МАТЕМАТИЧКИ ВЕСНИК
48 (1996), 1–9

UDK 512.643
оригинални научни рад
research paper

GENERAL DETERMINANTAL REPRESENTATION OF PSEUDOINVERSES OF MATRICES

Predrag Stanimirović

Abstract. In this paper we establish general determinantal representation of generalized inverses and general form of different definitions of rectangular determinants and induced general inverses, in terms of minors of a matrix, satisfying certain conditions. Using this representation we obtain a general algorithm for exact computation of different classes of pseudoinverses: Moore-Penrose and weighted Moore-Penrose inverse, group inverse, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2\}$ inverses, left/right inverses, Radić's and Stojaković's generalized inverses.

We also give some examples which illustrate our results.

1. Introduction

Let $\mathbf{C}_r^{m\times n}$ be the set of $m\times n$ complex/rational matrices whose rank is r. Conjugate, transpose and conjugate-transpose matrix of A will be denoted by $\overline{A}$, A^T and A^* respectively. Minor of A containing rows $\alpha_1,\ \dots,\ \alpha_t$ and colums $\beta_1,\ \dots,\ \beta_t$ will be denoted by $A\begin{pmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{pmatrix}$, while its *algebraic complement* corresponding to the element a_{ij} is defined by $A_{ij}\begin{pmatrix}\alpha_1 & \dots & \alpha_{p-1} & i & \alpha_{p+1} & \dots & \alpha_t\\ \beta_1 & \dots & \beta_{q-1} & j & \beta_{q+1} & \dots & \beta_t\end{pmatrix} = (-1)^{p+q}A\begin{pmatrix}\alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_t\\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_t\end{pmatrix}$.

For any matrix $A\in\mathbf{C}^{m\times n}$ consider the following equations in X:

$$(1)\ AXA=A,\quad (2)\ XAX=X,\quad (3)\ (AX)^*=AX,\quad (4)\ (XA)^*=XA$$

and if $m=n$ also

$$(5)\quad AX=XA.$$

For a subset $\mathcal{S}$ od $\{1,2,3,4,5\}$ the set of matrices G obeying the conditions represented in $\mathcal{S}$ will be denoted by $A(\mathcal{S})$. A matrix $G\in A(\mathcal{S})$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. In particular for any $A\in\mathbf{C}^{m\times n}$ the set $A\{1,2,3,4\}$ consists of a single element, the *Moore-Penrose inverse* of A, denoted by $A^\dagger$ [13]. In the case $m=n$, the *group inverse*, denoted as $A^\sharp$, of A is the unique $\{1,2,5\}$ inverse, and exists if and only if $\operatorname{ind}(A)=\min\{k : k>0 \text{ and } \operatorname{rank}(A^{k+1})=\operatorname{rank}(A^k)\}=1$.

AMS Subject Classification: 15A09, 65F05

The starting point of the investigations of this paper is the determinantal representation of *Moore-Penrose inverse*, studied in [1], [4], [6], [7], [8], [11]. The main result of these papers is:

THEOREM 1.1 *Element $a^{\dagger}_{ij}$ in the i-row and j-column of the Moore-Penrose pseudoinverse of a given matrix $A \in \mathbf{C}^{m\times n}_r$ is given by*

$$a^{\dagger}_{ij} = \frac{\sum\limits_{\substack{1\leqslant\beta_1<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m}} \overline{A}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_r\\ \beta_1 & \dots & i & \dots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\leqslant\delta_1<\cdots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_r\leqslant m}} \overline{A}\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix} A\begin{pmatrix}\gamma_1 & \dots & \gamma_r\\ \delta_1 & \dots & \delta_r\end{pmatrix}}, \qquad \begin{pmatrix}1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m\end{pmatrix}.$$

For the sake of completeness in the following definition we unify definitions of generalized inverses introduced by M. Radić [16], [17], M. Stojaković [20] and V.N.Joshi [9].

DEFINITION 1.1 Let i, j be integers, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Then the (i, j)-th entry of Radić's, Stojaković's and Joshi's generalized inverse $A \in \mathbf{C}^{m\times n}_r$ is defined by

$$\frac{\sum\limits_{\substack{1\leqslant j_1<\cdots<j<\cdots<j_r\leqslant n\\ 1\leqslant i_1<\cdots<i<\cdots<i_r\leqslant m}} \varepsilon^{(i_1+\cdots+i_r)+(j_1+\cdots+j_r)} A_{ji}\begin{pmatrix}i_1 & \dots & j & \dots & i_r\\ j_1 & \dots & i & \dots & j_r\end{pmatrix}}{\sum\limits_{\substack{1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m\\ 1\leqslant\beta_1<\cdots<\beta_r\leqslant n}} \varepsilon^{(\alpha_1+\cdots+\alpha_r)+(\beta_1+\cdots+\beta_r)} A\begin{pmatrix}\alpha_1 & \dots & \alpha_r\\ \beta_1 & \dots & \beta_r\end{pmatrix}}, \qquad \text{where } \varepsilon \in \{-1, 1\}.$$

For $\varepsilon = 1$ we get Stojaković's definition, and for $\varepsilon = -1$, we get Radić's definition.

The following theorem [15] has described an useful representation of $\{i, j, k\}$ generalized inverses:

THEOREM 1.2 *If $A \in \mathbf{C}^{m\times n}_r$ has a full-rank factorization $A = PQ$, $P \in \mathbf{C}^{m\times r}_r$, $Q \in \mathbf{C}^{r\times n}_r$, $W_1 \in \mathbf{C}^{n\times r}$ and $W_2 \in \mathbf{C}^{r\times m}$ are some matrices such that* $\operatorname{rank}(QW_1) = \operatorname{rank}(W_2P) = \operatorname{rank}(A)$*, then:*

$A^{\dagger} = Q^{\dagger}P^{\dagger} = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$;

the generalized solution of the equations (1), (2) is given by $W_1(QW_1)^{-1}(W_2P)^{-1}W_2$;

the generalized solution of the equations (1), (2), (3) is given by $W_1(QW_1)^{-1}(P^*P)^{-1}P^*$;

the generalized solution of the equations (1), (2), (4) is given by $Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2$.

In [19] we developed determinantal representation of class of $\{1, 2\}$ inverses and the *weighted Moore-Penrose inverse*, as follows in the following two theorems:

THEOREM 1.3 *If $A = PQ$ is a full rank factorization of $A \in \mathbf{C}_r^{m\times n}$ and $W_1 \in \mathbf{C}^{n\times r}$, $W_2 \in \mathbf{C}^{r\times m}$ are some matrices such that* $\mathrm{rank}(QW_1) = \mathrm{rank}(W_2P) = \mathrm{rank}(A)$, *then an element $a_{ij}^{(1,2)} \in A^{(1,2)}$ is given by*

$$a_{ij}^{(1,2)} = \frac{\displaystyle\sum_{\substack{1\leqslant\beta_1<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m}} (W_1W_2)^T \begin{pmatrix} \alpha_1 & \ldots & i & \ldots & \alpha_r \\ \beta_1 & \ldots & j & \ldots & \beta_r \end{pmatrix} A_{ji} \begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r \end{pmatrix}}{\displaystyle\sum_{\substack{1\leqslant\delta_1<\cdots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_r\leqslant m}} A\begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix} (W_1W_2)^T \begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix}}.$$

THEOREM 1.4 *Let $M \in \mathbf{C}^{m\times m}$, $N \in \mathbf{C}^{n\times n}$ be positive definite, and suppose that $A = PQ$ is a full rank factorization of A, such that* $\mathrm{rank}(P^*MP) = \mathrm{rank}(QNQ^*) = \mathrm{rank}(MAN) = r$. *Element of the weighted Moore-Penrose inverse $A^{\dagger}_{M\bullet,\bullet N}$, lying on the i-th row and j-th column, can be represented in terms of square minors as follows:*

$$(a^{\dagger}_{M\bullet,\bullet N})_{ij} = \frac{\displaystyle\sum_{\substack{1\leqslant\beta_1<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m}} \overline{(MAN)} \begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r \end{pmatrix} A_{ji} \begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r \end{pmatrix}}{\displaystyle\sum_{\substack{1\leqslant\delta_1<\cdots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_r\leqslant m}} A\begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix} \overline{(MAN)} \begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix}}$$

Now we describe the main results of the paper. First we obtain determinantal representation of $\{i, j, k\}$ generalized inverses and the group inverse. Then we show that determinantal representation of the Moore-Penrose and the weighted Moore-Penrose inverse, $\{1, 2\}$, $\{i, j, k\}$ inverses are only partial cases of the common general determinantal inverses, generalization of the Arghiriade-Dragomir representation of the Moore-Penrose inverse [18], possesses the same general form. We also show that the class of left/right generalized inverses can be obtained from this general determinantal representation.

2. General determinantal representation

Applying method used in [19, Theorem 2.1] and using general form of $\{i, j, k\}$ inverses (Theorem 1.2), the following lemma can be proved.

LEMMA 2.1. *If $A \in \mathbf{C}_r^{m\times n}$ and $A = PQ$ is a full rank factorization of A, and if $W_1 \in \mathbf{C}^{m\times r}$ and $W_2 \in \mathbf{C}^{r\times n}$ are some matrices such that* $\mathrm{rank}(QW_1) = \mathrm{rank}(W_2P) = \mathrm{rank}(A)$, *then $a_{ij}^{(1,2,3)} \in A^{(1,2,3)}$ and $a_{ij}^{(1,2,4)} \in A^{(1,2,4)}$ can be represented in this way:*

$$a_{ij}^{(1,2,3)} = \frac{\displaystyle\sum_{\substack{1\leqslant\beta_1<\cdots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_r\leqslant m}} (W_1P^*)^T \begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r \end{pmatrix} A_{ji} \begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r \end{pmatrix}}{\displaystyle\sum_{\substack{1\leqslant\delta_1<\cdots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_r\leqslant m}} A\begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix} (W_1P^*)^T \begin{pmatrix} \gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r \end{pmatrix}}$$

$$a_{ij}^{(1,2,4)}=\frac{\sum\limits_{\substack{1\leqslant\beta_1<\dots<\beta_r\leqslant n\\ 1\leqslant\alpha_1<\dots<\alpha_r\leqslant m}}(Q^*W_2)^T\left(\begin{smallmatrix}\alpha_1&\dots&j&\dots&\alpha_r\\ \beta_1&\dots&i&\dots&\beta_r\end{smallmatrix}\right)A_{ji}\left(\begin{smallmatrix}\alpha_1&\dots&j&\dots&\alpha_r\\ \beta_1&\dots&i&\dots&\beta_r\end{smallmatrix}\right)}{\sum\limits_{\substack{1\leqslant\delta_1<\dots<\delta_r\leqslant n\\ 1\leqslant\gamma_1<\dots<\gamma_r\leqslant m}}A\left(\begin{smallmatrix}\gamma_1&\dots&\gamma_r\\ \delta_1&\dots&\delta_r\end{smallmatrix}\right)(Q^*W_2)^T\left(\begin{smallmatrix}\gamma_1&\dots&\gamma_r\\ \delta_1&\dots&\delta_r\end{smallmatrix}\right)}.$$

Note that conditions $\text{rank}(QW_1)=\text{rank}(W_2P)=\text{rank}(A)$ are satisfied if and only if $\text{rank}(W_1)=\text{rank}(W_2)=\text{rank}(A)$.

Determinantal representation of the group inverse of a singular n by n matrix can be obtained from [2] and [10]:

LEMMA 2.2. *The group inverse* $A^\sharp=(a_{ij}^\sharp)$ *of* $A\in\mathbf{C}_r^{n\times n}$ *exists if*

$$u=\sum_{1\leqslant\mu_1<\dots<\mu_r\leqslant n}A\begin{pmatrix}\mu_1&\dots&\mu_r\\ \mu_1&\dots&\mu_r\end{pmatrix}\neq 0.$$

Then $A^\sharp$ *has the following determinantal representation:*

$$a_{ij}^\sharp=\frac{\sum\limits_{\substack{1\leqslant\alpha_1<\dots<\alpha_r\leqslant n\\ 1\leqslant\beta_1<\dots<\beta_r\leqslant n}}A^T\begin{pmatrix}\alpha_1&\dots&j&\dots&\alpha_r\\ \beta_1&\dots&i&\dots&\beta_r\end{pmatrix}A_{ji}\begin{pmatrix}\alpha_1&\dots&j&\dots&\alpha_r\\ \beta_1&\dots&i&\dots&\beta_r\end{pmatrix}}{\sum\limits_{\substack{1\leqslant\gamma_1<\dots<\gamma_r\leqslant n\\ 1\leqslant\delta_1<\dots<\delta_r\leqslant n}}A^T\left(\begin{smallmatrix}\gamma_1&\dots&\gamma_r\\ \delta_1&\dots&\delta_r\end{smallmatrix}\right)A\left(\begin{smallmatrix}\gamma_1&\dots&\gamma_r\\ \delta_1&\dots&\delta_r\end{smallmatrix}\right)}.$$

Proof. In view of the supposition $u\neq 0$ from [10, Theorem 8] we get the following determinantal representation for the (i,j)-th entry of $A^\sharp$:

$$a_{ij}^\sharp=\frac{\sum\limits_{\substack{1\leqslant\alpha_1<\dots<\alpha_r\leqslant n\\ 1\leqslant\beta_1<\dots<\beta_r\leqslant n}}A\begin{pmatrix}\beta_1&\dots&i&\dots&\beta_r\\ \alpha_1&\dots&j&\dots&\alpha_r\end{pmatrix}A_{ji}\begin{pmatrix}\alpha_1&\dots&j&\dots&\alpha_r\\ \beta_1&\dots&i&\dots&\beta_r\end{pmatrix}}{u^2}.$$

Now the proof of the lemma can be completed using the following relation [10, Theorem 8]

$$u^{-2}\cdot A\begin{pmatrix}\alpha_1&\dots&\alpha_r\\ \beta_1&\dots&\beta_r\end{pmatrix}A\begin{pmatrix}\beta_1&\dots&\beta_r\\ \alpha_1&\dots&\alpha_r\end{pmatrix}=1$$

and the following, evident relation: $A\begin{pmatrix}\alpha_1&\dots&\alpha_r\\ \beta_1&\dots&\beta_r\end{pmatrix}=A^T\begin{pmatrix}\beta_1&\dots&\beta_r\\ \alpha_1&\dots&\alpha_r\end{pmatrix}$. ∎

Now we introduce a general determinantal representation which include all of presented determinantal representations, generlaized inverses introduced by M. Stojaković, M. Radić and V.N. Joshi and the well known concept of the inversion of square regular matrices.

THEOREM 2.1. *Let* $A\in\mathbf{C}_r^{m\times n}$ *and matrix* R *satisfies condition*

$$(U_1)\qquad\begin{cases}\text{rank}(AR^*)=\text{rank}(A), & \text{if } m\leqslant n\\ \text{rank}(R^*A)=\text{rank}(A), & \text{if } n\leqslant m.\end{cases}$$

An arbitrary (i,j)-th entry of all the above mentioned generalized inverses has the following determinantal representation:

$$g_{ij} = \frac{\sum\limits_{\substack{1\leqslant\beta_1<\cdots<\beta_t\leqslant n\\ 1\leqslant\alpha_1<\cdots<\alpha_t\leqslant m}} \overline{R}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_t\\ \beta_1 & \dots & i & \dots & \beta_t\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \dots & j & \dots & \alpha_t\\ \beta_1 & \dots & i & \dots & \beta_t\end{pmatrix}}{\sum\limits_{\substack{1\leqslant\delta_1<\cdots<\delta_t\leqslant n\\ 1\leqslant\gamma_1<\cdots<\gamma_t\leqslant m}} A\begin{pmatrix}\gamma_1 & \dots & \gamma_t\\ \delta_1 & \dots & \delta_t\end{pmatrix} \overline{R}\begin{pmatrix}\gamma_1 & \dots & \gamma_t\\ \delta_1 & \dots & \delta_t\end{pmatrix}}. \tag{2.1}$$

The numerator in (2.1) we simly denote by $A_{ij}^{(R,t)}$ and denominator by $\mathrm{DET}_{(R,T)}(A)$. The expression $A_{ij}^{(R,t)}$ will be called generalized albebraic complement corresponding to element a_{ji}, and $\mathrm{DET}_{(R,T)}(A)$ will be called generalized determinant.

Notation $r_c(A) = t \leqslant r \leqslant \min\{m,n\}$ denotes the greates integer which ensures $\mathrm{DET}_{(R,t)}(A) \neq 0$.

Proof. Consider the following cases:

1. Suppose that $t = m \leqslant n$. Using Laplace's development for the square minors of A, we get

$$\mathrm{DET}_{(R,m)}(A) = \det(AR^*) = \sum_{j_1<\cdots<j_m} \overline{R}\begin{pmatrix}1 & \dots & m\\ j_1 & \dots & j_m\end{pmatrix}\left[\sum_{k=1}^{r} a_{ij_k} A_{ij_k}\begin{pmatrix}1 & \dots & m\\ j_1 & \dots & j_m\end{pmatrix}\right]$$
$$= \sum_{l=1}^{n} a_{il}\left[\sum_{j_1<\cdots<j_m} \overline{R}\begin{pmatrix}1 & \dots & \dots & \dots & m\\ j_1 & \dots & l & \dots & j_m\end{pmatrix} A_{il}\begin{pmatrix}1 & \dots & \dots & \dots & m\\ j_1 & \dots & l & \dots & j_m\end{pmatrix}\right] = \sum_{l=1}^{n} a_{il}A_{li}^{(R,m)}.$$

For two integers $p \neq q$, $1 \leqslant p,q \leqslant m$, substituting in the minors of A in the expression

$$\mathrm{DET}_{(R,m)}(A) = \sum_{j_1<\cdots<j_m} \overline{R}\begin{pmatrix}1 & \dots & m\\ j_1 & \dots & j_m\end{pmatrix} A\begin{pmatrix}1 & \dots & m\\ j_1 & \dots & j_m\end{pmatrix}$$

the q-row by the p-row, starting from $\mathrm{DET}_{(R,m)}(A) = 0$, in a similar way it can be proved $\sum\limits_{l=1}^{n} a_{pl}A_{lq}^{(R,m)} = 0$.

In this way, $g_{ij} = \delta_{ij}\,\mathrm{DET}_{(R,m)}(A)$, and consequently $A \cdot A_{(R,m)}^{-1} = I_m$, which means that $A_{(R,m)}^{-1}$ represents a right inverse of a full-rank matrix A.

In a similar way it can be proved that $A_{(R,n)}^{-1}$ represents a left inverse in the case $t = n \leqslant m$. Now it is evident that (2.1) represents general determinantal representation of right/left inverses of a full rank matrix A.

2. For $R = A$ we obtain determinantal representation of $A^\dagger$, presented in Theorem 1.3. Then it is trivial to conclude $r_c(A) = r$, which represents a known result in [8].

3. If $m = n$, $\mathrm{ind}(A) = 1$ and $R = A^*$, the determinantal representation of the group inverse is obtained (Lemma 2.2).

4. If $r=r_c(A)$ and the matrix R satisfies condition

$$(U_2)\qquad \overline{R}\begin{pmatrix} i_1 & \dots & i_r \\ j_1 & \dots & j_r \end{pmatrix} = K\cdot\varepsilon^{(i_1+\dots+i_r)+(j_1+\dots+j_r)},\ \text{where } K\in\mathbf{C} \text{ and } \varepsilon\in\{-1,1\},$$
$$\text{for all combinations } 1\leqslant i_1<\dots<i_r\leqslant m;\ 1\leqslant j_1<\dots<j_r\leqslant n,$$

then, in the case $\varepsilon=1$, $A^{-1}_{(R,r)}$ is equal to Stojaković's inverse and the Radić's inverse in the case $\varepsilon=-1$ (Definition 1.1)

5. If $A=PQ$ is a full-rank factorization of $A\in\mathbf{C}_r^{m\times n}$ and if $W_1\in\mathbf{C}^{n\times r}$ and $W_2\in\mathbf{C}^{r\times m}$ are some matrices such that $\operatorname{rank}(QW_1)=\operatorname{rank}(W_2P)=\operatorname{rank}(A)=r$, then:

– for $R=PW_1^*$ we obtain determinantal representation of $A\{1,2,3\}$ (Lemma 2.1);

– for $R=W_2^*Q$ we obtain determinantal representation of $A\{1,2,4\}$ (Lemma 2.1);

– for $R=(W_1W_2)^*$ we obtain determinantal representation of $A\{1,2\}$ (Lemma 1.3).

6. For $R=MAN$, where M and N are positive definite matrices of the appropriate sizes, $A^{-1}_{(R,r)}$ reduces to the weighted Moore-Penrose inverse (Theorem 1.4).

7. For a regular matrix A formula (2.1) is transformed into the well known inversion of regular square matrices, for an arbitrary n by n matrix R.

8. If A is a full-rank matrix and each minor of R is the degree of the corresponding minor of A with the exponent $\frac{1}{2k-1}$, for some positive integer k, we obtain Radić's generalization of the Arghiriade-Dragomir representation of the Moore-Penrose inverse [18].

3. Examples

If a matrix R runs over the set of m by n matrices, in (2.1) we get various definitions of generalized inverses.

1. If $r=r_c(A)$ and a matrix R satisfies condiiton (U_1), then $A^{-1}_{(R,r)}$ is equal to Stojaković's inverse, i.e. equivalent Joshi's inverse, in the case $\varepsilon=1$ and the Radić's inverse, in the case $\varepsilon=-1$.

For example, consider matrix $A=\begin{pmatrix} \frac{11}{2} & \frac{22}{15} & 1 \\ \frac{3}{20} & \frac{-2}{7} & \frac{234}{233} \end{pmatrix}$. Using $R=\begin{pmatrix} 2 & 0 & -2 \\ 1 & 1 & 0 \end{pmatrix}$ we get the following Stojaković's inverse of A: $A^{-1}_{(R,2)}=\begin{pmatrix} \frac{58600}{440191} & \frac{619780}{1320573} \\ \frac{139335}{880382} & \frac{366975}{440191} \\ \frac{22135}{880382} & \frac{1720705}{1320573} \end{pmatrix}$. Now, using fixed point representation for the elements in A and the same matrix R we get

$$A=\begin{pmatrix} 5.50000000000000000000 & 1.53333333333333344000 & 1.00000000000000000000 \\ 0.14999999999999999400 & -0.28571428571428569800 & 1.00429184549356232000 \end{pmatrix},$$

and the following Stojaković's inverse of A:

$$A^{-1}_{(R,2)}=\begin{pmatrix}0.13312403025050492700 & -0.46932657263172883300\\ 0.15826652521291895100 & 0.83367220138530784300\\ 0.02514249496241404220 & 1.30299877401703657000\end{pmatrix}.$$

2. Furthermore, if $R=A$ satisfies (U_1), then $A^{-1}_{(R,r)}=A^{\dagger}$, and both generalized inverses are identical to the Stojaković's or Radić's generalized inverse.

Concretely for $R=A=\begin{pmatrix}\frac{5729}{327} & \frac{5729}{327} & 0\\ 0 & \frac{5729}{327} & \frac{5729}{327}\\ -\frac{5729}{327} & 0 & -\frac{5729}{327}\end{pmatrix}$ we get the following Moore-Penrose inverse of A:

$$A^{-1}_{(R,2)}=A^{\dagger}=\begin{pmatrix}\frac{2008044837}{256295929} & 0 & -\frac{2008044837}{256295929}\\ \frac{2008044837}{256295929} & \frac{2008044837}{256295929} & 0\\ 0 & \frac{2008044837}{256295929} & \frac{2008044837}{256295929}\end{pmatrix},$$

which is identical to the Stojaković's generalized inverse.

3. If $A\in\mathbf{C}^{m\times n}_r$ and $R=A$ we get $A^{-1}_{(R,r)}=A^{\dagger}$. For example, if we use $R=A=\begin{pmatrix}\frac{175}{23} & 0 & \frac{175}{23}\\ 0 & \frac{1}{13} & \frac{175}{23}\\ \frac{175}{23} & \frac{1}{13} & \frac{525}{46}\\ 0 & \frac{1}{13} & \frac{175}{23}\end{pmatrix}$, then we obtain

$$A^{-1}_{(R,2)}=A^{\dagger}=\begin{pmatrix}\frac{192878339}{497627891} & \frac{-201395239}{995255782} & \frac{-4258450}{497627891} & \frac{-201395239}{995255782}\\ \frac{1684865000}{497627891} & \frac{-1263648750}{497627891} & \frac{-421216250}{497627891} & \frac{-2263648750}{497627891}\\ \frac{-655721205}{497627891} & \frac{-1075979571}{995255782} & \frac{1281633260}{995255782} & \frac{-1075979571}{995255782}\end{pmatrix}.$$

4. For a square matrix A, such that $\mathrm{ind}(A)\leqslant 1$ and $R=A^*$ we get $A^{-1}_{(R,m)}=A^{\sharp}$. For example, let $A=\begin{pmatrix}21.93-3i & 4 & \frac{275}{35917} & 9.13570+2950.844725i\\ 11.35 & 35.75-2i & 0 & 1257420\\ 257384 & \frac{91584}{23} & 12+15i & \frac{213574}{5762403}\\ 159384-135i & 109825.23 & \frac{183294}{7359} & 0.000579\end{pmatrix}$, where $i=\sqrt{-1}$. Using $R=A^*$, we get

$$A^{\sharp}=A^{\dagger}=A^{-1}=$$

$$\begin{pmatrix}-0.005992-0.010995i & -0.000026+0.000014i & 0.000004+0.000001i & 0\\ -0.029221+0.010681i & 0.000025+0.000068i & -0.000004+0.000001i & 0.000010\\ 167.244553-23.230652i & 0.053301-0.392650i & -0.0109+0.000438i & -0.005703-0.000733i\\ 0.000001 & 0.000001 & 0 & 0\end{pmatrix}.$$

5. Consider the following matrix of rank 2: $A=\begin{pmatrix}2 & 0 & -5 & 4\\ 7 & -4 & -9 & \frac{3}{2}\\ 3 & -4 & 1 & -\frac{13}{2}\end{pmatrix}$. Matrices $P=\begin{pmatrix}2 & 0\\ 7 & -4\\ 3 & -4\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 0 & -\frac{5}{2} & 2\\ 0 & 1 & -\frac{17}{8} & \frac{25}{8}\end{pmatrix}$ form a full-rank factorization of A. If we use matrix $W_1=\begin{pmatrix}0 & 11\\ -2 & 2\\ 4 & 3\\ 2 & 7\end{pmatrix}$ satisfying $\mathrm{rank}(QW_1)=\mathrm{rank}(A)$ and $R=PW_1^T$, then the following $\{1,2,3\}$ inverse of A can be obtained:

$$A^{-1}_{(R,2)} = \begin{pmatrix} \frac{143}{735} & -\frac{11}{147} & -\frac{341}{735} \\ -\frac{3}{245} & \frac{20}{245} & \frac{26}{245} \\ \frac{109}{735} & -\frac{31}{147} & -\frac{373}{735} \\ \frac{6}{35} & -\frac{1}{7} & -\frac{17}{35} \end{pmatrix}.$$

6. In this example we show necessity of the condition $\mathrm{rank}(QW_1) = \mathrm{rank}(A)$. Given matrix $A = \begin{pmatrix} 2 & 0 & -5 & 4 \\ 7 & -4 & -9 & \frac{3}{2} \\ 3 & -4 & 7 & -\frac{13}{2} \end{pmatrix}$ of rank 3, full-rank factorization of A can be obtained from matrices $P = \begin{pmatrix} 2 & 0 & -5 \\ 7 & -4 & -9 \\ 3 & -4 & 7 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 & 0 & \frac{21}{8} \\ 0 & 1 & 0 & \frac{117}{32} \\ 0 & 0 & 1 & \frac{1}{4} \end{pmatrix}$. Now we use the matrix $W_1 = \begin{pmatrix} 0 & 11 & 0 \\ -2 & 2 & 4 \\ 4 & 3 & -8 \\ 2 & 7 & -4 \end{pmatrix}$ such that $\mathrm{rank}(QW_1) = 2$. For $R = PW_1^T$ we get:

$$X = A^{-1}_{(R,2)} = \begin{pmatrix} \frac{17}{5717} & \frac{990}{5717} & \frac{957}{5717} \\ \frac{129}{5717} & \frac{176}{5717} & -\frac{211}{5717} \\ -\frac{251}{5717} & \frac{278}{5717} & \frac{1031}{5717} \\ -\frac{120}{5717} & \frac{634}{5717} & \frac{994}{5717} \end{pmatrix},$$

and it is trivial to verify that the equations (1), (2) and (3) are not satisfied.

7. For $A \in \mathbf{C}^{m\times n}$ using $R = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $R \in \mathbf{C}^{m\times n}$, we obtain $\mathrm{DET}_{(R,r)}(A) = A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix}$ and the following algebraic complement of the element a_{ij}

$$A^{(R,r)}_{ij} = \begin{cases} 0, & \text{for } j > r \text{ or } i > r \\ A_{ji}\begin{pmatrix} 1 & \dots & i & \dots & r \\ 1 & \dots & j & \dots & r \end{pmatrix}, & \text{for } j, i \leqslant r. \end{cases}$$

Generalized inverse of $A \in \mathbf{C}^{m\times n}$, is equal to

$$A^{-1}_{(R,r)} = \frac{1}{A\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix}} \begin{pmatrix} A_{11}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{r1}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_{1r}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & \dots & A_{rr}\begin{pmatrix} 1 & \dots & r \\ 1 & \dots & r \end{pmatrix} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Concretely, for $A = \begin{pmatrix} \frac{13}{56} & 115 & \frac{476}{13} \\ \frac{1}{3} & -372 & \frac{23}{26} \\ -3 & \frac{14}{3} & \frac{21}{17} \\ \frac{12}{13} & 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the following right generalized inverse of A can be obtained:

$$A^{-1}_{(R,2)} = \begin{pmatrix} \frac{10652600}{6188751} & \frac{-8558144}{80453763} & \frac{-1364947612}{26817921} & 0 \\ \frac{35980}{2062917} & \frac{-3615752}{8939307} & \frac{-7448669}{160907526} & 0 \\ \frac{76388480}{18566253} & \frac{7857808}{6188751} & \frac{-1097248}{6188751} & 0 \end{pmatrix}.$$

Problem. Find determinantal representation of the Drazin inverse, i.e. determine the value of the matrix R corresponding to the Drazin inverse.

REFERENCES

[**1**] Arghiriade, E. et Dragomir, A., *Une nouvelle definition de l'inverse generalisee d'une matrice*, Lincei - Rend. Sc. fis. mat. e nat. **XXXV** (1963), 158–165

[**2**] Bapat, R.B., Bhaskara K.P.S. and Manjunatha Prasad, K., *General Inverses Over Integral Domains*, Linear Algebra Appl. **140** (1990), 181–196

[**3**] Ben-Israel, A. and Grevile, T.N.E., *Generalized inverses: Theory and applications*, Wiley-Interscience, New York 1974

[**4**] Ben-Israel, A., *Generalized inverses of matrices: a perspective of the work of Penrose*, Math. Proc. Camb. Phil. Soc. **100** (1986), 407–425

[**5**] Duane Pyle, L., *The Weighted Generalized Inverse in Nonlinear Programming-Active Set Selection Using a Variable-metric Generalization of the Simplex Algorithm*, International simposium on extremal methods and systems analysis, Lecture Notes in Economics and Mathematical Systems **174** (1977), 197–231

[**6**] Gabriel, R., *Extinderea Complementilor Algebrici Generalizati la Matrici Oarecare*, Studii si cercetari matematice **17**, 10 (1965), 1566–1581

[**7**] Gabriel, R., *Das verallgemeinerte Inverse einer Matrix, deren Elemente einem beliebigen Körper angehören*, J. Reine angew Math. **234** (1967), 107–122

[**8**] Gabriel, R., *Das verallgemeinerte Inverse einer Matrix, über einem beliebigen Körper - analytish betrachtet*, J. Reine angew Math. **244(V)** (1970), 83–93

[**9**] Joshi, V.N., *A determinant for rectangular matrices*, Bull. Australl. Math. Soc. **21** (1980), 137–146

[**10**] Manjunatha Prasad, K., Bhaskara Rao K.P.S. and Bapat, R.B., *General Inverses Over Integral Domains II. Group Inverses and Drazin Inverses*, Linear Algebra Appl. **146** (1991), 31–47

[**11**] Moore, E.H., *General Analysis, Part I. The Algebra of Matrices*, (compiled and edited by R.W. Barnard), The Amer. Philos. Soc. 1935

[**12**] Moore, E.H., *On the reciprocal of the general algebraic matrix (Abstract)*, Bull. Amer. Math. Soc. **26** (1920), 394–395

[**13**] Penrose, R., *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413

[**14**] Radhakrishna Rao, C. and Kumar Mitra, S., *Generalized inverse of matrices and its applications*, John Wiley & Sons, Inc, New York, London, Sydney, Toronto 1971

[**15**] Radić, M., *Some contributions to the inversion of rectangular matrices*, Glasnik matematički **1(21)**, 1 (1966), 23–37

[**16**] Radić, M., *Inverzija Pravokutnih matrica*, Doktorska disertacija, 1964.

[**17**] Radić, M., *A definition of the determinant of a rectangular matrix*, Glasnik matematički **1(21)**, 1 (1966), 17–22

[**18**] Radić, M., *On a generalization of the Arghiriade-Dragomir representation of the Moore-Penrose inverse*, Lincei-Rend. Sc. fis. mat. e nat. **44** (1968), 333–336

[**19**] Stanimirović, P. and Stanković, M., *Determinantal representation of weighted Moore-Penrose inverse*, Matematički Vesnik **46** (1994), 41–50

[**20**] Stojaković, M., *Determinante nekvadratnih matrica*, Vesnik D.M.N.R.S. **1–2** (1952), 9–21

(received 07.03.1995.)

University of Niš, Faculty of Philosophy, Department of Mathematics

Ćirila i Metodija 2, 18000 Niš, Yugoslavia