{\vee}{kj}},$$

and this is the equation (1.7). Starting from (2.5a), one obtains the same. For, starting from one of the equations (2.5a,b), it follows (1.7), and from here the other of these equations follows, so we conclude that they are equivalent.

By virtue of the facts exposed above, we have

THEOREM 2.1 *a) A mapping $f\colon GR_N \to G\overline{R}_N$ is geodesic if and only together with the mapping f the system of local coordinates and the second kind Cristoffel symbols of these spaces satisfy* (2.3).

b) If the mapping f is geodesic, then the equations $(2.5a, b)$ *are satisfied. Conversely, if one of these equations is satisfied, this mapping is geodesic, and the other is satisfied too.*

COROLLARY. *If the mapping $f\colon GR_N \to G\overline{R}_N$ is geodesic, then the basic tensor $\overline{g}_{ij}$ of the space $G\overline{R}_N$ satisfies the relation*

$$\overline{g}_{ij\underset{1}{|}k} + \overline{g}_{ij\underset{2}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{\overline{1}}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{\overline{2}}{|}k} = 2\psi_i \overline{g}_{kj} + 2\psi_j \overline{g}_{ik} + 4\psi_k \overline{g}_{ij}, \tag{2.6}$$

where ψ_i is a covariant vector.

This relation is obtained by adding (2.5a,b).

Further, we have

THEOREM 2.2 *By a geodesic mapping $f\colon GR_N \to G\overline{G}_N$ the vector ψ_i is given in the form*

$$\psi_i = \frac{1}{N+1} \left(\ln \sqrt{\left| \frac{\underline{\overline{g}}}{\underline{g}} \right|} \right)_{,i}, \tag{2.7}$$

where $\underline{g} = \det(g_{\underline{ij}})$, $\underline{\overline{g}} = \det(\overline{g}_{\underline{ij}})$ and the comma denotes a partial derivative.

Proof. With respect to (2.1) one gets

$$\Gamma_{i.jk} + \Gamma_{j.ik} = g_{\underline{ij},k}, \quad \Gamma_{i.jk} + \Gamma_{k.ji} = g_{\underline{ik},j}. \tag{2.8a, b}$$

On the other hand, as in the case of Riemannian space, we have:

$$\underline{g}_{,i} = \underline{g} g^{\underline{jk}} g_{\underline{jk},i} = \underline{g} g^{\underline{jk}} (\Gamma_{j.ki} + \Gamma_{k.ji}) = \underline{g} (\Gamma^k_{ki} + \Gamma^j_{ji}) = 2\underline{g} \Gamma^p_{pi},$$

where we used (2.8a). Using (2.8b) we obtain $\underline{g}_{,i} = 2\underline{g}\Gamma^p_{ip}$. From these equations we have

$$\Gamma^p_{pi} = \Gamma^p_{ip} = (\ln \sqrt{|\underline{g}|})_{,i}. \tag{2.9}$$

Analogous equation is valid for $\overline{\Gamma}$, too, and from (1.9) one obtains

$$\psi_i = \frac{1}{N+1}(\overline{\Gamma}^p_{ip} - \Gamma^p_{ip}) = \frac{1}{N+1}[(\ln\sqrt{|\underline{\overline{g}}|})_{,i} - (\ln\sqrt{|\underline{g}|})_{,i}],$$

i.e. (2.7) is in effect. ■

REMARK. From (2.9) we see that in GR_N, Γ^p_{pi} is the gradient and that

$$\Gamma^p_{\underset{\vee}{pi}} = 0. \tag{2.10}$$

From (2.7) it follows that at the mapping $f\colon GR_N \to G\overline{R}_N$ the corresponding vector ψ_i is gradient, too.

3. Generalized Thomas's projective parameters

Putting P into (1.7′) in accordance with (1.1) we get

$$\begin{aligned}\overline{L}^i_{jk} - \overline{L}^i_{\underset{\vee}{jk}} &- \frac{1}{N+1}(\delta^i_j \overline{L}^p_{\underline{kp}} + \delta^i_k \overline{L}^p_{\underline{jp}}) \\ &= L^i_{jk} - L^i_{\underset{\vee}{jk}} - \frac{1}{N+1}(\delta^i_j L^p_{\underline{kp}} + \delta^i_k L^p_{\underline{jp}}),\end{aligned}$$

Denoting

$$T^i_{jk} = L^i_{\underline{jk}} - \frac{1}{N+1}(\delta^i_j L^p_{\underline{kp}} + \delta^i_k L^p_{\underline{jp}}) = T^i_{kj}, \tag{3.1}$$

we see that

$$\overline{T}^i_{jk} = T^i_{jk}. \tag{3.2}$$

The magnitudes T^i_{jk} we call *generalized Thomas's projective parameters* at the mapping $f\colon GA_N \to G\overline{A}_N$. Accordingly, these magnitudes are invariant at a geodesic mapping. Starting from (3.1) and (3.2), one obtains (1.7′), and we conclude that the next theorem is valid.

THEOREM 3.1 *A necessary and sufficient condition that a mapping $f\colon GA_N \to G\overline{A}_N$ be geodesic is that the generalized Thomas's projective parameters are invariant.*

Using the transformation law for connexion coefficients L^i_{jk}, from (3.1) we obtain *the transformation law for $T^i_{jk}(x)$* passing from coordinates x^i to coordinates $x^{i'}$ in GA_N:

$$T^{i'}_{j'k'}(x') = T^i_{jk}(x) x^{i'}_i x^j_{j'} x^k_{k'} - \frac{1}{N+1} x^{i'}_i \left[x^i_{j'}(\ln\Delta)_{,k'} + x^i_{k'}(\ln\Delta)_{,j'}\right] + x^{i'}_i x^i_{j'k'}, \tag{3.3}$$

where $\Delta = \det(x^i_{i'})$, $x^i_{i'} = \partial x^i/\partial x^{i'}$, $x^i_{j'k'} = \partial^2 x^i/\partial x^{j'}\partial x^{k'}$ and so on.

Let in GA_N local coordinates are x^i, in $G\overline{A}_N$, $x^{i'}$, and $f\colon GA_N \to G\overline{A}_N$. Passing from $x^{i'}$ in $G\overline{A}_N$ to x^i, we obtain an inverse relation with respect to (3.3), and (3.2) gives

$$T^i_{jk}(x) = \overline{T}^i_{jk}(x) = \overline{T}^{i'}_{j'k'}(x')x^i_{i'}x^{j'}_j x^{k'}_k - \frac{1}{N+1}x^i_{i'}\left[x^{i'}_j(\ln\Delta'_{,k} + x^{i'}_k(\ln\Delta')_{,j}\right] + x^i_{i'}x^{i'}_{jk}, \tag{3.4}$$

where $\Delta' = \det(x^{i'}_i)$.

Writing (3.4) in the form

$$T^i_{jk}(x)x^{i'}_i = \overline{T}^{i'}_{j'k'}(x')x^{j'}_j x^{k'}_k - \frac{1}{N+1}\left[x^{i'}_j(\ln\Delta')_{,k} + x^{i'}_k(\ln\Delta')_{,j}\right] + x^{i'}_{jk}, \tag{3.4$'$}$$

we see that the following is valid.

THEOREM 3.2 *A space GA_N, with connexion coefficients $L^i_{jk}(x)$ in local coordinates x^i, allows a geodesic mapping f on a space $G\overline{A}_N$ with connexion coefficients $\overline{L}^{i'}_{j'k'}(x')$ in local coordinates $x^{i'}$, if and only if there exist functions*

$$x^{i'} = x^{i'}(x^1,\dots,x^N),\ (i' = 1',\dots,N';\ \det(x^{i'}_i) \neq 0) \tag{3.5}$$

of the class C^r $(r > 2)$, satisfying the equation (3.4$'$), and by which the mapping $f\colon GA_N \to G\overline{A}_N$ is realized.

REMARK. The equations (3.4$'$) form a system of second order partial differential (nonlinear) equations with respect to unknown functions $x^{i'}$ (3.5). But, practical solving of this problem is very difficult in general.

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