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GENERALIZED EIGENVECTOR EXPANSION FOR WEAKLY PERTURBATED DISCRETE OPERATORS

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Abstract. In this paper we consider the expansion theorem in generalized eigenvectors of the operator $A = L + T$, where L is a discrete, positive selfadjoint operator in a separable Hilbert space, and T is a closed operator which is subordinated to L in a certain sense.

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$ and let L be a discrete, positive selfadjoint operator on $\mathcal{H}$. Vector $x \neq 0$ is a generalized eigenvector (for the eigenvalue λ) if for some $k \geqslant 1$ $(\lambda - L)^k x = 0$. Denote by $N(\cdot)$ the eigenvalue distribution function of L. Let $\mathcal{D}(L)$ and $\mathcal{D}(T)$ denote the domain of the operators L and T, respectively.

In this paper we consider the expansion theorem for the operator $A = L + T$, where T is a closed operator which is subordinated to L in a certain sense.

In the case when T is a bounded operator, $L = L^*$ is a discrete operator and $\lambda_{n+1}(L) - \lambda_n(L) \to \infty$ $(n \to \infty)$ the problem was solved in [3].

THEOREM 1. *Suppose that T is a closed operator on $\mathcal{H}$, $L = L^*$ is a positive discrete operator, $\mathcal{D}(L) \subset \mathcal{D}(T)$, $A = L + T$,*

$$\|Tx\| \leqslant C\|L^\beta x\|, \quad x \in \mathcal{D}(L), \tag{1}$$

and numbers α and β satisfy one of the following two conditions:
a) $0 < \beta < 1$, $0 < \alpha < \frac{2}{3}(1-\beta)$ and $N(t) = C_0 t^\alpha (1 + o(1))$ $(t \to +\infty)$;
b) $0 < \beta < 1$, $0 < \alpha < 1 - \beta$ and $N(t) = C_0 t^\alpha (1 + O(t^{-\delta}))$, $\alpha < \delta < 1$ $(t \to +\infty)$.
Then for every $f \in \mathcal{D}(L)$ we have

$$f = \sum_{k=1}^{\infty} \left(\sum_{s=1}^{n_k} c_{ks} x_{ks} \right), \tag{2}$$

where x_{ks} are generalized eigenvectors of A and $c_{ks} \in \mathbf{C}$.

Proof. Suppose that $\{e_n\}_{n=1}^\infty$ is the system of eigenvectors of L $(Le_n = \lambda_n e_n)$. Since $L = L^*$, $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. Then

$$(L - \lambda)^{-1} = \sum_{n=1}^{\infty} \frac{(\,\cdot\,, e_n) e_n}{\lambda_n - \lambda}$$

and

$$T(L-\lambda)^{-1} = \sum_{n=1}^{\infty} \frac{(\,\cdot\,, e_n)Te_n}{\lambda_n - \lambda}. \tag{3}$$

From (1) and (3), applying Cauchy's inequality, we conclude that

$$\|T(L-\lambda)^{-1}\| \leqslant C^{1/2}\left(\sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}}{|\lambda-\lambda_n|^2}\right)^{1/2}. \tag{4}$$

By the following Lemma, the righthandside of this inequality tends to zero if λ belongs to a certain sequence of circles with radii tending to infinity.

LEMMA. *If either of the conditions a) and b) of the Theorem 1 is satisfied, then there exists a sequence of circles* $\Gamma_k = \{\,\lambda : |\lambda| = r_k\,\}$, $\lim_{k\to\infty} r_k = \infty$, *such that*

$$\lim_{k\to\infty} \max_{\lambda\in\Gamma_k} \left(\sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{|\lambda-\lambda_\nu|^2}\right) = 0. \tag{5}$$

Since $\lim_{n\to\infty} \max_{\lambda\in\Gamma_n} \|T(\lambda-L)^{-1}\| = 0$ (follows from (4) and the Lemma), it follows from $(\lambda-A)^{-1} = (\lambda-L)^{-1}(I-T(\lambda-L)^{-1})^{-1}$ that the operator A is discrete and

$$\lim_{k\to\infty} \max_{\lambda\in\Gamma_k} \|(\lambda-A)^{-1}\| = 0. \tag{6}$$

From (6) and Naymark's theorem [4] we obtain the relation (2), for all $f \in \mathcal{D}(L)$, where x_{ks}, $s = 1, 2, \dots, n_k$, are the generalized eigenvectors corresponding to eigenvalues lying in the ring $\{\,\lambda : r_k < |\lambda| < r_{k+1}\,\}$. ■

REMARK. In the case when in each interval I of the fixed length l the number of eigenvalues λ of A with property $\operatorname{Re}\lambda \in I$ is uniformly bounded, the Riesz basis property of the generalized eigenvectors system was proved in [1] (under some aditional conditions).

Proof of the Lemma. Case a). It follows from $N(t) = C_0 t^\alpha(1+o(1))$ that $\lambda_n = C_0^{-1/\alpha} n^{1/\alpha}(1+o(1))$. Let q be a real number such that

$$0 < \alpha q < C_0^{-1/\alpha}. \tag{7}$$

Denote by S the set of natural numbers n such that $\lambda_{n+1} - \lambda_n \geqslant q n^{1/\alpha - 1}$. Suppose that S is finite, i.e. $S = \{n_1, n_2, \dots, n_s\}$. Then we have $\lambda_{n+1} - \lambda_n < q n^{1/\alpha-1}$ for all $n > n_s + 1$ and

$$\lambda_{N+1} - \lambda_{n_s+1} < q \sum_{\nu=n_s+1}^{N} \nu^{1/\alpha-1} < q \int_{n_s+1}^{N+1} x^{1/\alpha-1}\,dx = \alpha q[(N+1)^{1/\alpha} - (n_s+1)^{1/\alpha}],$$

i.e.

$$\frac{\lambda_{N+1} - \lambda_{n_s+1}}{N^{1/\alpha}} \leqslant \alpha q \frac{(N+1)^{1/\alpha} - (n_s+1)^{1/\alpha}}{N^{1/\alpha}}$$

for each $N > n_s$. When $N \to \infty$ we obtain $C_0^{-1/\alpha} \leqslant \alpha q$, i.e. a contradiction with (7). So, it follows that S is an infinite set.

Let $\Gamma_\nu = \{\, \lambda : |\lambda| = r_\nu = \frac{1}{2}(\lambda_{n_\nu+1} + \lambda_{n_\nu}) \,\}$. We will prove now the realtion (5). If $\lambda \in \Gamma_k$, then

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{|\lambda - \lambda_\nu|^2} \leqslant \sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2}$$
$$= \sum_{\nu=1}^{n_k-1} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} + \sum_{\nu=n_k+2}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} + \frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} + \frac{\lambda_{n_k+1}^{2\beta}}{(r_k - \lambda_{n_k+1})^2}.$$

As we have $0 < \alpha < \frac{2}{3}(1-\beta)$, by direct computation we get

$$\lim_{k\to\infty} \left[\frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} + \frac{\lambda_{n_k+1}^{2\beta}}{(r_k - \lambda_{n_k+1})^2} \right] = 0. \tag{8}$$

Since the function $\varphi(x) = x^\beta/(r_k - x)$ is nondecreasing on $[0, r_k)$, we obtain

$$\sum_{\nu=1}^{n_k-1} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} \leqslant \mathrm{const} \cdot n_k \frac{\lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k})^2} \leqslant \frac{\mathrm{const}}{n_k^{\frac{2}{3}-3-\frac{2\beta}{\alpha}}} \to 0 \ (k \to \infty). \tag{9}$$

Since

$$\sum_{\nu=n_k+2}^{\infty} \frac{\lambda_\nu^{2\beta}}{(r_k - \lambda_\nu)^2} = \int_{\lambda_{n_k+1}}^{\infty} \frac{t^{2\beta}}{(r_k - t)^2}\, dN(t)$$
$$= \frac{n_k \lambda_{n_k}^{2\beta}}{(r_k - \lambda_{n_k+1})^2} - \int_{\lambda_{n_k+1}}^{\infty} N(t) \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt,$$

it is enough to prove that

$$\lim_{k\to\infty} \int_{\lambda_{n_k+1}}^{\infty} t^\alpha \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt = 0. \tag{10}$$

The function $G(x) = \int_x^\infty [(\beta-1)u - \beta]/(u-1)^3\, du$ $(x > 1)$ has the following asymptotical behavior in the neighborhood of $x = 1$: $G(x) \sim \frac{1}{2}(x-1)^{-2}$. Then (10) follows from

$$\int_{\lambda_{n_k+1}}^{\infty} t^\alpha \left(\frac{t^{2\beta}}{(r_k - t)^2} \right)' dt = 2 r_k^{\alpha+2\beta-2} G(c_k) \sim \frac{r_k^{\alpha+2\beta}}{(\lambda_{n_k+1} - r_k)^2} \to 0 \ (k \to \infty),$$

where $c_k = \lambda_{n_k+1}/r_k$ $(\to 1)$. From (8), (9) and (10) we obtain (5).

Case b). It follows from b) that

$$\lambda_n = C_0^{-1/\alpha} n^{1/\alpha} (1 + O(n^{-\delta/\alpha})). \tag{11}$$

Let $\mu_n = C_0^{-1/\alpha} n^{1/\alpha}$ and $\Gamma_n = \{\, \lambda : |\lambda| = r_n = \frac{1}{2}(\mu_n + \mu_{n+1}) \,\}$. From (11) we get

$$\sup_{n,\nu} \left| \frac{\lambda_\nu - \mu_\nu}{r_n - \lambda_\nu} \right| < \infty. \tag{12}$$

If $\lambda \in \Gamma_n$, then from (12) we obtain

$$\sum_{\nu=1}^{\infty} \frac{\lambda_\nu^{2\beta}}{|\lambda - \lambda_\nu|^2} \leqslant \text{const} \sum_{\nu=1}^{\infty} \frac{\mu_\nu^{2\beta}}{(r_n - \mu_\nu)^2}.$$

As in the case a) it can be proved that

$$\sum_{\nu=1}^{\infty} \frac{\mu_\nu^{2\beta}}{(r_n - \mu_\nu)^2} \to 0 \quad (n \to \infty)$$

for $0 < \alpha < 1 - \beta$. The Lemma is proved. ∎

Example. Suppose m, n and r are integers, $m \geqslant 1$, $n \geqslant 2$, $0 < r < m$, Ω is a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary, L is a formal selfadjoint eliptic differential expression

$$L = (-1)^{m/2} \sum_{|k|=m} a_k(x) D^k$$

with smooth coefficients and T is a linear differential expression

$$T = \sum_{|k| \leqslant r} b_k(x) D^k$$

with smooth complex functions b_k. Let $A\colon \mathcal{D}(A) \to L^2(\Omega)$ $(\mathcal{D}(A) = W_2^m \cup \overset{\circ}{W}{}_2^{m/2})$ be a differential operator defined by $A = L + T$. Then we get

Theorem 2. *If $n/m < \frac{2}{3}(1 - r/m)$, the for $f \in \mathcal{D}(A)$ the expansion theorem in generalized eigenvectors of the operator A holds.*

Proof. The statement of the theorem is obtained from Theorem 1 for $\alpha = n/m$, $\beta = r/m$ (see [2]). ∎

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