МАТЕМАТИЧКИ ВЕСНИК
45 (1993), 29–33

UDC 517.982.2
оригинални научни рад
research paper

AN INEQUALITY RELATED TO THE UNIFORM CONVEXITY IN BANACH SPACES

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Abstract. We prove an inequality that implies that a 2-concave and p-convex Banach lattice is "more" uniformly convex than L^p.

1. Introduction

In this note we prove the following

THEOREM. *Let X be a 2-concave Banach lattice with 2-concavity constant equal to one, and let $1 \leqslant p \leqslant 2$. Then*

$$\|(|x+y|^p + |x-y|^p)^{1/p}\| \geqslant \left\{(\|x\|+\|y\|)^p + \big|\|x\|-\|y\|\big|^p\right\}^{1/p}, \tag{1}$$

for all $x, y \in X$. In particular, inequality (1) *holds in an arbitrary L^q space with $1 \leqslant q \leqslant 2$.*

For the definition of the expression $(|u|^p+|v|^p)^{1/p}$ and other notions concerning abstract Banach lattices we refer to [3], Ch. 1 (especially Theorem 1.d.1). In the case where $X = L^p$ $(1 < p < 2)$ inequality (1) becomes

$$\|x+y\|^p + \|x-y\|^p \geqslant (\|x\|+\|y\|)^p + \big|\|x\|-\|y\|\big|^p, \tag{2}$$

which was used by Hanner [2] to calculate the precise value of the modulus of convexity of L^p. Moreover, it follows from [4] that the validity of (2) in some normed spaces X implies that X is "more" uniformly convex than L^p (where L^p is at least two-dimensional). An immediate consequence of Theorem is that (1) holds in a large class (denoted by $\Delta(p,2)$; see Section 2) containing, for example, L^q for $p \leqslant q \leqslant 2$ as well as certain Orlicz and mixed normed Lebesgue spaces. Note that, in [4], the validity of (2) in L^q $(p \leqslant q \leqslant 2)$ was deduced from the case $q = p$ by using the fact that L^q can be embedded into $L^p(0,1)$ isometrically (see [3], pp. 181–182). The proof in the present note is quite elementary and lies on the fact that (for $1 \leqslant p \leqslant 2$) the function

$$F_p(\xi,\eta) := \left\{(\xi^{1/2}+\eta^{1/2})^p + |\xi^{1/2}-\eta^{1/2}|^p\right\}^{2/p} \qquad (\xi \geqslant 0,\ \eta \geqslant 0) \tag{3}$$

is convex. Before proving the result we mention a generalization of F_p that could be of some independent interest. Let r_j $(j = 0, 1, 2, \dots)$ denote the Rademacher functions,

$$r_j(t) = \operatorname{sign}(\sin(2^j \pi t)) \qquad (t \text{ real}).$$

Define the functions Φ_p on the positive cone l^1_+ of the sequence space l^1 by

$$\Phi_p(\xi) = \left\{ \int_0^1 \left| \sum_{j=0}^{\infty} r_j(t) \xi_j^{1/2} \right|^p dt \right\}^{2/p} \qquad (\xi = (\xi_j)_0^\infty \geqslant 0).$$

That the definition is correct follows from the well known fact that if $(a_j)_0^\infty \in l^2$, then the series $\sum a_j r_j(t)$ converges almost everywhere, and from Khintchine's inequality [3], Theorem 2.b.3, which says that

$$A_p \|\xi\|_{l^1} \leqslant \Phi_p(\xi) \leqslant B_p \|\xi\|_{l^1} \qquad (A_p, B_p = \text{const} > 0).$$

Starting from the observation that $\Phi(\xi_1, \xi_2, 0, 0, \dots) = \text{const}\ F_p(\xi_1, \xi_2)$ we conjecture that Φ_p is a convex function on l^1_+ (for $1 \leqslant p \leqslant 2$). (We shall also prove that if $p > 2$, then F_p is concave, and we conjecture that Φ_p is concave if $p > 2$).

This would lead to the inequality

$$\|\Phi_p(x_1, x_2, \dots)\| \geqslant \Phi_p(\|x_1\|, \|x_2\|, \dots),$$

where x_1, x_2, ... are elements of a Banach lattice whose 2-concavity constant is eual to one. Further remarks are at the end of the paper.

2. Definitions and examples

We denote by $\Delta(p, q)$, where $1 \leqslant p \leqslant q \leqslant +\infty$, the class of (real) Banach lattices X such that

$$\|(|u|^p + |v|^p)^{1/p}\| \leqslant (\|u\|^p + \|v\|^p)^{1/p} \tag{4}$$

and

$$\|(|u|^q + |v|^q)^{1/q}\| \geqslant (\|u\|^q + \|v\|^q)^{1/q} \tag{5}$$

for all $u, v \in X$. In other words, X is in $\Delta(p, q)$ if it is p-convex and q-concave and its p-convexity and q-concavity constants are equal to one. It is clear that $\Delta(1, \infty)$ is just the class of all Banach lattices. And by [3], Proposition 1.d.5, $\Delta(p, q)$ is contained in $\Delta(r, s)$ for $r \leqslant p \leqslant q \leqslant s$. In particular, $L^q \in \Delta(r, s)$ if $r \leqslant q \leqslant s$, a fact which can easily be verified by direct calculations.

It was proved by Figiel [1] (see also [3], pp. 80–81) that if $X \in \Delta(p, q)$ with $p > 1$ and $q < +\infty$, then X is uniformly convex in the sense that

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x - y\| = \varepsilon,\ \|x\| = \|y\| = 1 \right\} > 0$$

for $\varepsilon > 0$. The function δ_X is called the modulus of convexity of X. Let δ_p denote the modulus of convexity of L^p, $\dim(L^p) \geqslant 2$. (It follows from [2] that δ_p is independent of a particular choice of L^p.) As noted in Introduction, the following fact follows immediately from (1) and (4).

COROLLARY 1. *If* $X \in \Delta(p,2)$ *(in particular,* $X = L^q$ *for* $2 \geqslant q \geqslant p$*), then inequality* (2) *holds.*

As noted in Introduction, this implies the following

COROLLARY 2. *If* $X \in \Delta(p,2)$*, then* $\delta_X(\varepsilon) \geqslant \delta_p(\varepsilon)$ $(\varepsilon > 0)$.

Mixed normed spaces. For technical reasons we define only sequence spaces. Let $1 \leqslant r,s \leqslant 2$. The space $X = l^{r,s}$ consists of those scalar sequences $x = \{x_{j,k}\}_{j,k=0}^{\infty}$ such that

$$\|x\| = \left\{ \sum_{j=0}^{\infty} \left[\sum_{k=0}^{\infty} |x_{j,k}|^s \right]^{r/s} \right\}^{1/r} < \infty.$$

It is not hard to show that $l^{r,s} \in \Delta(p,q)$, where $p = \min(r,s)$ and $q = \max(r,s)$. Hence, by Corollary 2, $\delta_X(\varepsilon) \geqslant \delta_p(\varepsilon)$. Since $l^{r,s}$ contains an isometric copy of l^p, we conclude that $\delta_X = \delta_p$.

Orlicz spaces. Let M be a convex, strictly increasing function on the interval $[0,\infty)$ with $M(0) = 0$. The space l^M consists of the scalar sequnces $x = \{x_j\}_0^\infty$ for which

$$\|x\| = \|x\|_M = \inf\left\{ \lambda > 0 : \sum_{j=0}^{\infty} M\left(\frac{|x_j|}{\lambda}\right) \leqslant 1 \right\} < \infty.$$

One can prove that $l^M \in \Delta(p,q)$ provided that the function $M(t^{1/p})$ is convex and the function $M(t^{1/q})$ is concave. Therefore, inequality (1) holds in l^M if the function $M(t^{1/q})$ is concave. Estimates for the moduli of convexity of Orlicz spaces can be found in [1].

3. Proofs

Our proof is based on the following lemma.

LEMMA. *Let* F_p *be defined by* (3)*. Then, if* $1 \leqslant p \leqslant 2$*, the function* F_p *is convex, and if* $p > 2$*, it is concave. In all the cases* $F_p(\xi,\eta)$ *increases with* ξ *and* η.

Before proving the lemma we use it to prove the theorem. Let $x, y \in X$, where $X \in \Delta(1,2)$, and $1 \leqslant p \leqslant 2$. Then

$$(|x+y|^p + |x-y|^p)^{1/p} = \left((|x| + |y|)^p + \big||x| - |y|\big|^p\right)^{1/p}$$

(this is deduced from the case where x, y are real scalars, by using Theorem 1.d.1 of [3]) and we may assume that $x \geqslant 0$, $y \geqslant 0$. Assuming this we have

$$(|x+y|^p + |x-y|^p)^{1/p} = F_p(x^2, y^2)^{1/2}$$

(see [3], Theorem 1.d.1). Since F_p is convex, homogeneous and "increasing", there is a set $A \subset \{ (\alpha,\beta) : \alpha \geqslant 0, \beta \geqslant 0 \}$ such that

$$F_p(\xi,\eta) = \sup\{ \alpha\xi + \beta\eta : (\alpha,\beta) \in A \},$$

whence $F_p(x^2,y^2)^{1/2} \geqslant (\alpha x^2 + \beta y^2)^{1/2}$, $(\alpha,\beta) \in A$, and hence, by (5) with $q = 2$,

$$\|F_p(x^2,y^2)^{1/2}\| \geqslant (\alpha\|x\|^2 + \beta\|y\|^2)^{1/2}$$

for all $(\alpha,\beta) \in A$. Taking the supremum over $(\alpha,\beta) \in A$ we obtain

$$\|F_p(x^2,y^2)^{1/2}\| \geqslant F_p(\|x\|^2,\|y\|^2)^{1/2},$$

which concludes the proof. ■

Proof of Lemma. Let $1 < p \leqslant 2$. (The case $p = 1$ is similar.) Since $F_p(\lambda\xi,\lambda\eta) = \lambda F_p(\xi,\eta)$ for $\lambda \geqslant 0$, the convexity of F_p will follow from the convexity of the function $f(t) = F_p(1,t)$, $t > 0$. To prove that f is convex observe first that $f(t) = tf(1/t)$, whence $f''(t) = t^{-3}f''(1/t)$ for $t \neq 1$. And since $f'(1)$ exists, it remains to prove that $f''(t) \geqslant 0$ for $0 < t < 1$. To prove this write f as

$$f(t) = g(t^{1/2})^{2/p}, \quad g(t) = (1+t)^p + (1-t)^p \qquad (0 < t < 1).$$

We have

$$\begin{aligned}2pf''(t) = t^{-2/3}g(t^{1/2})^{(2/p)-2}\bigg[\left(\frac{2}{p}-1\right)g'(t^{1/2})^2t^{1/2} \\ + g(t^{1/2})g''(t^{1/2})t^{1/2} - g(t^{1/2})g'(t^{1/2})\bigg].\end{aligned}$$

Hence, $f''(t) > 0$ if and only if $A(t) > 0$, where

$$\begin{aligned}A(t) &= \frac{1}{p}\left[\left(\frac{2}{p}-1\right)g'(t)^2t + g(t)g''(t)t - g(t)g'(t)\right] \\ &= 4(p-1)t(1-t^2)^{p-2} - [(1+t)^{2p-2} - (1-t)^{2p-2}].\end{aligned}$$

If $3/2 \leqslant p \leqslant 2$, the function $\varphi(t) = (1+t)^{2p-2} - (1-t)^{2p-2}$ is concave and therefore

$$\varphi(t) \leqslant \varphi(0) + \varphi'(0)t = 4(p-1)t \leqslant 4(p-1)t(1-t^2)^{p-2},$$

which implies $A(t) > 0$. If $1 < p \leqslant 3/2$, then

$$A'(t) = 4(p-1)(1-t^2)^{p-3}[1+(3-2p)t^2] - 2(p-1)[(1+t)^{2p-3} + (1-t)^{2p-3}].$$

Since $0 \leqslant 3-2p \leqslant 1$, the function $t \mapsto t^{3-2p}$ is concave, hence

$$\frac{(1+t)^{2p-3} + (1-t)^{2p-3}}{2} = \frac{1}{2}\left[\left(\frac{1}{1+t}\right)^{3-2p} + \left(\frac{1}{1-t}\right)^{3-2p}\right] \leqslant (1-t^2)^{2p-3}.$$

Hence

$$A'(t) \geqslant 4(p-1)(1-t^2)^{2p-3}[1+(3-2p)t^2-1] \geqslant 0.$$

This implies $A(t) \geqslant A(0) = 0$, which concludes the proof in the case $1 < p \leqslant 2$. If $p > 2$, proving that F_p is concave reduces to proving that $A(t) \leqslant 0$ $(0 < t < 1)$. In this case the function φ is convex which implies that

$$\varphi(t) \geqslant \varphi(0) + \varphi'(0)t = 4(p-1)t \geqslant 4(p-1)t(1-t^2)^{p-2},$$

and this completes the proof. ■

Remark. The discussion of the case $1 < p \leqslant 2$ can be made simplier. Namely, it is easy to see that the function $g(t^{1/2})$ is convex $(0 < t < 1)$, which implies that $f(t) = g(t^{1/2})^{2/p}$ is convex (since $2/p > 1$). This trick can also be used if $2 < p < 3$, because then the function $g(t^{1/2})$ is concave. However, if $p > 3$, $g(t^{1/2})$ is convex.

4. Dual results

Using the case $p > 2$ of Lemma one proves that if $x, y \in X$, where $X \in \Delta(2, \infty)$ (which means that X satisfies (4) with $p = 2$), then there holds the reverse of (1). A consequence is that the reverse of (2) is valid in every lattice of class $\Delta(2, p)$ $(p > 2)$ and, in particular, in L^q for $2 \leqslant q \leqslant p$. (The latter was proved in [4] by using the Riesz-Thorin interpolation theorem.) Combining this with Hanner's results we see that if $X \in \Delta(2, p)$, then X is "more" uniformly convex than L^p (dimension $\geqslant 2$) in the sense that $\rho_X(\tau) \leqslant \rho_p(\tau)$, where

$$\rho_X(\tau) = \sup\left\{ \frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = 1,\ \|y\| = 1 \right\},$$

and $\rho_p = \rho_{L^p}$. The function ρ_X is called the modulus of smoothness of X (see [3], Ch. 1, for further information).

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(received 05.03.1993)

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