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ON UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS AND THEIR DERIVATIVES CORRESPONDING TO SELF-ADJOINT EXTENSIONS OF SCHRÖDINGER OPERATOR

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Abstract. In this paper we consider problem of the global uniform convergence of spectral expansions and their derivatives generated by arbitrary non-negative self-adjoint extensions of the Schrödinger operator

$$\mathcal{L}(u)(x) = -u''(x) + q(x)u(x) \tag{1}$$

with discrete spectrum, for functions in the Sobolev class $\overset{\circ}{W}{}_p^{(k)}(G)$ $(p > 1)$ defined on a finite interval $G \subset \mathbf{R}$.

Assuming that the potential $q(x)$ of the operator $\mathcal{L}$ belongs to the class $L_p(G)$ $(1 < p \le 2)$, we establish conditions ensuring the absolute and uniform convergence on the entire closed interval $\overline{G}$ of the series

$$\sum_{n=1}^{\infty}(f, u_n)_{L_2(G)} u_n(x), \qquad \sum_{n=1}^{\infty}(f, u_n)_{L_2(G)} u_n'(x)$$

if $f \in \overset{\circ}{W}{}_p^{(1)}(G)$ or $f \in \overset{\circ}{W}{}_p^{(2)}(G)$ respectively, where $\{u_n(x)\}_1^\infty$ is the orthonormal system of eigenfunctions corresponding to one of the mentioned extensions of operator (1). Also, increasing the smoothness of the functions $f(x)$ and $q(x)$ correspondingly, we prove a theorem concerning the absolute and uniform convergence on the entire closed interval $\overline{G}$ of the series

$$\sum_{n=1}^{\infty}(f, u_n)_{L_2(G)} u_n^{(2k)}(x), \qquad \sum_{n=1}^{\infty}(f, u_n)_{L_2(G)} u_n^{(2k+1)}(x), \qquad k \ge 1.$$

1. Introduction

1. Let $G = (a, b)$ be a finite interval of the real axis $\mathbf{R}$. Consider an arbitrary non-negative self-adjoint extension of the operator (1) with the potential $q(x) \in L_p(G)$ allowing the discrete spectrum; denote by $\{u_n(x)\}_1^\infty$ the orthonormal (and complete in $L_2(G)$) system of eigenfunctions corresponding to this extension, and by $\{\lambda_n\}_1^\infty$ the corresponding system of non-negative eigenvalues enumerated in nondecreasing order. (By definition, $u_n(x)$ is continuously differentiable and $u_n'(x)$

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is absolutely continuous on the closed interval $\overline{G}, u_n(x)$ satisfies the differential equation

$$-u_n''(x) + q(x)u_n(x) = \lambda_n u_n(x) \tag{2}$$

almost everywhere on (a, b), and this function satisfies the corresponding boundary conditions.)

Let $f(x) \in L_1(G)$ and let μ be an arbitrary positive number. We form the partial sum of order μ of the expansion of $f(x)$ in terms of the system $\{u_n(x)\}_1^\infty$:

$$\sigma_\mu(x, f) \stackrel{\text{def}}{=} \sum_{\sqrt{\lambda_n}<\mu} f_n u_n(x),$$

where $f_n \stackrel{\text{def}}{=} (f, u_n)_{L_2(G)}$ is the Fourier coefficient of $f(x)$ relative to that system.

2. We denote by $\overset{\circ}{W}{}_p^{(k)}(G)$ the set of functions $f(x)$ in the class $W_p^{(k)}(G)$ such that

$$f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0 = f(b) = f'(b) = \cdots = f^{(k-1)}(b).$$

(By definition, $f(x) \in W_p^{(k)}(G)$ if functions $f(x)$, $f'(x)$, ... , $f^{(k-2)}(x)$ are continuously differentiable on $[a, b]$, function $f^{(k-1)}(x)$ is absolutely continuous on $[a, b]$ and $f^{(k)}(x) \in L_p(G)$.)

Let $\mathcal{L}^k(f) \stackrel{\text{def}}{=} \mathcal{L}(\mathcal{L}(\cdots(\mathcal{L}(f))\cdots))$, with k appearences of $\mathcal{L}$. If $q(x)$ is in $W_p^{(2k-1)}(G)$ and $f(x) \in \overset{\circ}{W}{}_p^{(2k+1)}(G)$, then $\mathcal{L}^k(f)(x) \in \overset{\circ}{W}{}_p^{(1)}(G)$.

3. The following assertions are valid.

Theorem 1. *(a) If $q(x) \in L_p(G)$, $f(x) \in \overset{\circ}{W}{}_p^{(1)}(G)$ $(1 < p \le 2)$ and $f'(x)$ is a piecewise monotone function on $\overline{G}$, then the equality*

$$f(x) = \lim_{\mu\to+\infty} \sigma_\mu(x, f)$$

holds uniformly on $\overline{G}$.

(b) If $q(x) \in L_p(G)$ and $f(x) \in \overset{\circ}{W}{}_p^{(2)}(G)$ $(1 < p \le 2)$, then the equality

$$f'(x) = \lim_{\mu\to+\infty} \frac{d}{dx}\sigma_\mu(x, f)$$

holds uniformly on the entire closed interval $\overline{G}$.

Theorem 2. *(a) If $q(x) \in W_p^{(2k-1)}(G)$, $f(x) \in \overset{\circ}{W}{}_p^{(2k+1)}(G)$ $(1 < p \le 2$, $k \ge 1)$ and $\mathcal{L}^k(f)'(x)$ is a piecewise monotone function on $\overline{G}$, then*

$$f^{(j)}(x) = \lim_{\mu\to+\infty} \frac{d^j}{dx^j}\sigma_\mu(x, f), \qquad 0 \le j \le 2k,$$

uniformly on $\overline{G}$.

(b) If $q(x) \in W_p^{(2k)}(G)$ *and* $f(x) \in \overset{\circ}{W}{}_p^{(2k+2)}(G)$ $(1 < p \le 2,\ k \ge 1)$, *then the equalities*

$$f^{(j)}(x) = \lim_{\mu\to+\infty} \frac{d^j}{dx^j}\,\sigma_\mu(x,f), \qquad 0 \le j \le 2k+1,$$

hold uniformly on $\overline{G}$.

Remark 1. It will be shown, under the assumptions of Theorems 1–2, that the coresponding series

$$\sum_{n=1}^{\infty} f_n u_n(x), \quad \sum_{n=1}^{\infty} f_n u_n'(x), \quad \ldots, \quad \sum_{n=1}^{\infty} f_n u_n^{(2k-1)}(x)$$

converge absolutely on the closed interval $\overline{G}$.

Remark 2. The assertions of Theorems 1–2 are in "well accordance" with the corresponding classical results for the global uniform convergence of the trigonometrical Fourier series.

As far as the uniform convergence on compact subsets of G concerned, the exact conditions for that covergence were obtained by means of uniform equiconvergence theorems in [3], [4] and [6].

2. Proof of theorem 1

1. The idea of the proof is very simple. It is based on some upper-bound estimates for f_n, $u_n(x)$, $u_n'(x)$, $u_n''(x)$, ... , with respect to λ_n. Thus, we first list the necessary estimates.

Let $\{u_n(x)\}_1^\infty$ be the orthonormal system of eigenfunctions corresponding to an arbitrary non-negative self-adjoint extension of the operator (1), and let $\{\lambda_n\}_1^\infty$ be the corresponding system of eigenvalues enumerated in nondecreasing order. Then the following assertions hold.

(a) If $q(x) \in L_1(G)$, then there exists a constant $C > 0$, independent of $n \in \mathbf{N}$, such that

$$\max_{x\in\overline{G}} |u_n(x)| \le C, \qquad n \in \mathbf{N}. \tag{3}$$

(b) If $q(x) \in L_p(G)$ $(p > 1)$ then there exists a constant $A > 0$ such that

$$\sum_{t\le\sqrt{\lambda_n}\le t+1} 1 \le A \tag{4}$$

for every $t \ge 0$, where A does not depend on t.

(c) If $q(x) \in L_1(G)$, then there exists a constant $C_1 > 0$, not depending on $n \in \mathbf{N}$, such that

$$\max_{x\in\overline{G}} |u_n'(x)| \le \begin{cases} C_1\sqrt{\lambda_n}, & \text{if } \lambda_n > 1, \\ C_1, & \text{if } 0 \le \lambda_n \le 1. \end{cases} \tag{5}$$

(d) Suppose $q(x) \in L_1(G) \cap C^{(j-2)}(G)$ $(j \geq 2)$, and the derivatives of $q(x)$ are bounded on G. Then the eigenfunction $u_n(x)$ has bounded continuous derivatives up to the j-th order, and there exists a constant $C_j > 0$, independent of $n \in \mathbf{N}$, such that

$$\max_{x\in\overline{G}} |u_n^{(j)}(x)| \leq \begin{cases} C_j \lambda_n^{j/2}, & \text{if } \lambda_n > 1, \\ C_j, & \text{if } 0 \leq \lambda_n \leq 1. \end{cases} \tag{6}$$

The propositions (a)–(b) were proved in [2], and (c)–(d) in [5].

2. We will also use an inequality of Riesz. Let $\{\varphi_n(x)\}_1^\infty$ be an orthonormal on G system of (complex-valued) functions such that there exists a constant $M > 0$, not depending on $n \in \mathbf{N}$, with $\sup_{x\in G} |\varphi_n(x)| \leq M$ for every $n \in \mathbf{N}$. If $g(x) \in L_p(G)$ $(1 < p \leq 2)$, then the Fourier coefficients $g_n \overset{\text{def}}{=} \int_a^b g(x)\,\overline{\varphi_n(x)}\,dx$ satisfy the inequality

$$\left(\sum_{n=1}^{\infty} |g_n|^r\right)^{1/r} \leq M^{(2/p-1)} \|g\|_{L_p(G)}, \tag{7}$$

where $1/p + 1/r = 1$ (see [1], p. 154).

3. Now we can prove Theorem 1. Let $f(x) \in \overset{\circ}{W}{}_p^{(1)}(G)$ and let $f'(x)$ be a monotone function on the closed intervals $[x_{i-1}, x_i]$ $(1 \leq i \leq l)$, where $a = x_0 < x_1 < \cdots < x_{l-1} < x_l = b$. If $\lambda_n \neq 0$, then using equation (2), the boundary conditions imposed on the function $f(x)$ and the partial diferentiation, we have

$$\begin{aligned} f_n &= \int_a^b f(x) u_n(x)\,dx = \frac{1}{\lambda_n} \int_a^b f(x)\bigl[-u_n''(x) + q(x)u_n(x)\bigr]\,dx \\ &= \frac{1}{\lambda_n} \int_a^b f'(x) u_n'(x)\,dx + \frac{1}{\lambda_n} \int_a^b q(x) f(x) u_n(x)\,dx. \end{aligned} \tag{8}$$

By the Bonnet formula we get

$$\begin{aligned} \int_a^b f'(x)u_n'(x)\,dx &= \sum_{i=1}^{l} \int_{x_{i-1}}^{x_i} f'(x)u_n'(x)\,dx \\ &= \sum_{i=1}^{l} \bigl[f'(x_{i-1}+0)\bigl(u_n(\xi_i) - u_n(x_{i-1})\bigr) + f'(x_i - 0)\bigl(u_n(x_i) - u_n(\xi_i)\bigr)\bigr] \end{aligned} \tag{9}$$

for some point $\xi_i \in [x_{i-1}, x_i]$.

Denote by $s(f'; u_n)$ and h_n the sum in (9) and the last integral in (8) respectively. Then estimate (3) and the Hölder inequality give us the estimates

$$\begin{aligned} |s(f'; u_n)| &\leq 2C \cdot \sum_{i=1}^{l} \bigl(|f'(x_{i-1}+0)| + |f'(x_i - 0)|\bigr) \overset{\text{def}}{=} C_f, \\ |h_n| &\leq C\|f\|_{L_r(G)} \cdot \|q\|_{L_p(G)}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{r} = 1. \end{aligned} \tag{10}$$

The absolute and uniform convergence of the series $\sum_{n=1}^{\infty} f_n u_n(x)$ on the closed interval $\overline{G}$ results now from the following formal chain of inequalities, obtained by (8) and the estimates (3)–(4) and (10):

$$\sum_{n=1}^{\infty} |f_n|\,|u_n(x)| \le$$
$$\le \sum_{0\le\sqrt{\lambda_n}\le 1} |f_n|\,|u_n(x)| + \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n}\big(|s(f';u_n)| + |h_n|\big)|u_n(x)|$$
$$\le AC^2\|f\|_{L_1(G)} + C\big(C_f + C\|f\|_{L_r(G)}\cdot\|q\|_{L_p(G)}\big)\cdot\sum_{k=1}^{\infty}\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1}\frac{1}{\lambda_n}\Bigg)$$
$$\le D_1 + D_2\sum_{k=1}^{\infty}\frac{1}{k^2}\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1} 1\Bigg) \le D_1 + AD_2\sum_{k=1}^{\infty}\frac{1}{k^2},$$

where D_1, D_2 have the obvious meaning.

4. In order to prove the proposition (b) of theorem 1, we should transforme the Fourier coefficient f_n in the following way:

$$f_n = \int_a^b f(x)u_n(x)\,dx = \frac{1}{\lambda_n}\int_a^b f(x)\big[-u_n''(x) + q(x)u_n(x)\big]\,dx$$
$$= \frac{1}{\lambda_n}\int_a^b \big[-f''(x) + q(x)f(x)\big]u_n(x)\,dx = \frac{1}{\lambda_n}\mathcal{L}(f)_n, \quad \lambda_n \ne 0, \qquad (11)$$

where $\mathcal{L}(f)_n$ $(n\in\mathbf{N})$ denote the Fourier coefficients of function $\mathcal{L}(f)(x)\in L_p(G)$ relative to the system $\{u_n(x)\}_1^{\infty}$.

Now, using (11), the estimates (3)–(5), (7) and the Hölder inequality, we formally get

$$\sum_{n=1}^{\infty} |f_n|\,|u_n'(x)| =$$
$$= \sum_{0\le\sqrt{\lambda_n}\le 1} |f_n|\,|u_n'(x)| + \sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}|\mathcal{L}(f)_n|\,|u_n'(x)| \le$$
$$\le ACC_1\|f\|_{L_1(G)} + C_1\sum_{k=1}^{\infty}\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1}\frac{1}{\sqrt{\lambda_n}}|\mathcal{L}(f)_n|\Bigg)$$
$$\le \tilde{D}_1 + C_1\cdot\sum_{k=1}^{\infty}\frac{1}{k}\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1}|\mathcal{L}(f)_n|^r\Bigg)^{1/r}\cdot\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1} 1\Bigg)^{1/p}$$
$$\le \tilde{D}_1 + A^{1/p}C_1\Bigg(\sum_{k=1}^{\infty}\frac{1}{k^p}\Bigg)^{1/p}\cdot\Bigg(\sum_{k=1}^{\infty}\Bigg(\sum_{k<\sqrt{\lambda_n}\le k+1}|\mathcal{L}(f)_n|^r\Bigg)\Bigg)^{1/r}$$
$$\le \tilde{D}_1 + \tilde{D}_2 C^{(2/p-1)}\cdot\|\mathcal{L}(f)\|_{L_p(G)}, \qquad (12)$$

wherefrom we conclude that the series $\sum_{n=1}^{\infty} f_n u_n'(x)$ converges absolutely and uniformly on the interval $\overline{G}$.

Further, it is not difficult to see that, replacing $u_n'(x)$ in (12) by $u_n(x)$, the corresponding chain of inequalities gives us the absolute and uniform convergence on $\overline{G}$ of the series $\sum_{n=1}^{\infty} f_n u_n(x)$ under assumptions from the proposition (b) of theorem 1.

5. By the completeness and orthonormality of the system $\{u_n(x)\}_1^{\infty}$, using the standard "uniform convergence" arguments and the previously obtained results, we can prove that the equalities

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x), \qquad f'(x) = \sum_{n=1}^{\infty} f_n u_n'(x)$$

hold for every $x \in \overline{G}$.

Proof of Theorem 1 is completed.

3. Proof of theorem 2

1. All the important elements of the proof are actually clarified in the previous section. That is why we will consider in detail the proposition (a) only.

As it was already mentioned, if $q(x) \in W_p^{(2k-1)}(G)$ and $f(x) \in \overset{\circ}{W}{}_p^{(2k+1)}(G)$, then $\mathcal{L}^k(f)(x) \in \overset{\circ}{W}{}_p^{(1)}(G)$, and all the functions $\mathcal{L}^j(f)(x)$ $(1 \le j \le k)$ and their first derivatives take the zerovalues at the points a and b. Therefore, if $\lambda_n \ne 0$, then we have

$$\begin{aligned} f_n &= \int_a^b f(x)u_n(x)\,dx = \frac{1}{\lambda_n}\int_a^b \mathcal{L}(f)(x)u_n(x)\,dx \\ &= \frac{1}{\lambda_n^2}\int_a^b \mathcal{L}^2(f)(x)u_n(x)\,dx = \cdots = \frac{1}{\lambda_n^k}\int_a^b \mathcal{L}^k(f)(x)u_n(x)\,dx \\ &= \frac{1}{\lambda_n^{k+1}}\int_a^b \mathcal{L}^k(f)'(x)u_n'(x)\,dx + \frac{1}{\lambda_n^{k+1}}\int_a^b \mathcal{L}^k(f)(x)q(x)u_n(x)\,dx. \end{aligned} \tag{13}$$

Let $\mathcal{L}^k(f)'(x)$ be a piecewise monotone function on $\overline{G}$; there exist closed intervals $[t_{i-1}, t_i]$ $(1 \le i \le m)$ such that $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$, and $\mathcal{L}^k(f)'(x)$ is monotone on every $[t_{i-1}, t_i]$. Using the Bonnet formula, we have

$$\int_a^b \mathcal{L}(f)'(x)u_n'(x)\,dx = \sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} \mathcal{L}^k(f)'(x)u_n'(x)\,dx$$

$$= \sum_{i=1}^{m}\big[\mathcal{L}^k(f)'(t_{i-1}+0)\big(u_n(\xi_i) - u_n(t_{i-1})\big) + \mathcal{L}^k(f)'(t_i - 0)\big(u_n(t_i) - u_n(\xi_i)\big)\big]$$

for some points $\xi_i \in [t_{i-1}, t_i]$. Thus, denoting by $s(\mathcal{L}^k(f)'; u_n), h_n$ the above sum and the last integral in (13) respectively, we get the equality

$$f_n = \frac{1}{\lambda_n^{k+1}} s(\mathcal{L}^k(f)'; u_n) + \frac{1}{\lambda_n^{k+1}} h_n \qquad (\lambda_n \neq 0). \tag{14}$$

It results from estimate (3) and the Hölder inequality that

$$|s(\mathcal{L}^k(f)'; u_n)| \le 2C \sum_{i=1}^{m} \big(|\mathcal{L}^k(f)'(t_{i-1}+0)| + |\mathcal{L}^k(f)'(t_i - 0)| \big) \stackrel{\text{def}}{=} C_{f,q},$$

$$|h_n| \le C \|\mathcal{L}^k(f)\|_{L_p(G)} \cdot \|q\|_{L_r(G)}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{r} = 1.$$

Now, using (14), the estimates (3)–(6) and the above estimate, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} |f_n| \, |u_n^{(2k)}(x)| \le \\
\le & \sum_{0 \le \sqrt{\lambda_n} \le 1} |f_n| \, |u_n^{(2k)}(x)| + \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{k+1}} \big(|s(\mathcal{L}^k(f)'; u_n)| + |h_n| \big) |u_n^{(2k)}(x)| \\
& \le ACC_{2k} \|f\|_{L_1(G)} + C_{2k} \big(C_{f,q} + C \|\mathcal{L}^k(f)\|_{L_p(G)} \cdot \|q\|_{L_r(G)} \big) \times \\
& \times \sum_{k=1}^{\infty} \Bigg(\sum_{k < \sqrt{\lambda_n} \le k+1} \frac{1}{\lambda_n} \Bigg) \le D_3 + AD_4 \sum_{k=1}^{\infty} \frac{1}{k^2},
\end{aligned} \tag{15}$$

where D_3 and D_4 have the obvious meaning. It results from (15) that the series $\sum_{n=1}^{\infty} f_n u_n^{(2k)}(x)$ converges absolutely and uniformly on $\overline{G}$.

Also, replacing $u_n^{(2k)}(x)$ in (15) by $u_n(x)$, $u_n'(x)$, ... , $u_n^{(2k-1)}(x)$ respectively, and using the corresponding estimates (5)–(6), we can conclude that the series $\sum_{n=1}^{\infty} f_n u_n^{(j)}(x)$ $(j = 0, 1, \ldots, 2k-1)$ converge absolutely and uniformly on $\overline{G}$ (under the assumptions from the proposition (a)).

2. It follows then by the orthonormality and completeness of the system $\{u_n(x)\}_1^{\infty}$, and by the known rules for differentiation of uniformly convergent series, that the equalities $f^{(j)}(x) = \sum_{n=1}^{\infty} f_n u_n^{(j)}(x)$ $(j = 0, 1, \ldots, 2k)$ hold uniformly on the closed interval $\overline{G}$.

The proposition (a) is proved.

3. Consider now the series $\sum_{n=1}^{\infty} f_n u_n^{(2k+1)}(x)$. It is not difficult to see, under conditions imposed on $q(x)$ and $f(x)$ in the proposition (b), that for $\lambda_n \neq 0$ the equality

$$f_n = \frac{1}{\lambda_n^{k+1}} \mathcal{L}^{k+1}(f)_n \tag{16}$$

holds, where $\mathcal{L}^{k+1}(f)_n$ $(n \in \mathbf{N})$ denote the Fourier coefficients of function $\mathcal{L}^{k+1}(f)(x) \in L_p(G)$.

Using (16), the estimates (3)–(4), (6)–(7) and the Hölder inequality, we can obtain a formal chain of inequalities for the series $\sum_{n=1}^{\infty} |f_n|\,|u_n^{(2k+1)}(x)|$ which is completely analogous to the inequalities (12). By virtue of this chain one can prove that the series $\sum_{n=1}^{\infty} f_n u_n^{(j)}(x)$ $(j = 0, 1, \ldots, 2k+1)$ converge absolutely and uniformly on $\overline{G}$, and that the equalities $f^{(j)}(x) = \sum_{n=1}^{\infty} f_n u_n^{(j)}(x)$ hold everywhere on $\overline{G}$, for $j = 0, 1, \ldots, 2k+1$. The details are left to the reader.

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