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ON gr-C-2^A-SECONDARY SUBMODULES

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Abstract. Let Ω be a group with identity e, Γ be a Ω-graded commutative ring and $\Im$ a graded Γ-module. In this article, we introduce the concept of gr-C-2^A-secondary submodules and investigate some properties of this new class of graded submodules. A non-zero graded submodule S of $\Im$ is said to be a gr-C-2^A-secondary submodule if whenever $r, s \in h(\Gamma)$, L is a graded submodule of $\Im$, and $rs\,S \subseteq L$, then either $r\,S \subseteq L$ or $s\,S \subseteq L$ or $rs \in Gr(Ann_\Gamma(S))$.

1. Introduction

In this article we assume that Γ is a commutative Ω-graded ring with identity and $\Im$ is a unitary graded Γ-module.

Let Ω be a group with identity e and Γ a commutative ring with identity 1_Γ. Then Γ is an Ω-*graded ring* if there exist additive subgroups Γ_g of Γ such that $\Gamma = \bigoplus_{g\in\Omega} \Gamma_g$ and $\Gamma_g\Gamma_h \subseteq \Gamma_{gh}$ for all $g, h \in \Omega$. Furthermore, $h(\Gamma) = \bigcup_{g\in\Omega} \Gamma_g$, (see [13]).

A left Γ-module $\Im$ is called Ω-*graded* Γ-*module* if there exists a family of additive subgroups $\{\Im_\alpha\}_{\alpha\in\Omega}$ of $\Im$ such that $\Im = \bigoplus_{\alpha\in\Omega} \Im_\alpha$ and $\Gamma_\alpha\Im_\beta \subseteq \Im_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$. Even if an element of $\Im$ belongs to $\cup_{\alpha\in\Omega}\Im_\alpha = h(\Im)$, it is called homogeneous. We refer to [9, 11–13] for basic properties and more information about graded rings and graded modules. By $L \leq_\Omega \Im$ we mean that L is a Ω-graded submodule of $\Im$.

Let Γ be a Ω-graded ring, $\Im$ a graded Γ-module and S a graded submodule of $\Im$. Then $(S :_\Gamma \Im)$ is defined as $(S :_\Gamma \Im) = \{a \in \Gamma | a\Im \subseteq S\}$. The annihilator of $\Im$ is defined as $(0 :_\Gamma \Im)$ and is denoted by $Ann_\Gamma(\Im)$. Let Γ be an Ω-graded ring. The *graded radical* of a graded ideal L, denoted by $Gr(L)$, is the set of all $t = \sum_{\alpha\in\Omega} t_\alpha \in \Gamma$, so that for every $\alpha \in \Omega$ there exists $n_\alpha > 0$ with $t_\alpha^{n_\alpha} \in L$, (see [15]). A proper graded submodule S of $\Im$ is called *a completely graded irreducible* if $S = \cap_{\alpha\in\Delta} S_\alpha$, where $\{S_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of $\Im$, then $S = S_\beta$ for some $\beta \in \Delta$.

The study of graded rings and modules has long attracted the attention of many researchers, as they have important applications in many fields such as geometry and

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physics. For example, graded Lie algebra plays an important role in differential geometry, such as the Frolicher-Nijenhuis and Nijenhuis-Richardson brackets (see [10]). In addition, they solve many physical problems related to supermanifolds, supersymmetries and quantizations of systems with symmetry (see [8, 17]).

The notion of graded 2-absorbing ideals was introduced and studied in [1]. Al-Zoubi and Abu-Dawwas in [3] extended graded 2-absorbing ideals to graded 2-absorbing submodules. In [2], the authors introduced the concept of the graded 2-absorbing primary ideal, which is a generalization of the graded primary ideal. The notion of graded 2-absorbing primary submodules as a generalization of graded 2-absorbing primary ideals was introduced and studied in [7]. In [4, 16], the authors introduced the dual notion of graded 2-absorbing submodules (i.e. graded 2-absorbing (resp., graded strongly 2-absorbing) second submodules) of $\Im$ and investigated some properties of these classes of graded modules. In this paper, we introduce the concept of graded classical 2-absorbing secondary submodules as a dual notion of graded 2-absorbing primary submodules. We investigate the basic properties and characteristics of graded classical 2-absorbing secondary submodules.

2. Results

Definition 2.1. Let Γ be a Ω-graded ring and $\Im$ a graded Γ-module. A non-zero graded submodule S of $\Im$ is said to be graded classical 2-absorbing secondary (Abbreviated, gr-C-2^A-secondary) submodule of $\Im$ if whenever $r, s \in h(\Gamma)$, $L \leq_\Omega \Im$, and $rs\, S \subseteq L$, then $r\, S \subseteq L$ or $s\, S \subseteq L$ or $rs \in Gr(Ann_\Gamma(S))$.

We say that $\Im$ is a gr-C-2^A-secondary module if $\Im$ is a gr-C-2^A-secondary submodule of itself.

Theorem 2.2. *Let S be a gr-C-2^A-secondary submodule of $\Im$, let $I = \bigoplus_{\alpha\in\Omega} I_\alpha$ and $J = \bigoplus_{\alpha\in\Omega} J_\alpha$ be a graded ideals of Γ. Then for every $\alpha, \beta \in \Omega$ and $L \leq_\Omega \Im$, with $I_\alpha J_\beta S \subseteq L$ either $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$ or $I_\alpha J_\beta \subseteq Gr(Ann_\Gamma(S))$.*

Proof. Let $\alpha, \beta \in \Omega$ such that $I_\alpha J_\beta S \subseteq L$ for some $L \leq_\Omega \Im$. Assume that $I_\alpha J_\beta \nsubseteq Gr(Ann_\Gamma(S)$. Then there exist $r_\alpha \in I_\alpha$ and $s_\beta \in J_\beta$ such that $r_\alpha s_\beta \notin Gr(Ann_\Gamma(S)$. Now since $r_\alpha s_\beta\, S \subseteq L$, we get $r_\alpha\, S \subseteq L$ or $s_\beta\, S \subseteq L$. We show that either $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$. On contrary, we suppose that $I_\alpha S \nsubseteq L$ and $J_\beta S \nsubseteq L$. Then there exist $r'_\alpha \in I_\alpha$ and $s'_\beta \in J_\beta$ such that $r'_\alpha\, S \nsubseteq L$ and $s'_\beta\, S \nsubseteq L$. Since $r'_\alpha s'_\beta\, S \subseteq L$ and S be a gr-C-2^A-secondary submodule of $\Im$, $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S)$. We have three cases:
Case I: Suppose that $r_\alpha\, S \subseteq L$ but $s_\beta\, S \nsubseteq L$. Since $r'_\alpha s_\beta\, S \subseteq L$ and $s_\beta\, S \nsubseteq L$ and $r'_\alpha\, S \nsubseteq L$, this implies $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$. Since $r_\alpha\, S \subseteq L$ and $r'_\alpha\, S \nsubseteq L$, we get $(r_\alpha + r'_\alpha)\, S \nsubseteq L$. As $(r_\alpha + r'_\alpha)s_\beta\, S \subseteq L$ and $s_\beta\, S \nsubseteq L$, then $(r_\alpha + r'_\alpha)\, S \nsubseteq L$ implies $(r_\alpha + r'_\alpha)s_\beta \in Gr(Ann_\Gamma(S)$. Since $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, we get $r_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, a contradiction.

Case II: Suppose $s_\beta\, S \subseteq L$ but $r_\alpha\, S \nsubseteq L$. Then similar to the Case I, we get a contradiction.

Case III: Suppose $r_\alpha S \subseteq L$ and $s_\beta S \subseteq L$. Now $s_\beta S \subseteq L$ and $s'_\beta S \nsubseteq L$ imply $(s_\beta + s'_\beta) S \nsubseteq L$. Since $r'_\alpha(s_\beta + s'_\beta) S \subseteq L$ and $(s_\beta + s'_\beta) S \nsubseteq L$ and $r'_\alpha S \nsubseteq L$, we get $r'_\alpha(s_\beta + s'_\beta) \in Gr(Ann_\Gamma(S))$. Now as $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we get $r'_\alpha s_\beta \in Gr(Ann_\Gamma(S))$. Again $r_\alpha S \subseteq L$ and $r'_\alpha S \nsubseteq L$ imply $(r_\alpha + r'_\alpha) S \nsubseteq L$. Since $(r_\alpha + r'_\alpha)s'_\beta S \subseteq L$ and $(r_\alpha + r'_\alpha) S \nsubseteq L$ and $s'_\beta S \nsubseteq L$, we have $(r_\alpha + r'_\alpha)s'_\beta \in Gr(Ann_\Gamma(S))$. Since $r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we get $r_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$. Since $(r_\alpha + r'_\alpha)(s_\beta + s'_\beta) S \subseteq L$ and $(r_\alpha + r'_\alpha) S \nsubseteq L$ and $(s_\beta + s'_\beta) S \nsubseteq L$, we get $(r_\alpha + r'_\alpha)(s_\beta + s'_\beta) \in Gr(Ann_\Gamma(S))$. Since $r_\alpha s'_\beta, r'_\alpha s_\beta, r'_\alpha s'_\beta \in Gr(Ann_\Gamma(S))$, we have $r_\alpha s_\beta \in Gr(Ann_\Gamma(S))$, a contradiction. Thus $I_\alpha S \subseteq L$ or $J_\beta S \subseteq L$. □

THEOREM 2.3. *Let S be a gr-C-2^A-secondary submodule of $\Im$, then for each $a, b \in h(\Gamma)$ we have $abS = a\,S$ or $ab\,S = bS$ or $ab \in Gr(Ann_\Gamma(S))$.*

Proof. Let $a, b \in h(\Gamma)$, then $abS \subseteq abS$ implies that $aS \subseteq abS$ or $aS \subseteq abS$ or $ab \in Gr(Ann_\Gamma(S))$. Clearly, $abS \subseteq aS$ and $abS \subseteq bS$, so we have $abS = aS$ or $abS = bS$ or $ab \in Gr(Ann_\Gamma(S))$. □

Let U and P be two graded submodules of a graded Γ-module. To prove that $U \subseteq P$, it suffices to show that if V is a completely graded irreducible submodule of $\Im$ such that $P \subseteq V$, then $U \subseteq V$ (see [4]). A proper graded ideal L of Γ is called a graded 2-absorbing primary (abbreviated, gr-2^A-primary) ideal if whenever $a, b, c \in h(\Gamma)$ with $abc \in L$, then $ab \in L$ or $ac \in Gr(L)$ or $bc \in Gr(L)$.

THEOREM 2.4. *Let S be a gr-C-2^A-secondary submodule of a graded Γ-module $\Im$. Then $Ann_\Gamma(S)$ is a gr-2^A-primary ideal of Γ.*

Proof. Let $r, s, t \in h(\Gamma)$ wit $rst \in Ann_\Gamma(S)$. Assume that $rs \notin Ann_\Gamma(S)$ and $rt \notin Gr(Ann_\Gamma(S))$. We show that $st \in Gr(Ann_\Gamma(S))$. There exist completely irreducible submodule J_1 and J_2 of $\Im$ such that $rs\,S \nsubseteq J_1$ and $rt\,S \nsubseteq J_2$. Since $rst\,S = 0 \subseteq J_1 \cap J_2$, $st\,S \subseteq (J_1 \cap J_2 :_\Im r)$. Since S is gr-C-2^A-secondary submodule of $\Im$, we have $rs\,S \subseteq J_1 \cap J_2$ or $rt\,S \subseteq J_1 \cap J_2$ or $st \in Gr(Ann_\Gamma(S))$. If $rs\,S \subseteq J_1 \cap J_2$ or $rt\,S \subseteq J_1 \cap J_2$, then $rs\,S \subseteq J_1$ or $rt\,S \subseteq J_2$ which are contradictions. Therefore $st \in Gr(Ann_\Gamma(S))$. □

A proper graded ideal L of Γ is a graded 2-absorbing (abbreviated, gr-2^A) ideal of Γ if whenever $a, b, c \in h(\Gamma)$ with $abc \in L$, then $ab \in L$ or $ac \in L$ or $bc \in L$ (see [1]).

COROLLARY 2.5. *Let S be a gr-C-2^A-secondary submodule of a graded Γ-module $\Im$. Then $Gr(Ann_\Gamma(S))$ is a gr-2^A ideal of Γ.*

Proof. By Theorem 2.4, $Ann_\Gamma(S)$ is gr-2^A-primary ideal of Γ. So by [2, Theorem 2.3], $Gr(Ann_\Gamma(S))$ is gr-2^A ideal of Γ. □

The following example shows that the converse of Theorem 2.4 is not true in general.

EXAMPLE 2.6. Let $\Gamma = \mathbb{Z}$ and $\Omega = \mathbb{Z}_2$, then Γ is a Ω-graded ring with $\Gamma_0 = \mathbb{Z}$ and $\Gamma_1 = \{0\}$. Consider $\Im = \mathbb{Z}_{pq} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, where p, q are two prime integers, $\Im$

is a Ω-graded module with $\Im_0 = \mathbb{Z}_{pq} \oplus \{0\}$ and $\Im_1 = \{\bar{0}\} \oplus \mathbb{Q}$. Then $Ann_\Gamma(\Im) = \{0\}$ is a gr-2^A-primary ideal of $\mathbb{Z}$. But $\Im$ is not gr-C-2^A-secondary $\mathbb{Z}$-module, since $pq\Im \subseteq \{\bar{0}\} \oplus \mathbb{Q}$, but $pM = p\mathbb{Z}_{pq} \oplus \mathbb{Q} \nsubseteq \{\bar{0}\} \oplus \mathbb{Q}$ and $q\Im = q\mathbb{Z}_{pq} \oplus \mathbb{Q} \nsubseteq \{\bar{0}\} \oplus \mathbb{Q}$ and $pq \notin Gr(Ann_\Gamma(\Im))$.

A graded domain Γ is called a gr-Dedekind ring if every graded ideal of Γ factorises into a product of graded prime ideals (see [19]).

A graded Γ-module $\Im$ is called a gr-comultiplication module if for every graded submodule S of $\Im$ there exists a graded ideal P of Γ such that $S = (0 :_\Im P)$, or, equivalently, for each graded submodule S of $\Im$, we have $S = (0 :_\Im Ann_\Gamma(S))$ (see [5]).

The gr-C-2^A-secondary submodules of a gr-comultiplication module over a gr-Dedekind domain are described in the following theorem.

Theorem 2.7. *Let Γ be a gr-Dedekind domain, and $\Im$ be a gr-comultiplication Γ-module, if S is gr-C-2^A-secondary submodule of $\Im$, then $S = (0 :_\Im Ann^n_\Gamma(L))$ or $S = (0 :_\Im Ann^n_\Gamma(L_1)Ann^m_\Gamma(L_2))$, where L, L_1, L_2 are graded minimal submodules of $\Im$ and n, m are positive integers.*

Proof. By Theorem 2.4, since S is gr-C-2^A-secondary submodule of $\Im$, then $Ann_\Gamma(S)$ is a gr-2^A-primary ideal of Γ. Using [18, Theorem 4.1] and [19, Lemma 1.1], we have either $Ann_\Gamma(S) = I^n$ or $Ann_\Gamma(S) = I_1^n I_2^m$, where I, I_1, I_2 are graded maximal ideals of Γ. First assume $Ann_\Gamma(S) = I^n$. If $(0 :_\Im I) = 0$, then $(0 :_\Im I^n) = 0$, and so we conclude that $S = 0$, a contradiction. Now by [5, Theorem 3.9], since I is graded maximal ideal of Γ, we have $(0 :_\Im I)$ is graded minimal submodule of $\Im$. This implies that $S = (0 :_\Im Ann^n_\Gamma(L))$, where $L = (0 :_\Im I)$. Now assume that $Ann_\Gamma(S) = I_1^n I_2^m$. If $(0 :_\Im I_1) = 0$ and $(0 :_\Im I_2) = 0$, then $S = 0$, a contradiction. Thus either $(0 :_\Im I_1) \neq 0$ or $(0 :_\Im I_2) \neq 0$. Hence one can see that either $S = (0 :_\Im Ann^n_\Gamma(L_1)Ann^m_\Gamma(L_2))$ or $S = (0 :_\Im Ann^n_\Gamma(L_1))$ or $S = (0 :_\Im Ann^m_\Gamma(L_2))$, where $L_1 = (0 :_\Im I_1)$ and $L_2 = (0 :_\Im I_2)$ are graded minimal submodules of $\Im$. □

For a graded Γ-submodule S of $\Im$, the graded second radical of S is defined as the sum of all gr-second Γ-submodules of $\Im$ contained in S, and is denoted by $GSec(S)$. If S does not contain any gr-second Γ-submodule, then $GSec(S) = \{0\}$. The graded second spectrum of $\Im$ is the collection of all gr-second Γ submodules and is represented by the symbol $GSpec^s(\Im)$. The set of all gr-prime Γ-submodules of $\Im$ is called the graded spectrum of $\Im$ and is denoted by $GSpec(\Im)$. The mapping $\psi : GSpec^s(\Im) \to GSpec(\Gamma/Ann_\Gamma(\Im))$ is defined by $\psi(S) = Ann_\Gamma(S)/Ann_\Gamma(\Im)$ is called the natural mapping of $GSpec^s(\Im)$, see [16]. A graded submodule S of $\Im$ is called a *graded strongly 2-absorbing second* (abbreviated, gr-S-2^A-second) submodule of $\Im$ if whenever a, $b \in h(\Gamma)$, S_1, S_2 are completely graded irreducible submodules of $\Im$, and $abS \subseteq S_1 \cap S_2$, then $aS \subseteq S_1 \cap S_2$ or $bS \subseteq S_1 \cap S_2$ or $ab \in Ann_\Gamma(S)$, see [4].

It is clear that every gr-S-2^A-second submodule is a gr-C-2^A-secondary submodule of $\Im$, but the converse is generally not true. This is illustrated by the following examples.

Example 2.8. Let $\Omega = \mathbb{Z}_2$ and $\Gamma = \mathbb{Z}$ be a Ω-graded ring with $\Gamma_0 = \mathbb{Z}$ and $\Gamma_1 = \{0\}$. Let $\Im = \mathbb{Z}_{p^\infty} = \{\frac{a}{p^n} + \mathbb{Z} : a, n \in \mathbb{Z},\ n \geqslant 0\}$ be a graded Γ-module with $\Im_0 = \mathbb{Z}_{p^\infty}$

and $\Im_1 = \{0_{\mathbb{Z}_{p^\infty}}\} = \{\mathbb{Z}\}$, where p is a fixed prime number. Consider the graded submodule $N = \langle \frac{1}{p^3} + \mathbb{Z} \rangle$ of $\Im$. Then N is gr-C-2^A-secondary submodule which is not a gr-S-2^A-second submodule.

THEOREM 2.9. *Let $\Im$ be a gr-comultiplication Γ-module, and the natural map ψ of $GSpec^s(S)$ is surjective, if S is a gr-C-2^A-secondary submodule of $\Im$, then $GSec(S)$ is a gr-S-2^A-second submodule of $\Im$.*

Proof. Let S be a gr-C-2^A-secondary submodule of $\Im$. By Corollary 2.5, $Gr(Ann_\Gamma(S))$ is gr-2^A ideal of Γ. By [16, Lemma 4.7], $Gr(Ann_\Gamma(S)) = Ann_\Gamma(GSec(S))$. Therefore, $Ann_\Gamma(GSec(S))$ is gr-2^A ideal of Γ. Using [16, Proposition 3.7], $GSec(S)$ is gr-S-2^A-second Γ-submodule of $\Im$. □

Let Γ be a Ω-graded ring, a graded Γ-module $\Im$ is a gr-sum-irreducible if $\Im \neq 0$ and the sum of any two proper graded submodule of $\Im$ is always a proper graded submodule (see [6]).

THEOREM 2.10. *Let S be a gr-C-2^A-secondary submodule of $\Im$. Then $r\,S = r^2\,S, \forall r \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. The converse hold, if S is a gr-sum-irreducible submodule of $\Im$.*

Proof. Let $r \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. Then $r^2 \in h(\Gamma)\backslash Gr(Ann_\Gamma(S))$. Thus by Theorem 2.3, we have $r\,S = r^2\,S$. Conversely, let S be a gr-sum-irreducible submodule of $\Im$ and $rs\,S \subseteq L$, for some $r, s \in h(\Gamma)$ and $L \leq_\Omega \Im$. Suppose that $rs \notin Gr(Ann_\Gamma(S))$. We show that $r\,S \subseteq L$ or $s\,S \subseteq L$. Since $rs \notin Gr(Ann_\Gamma(S))$, we have $r, s \notin Gr(Ann_\Gamma(S))$. Thus $r\,S = r^2\,S$ by assumption. Let $x \in S$, then $rx \in r\,S = r^2\,S$. So $\exists y \in S$ such that $rx = r^2 y$. This implies that $x - ry \in (0 :_S r) \subseteq (L :_S r)$. Thus $x = x - ry + ry \in (L :_S r) + (L :_S s)$. Hence $S \subseteq (L :_S r) + (L :_S s)$. Clearly, $(L :_S r) + (L :_S s) \subseteq S$, as S is gr-sum-irreducible submodule of $\Im$, $(L :_S r) = S$ or $(L :_S s) = S$, i.e $r\,S \subseteq L$ or $s\,S \subseteq L$, as needed. □

A graded Γ-module $\Im$ is called gr-multiplication, if for every graded submodule S of $\Im$, there exists a graded ideal K of Γ such that $S = K\Im$ (see [14]).

THEOREM 2.11. *Let $S \leq_\Omega \Im$. Then we have the following.*
(a) If S is a gr-C-2^A-secondary submodule of $\Im$, then IC is a gr-C-2^A-secondary submodule of $\Im$, for all graded ideal I of Γ, with $I \nsubseteq Ann_\Gamma(S)$.

(b) If $\Im$ is a gr-multiplication gr-C-2^A-secondary module, then every non-zero graded submodule of $\Im$ is a gr-C-2^A-secondary submodule of $\Im$.

Proof. (a) Let I be a graded ideal of Γ, with $I \nsubseteq Ann_\Gamma(S)$. Then IC is a non-zero graded submodule of $\Im$. Let $r, s \in h(\Gamma)$, L is graded submodule of $\Im$, and $rs\,IC \subseteq L$, then $rs\,S \subseteq (L :_\Im I)$, thus $r\,IC \subseteq L$ or $s\,IC \subseteq L$ or $rs \in Gr(Ann_\Gamma(S)) \subseteq Gr(Ann_\Gamma(IC))$, as desired.

(b) This follows from part (a). □

THEOREM 2.12. *Let Γ be Ω-graded ring and $\Im$, $\Im'$ be two graded Γ-module. Let $\psi : \Im \to \Im'$ be a graded monomorphism.*

(a) If S is a gr-C-2^A-secondary submodule of $\Im$, then $\psi(S)$ is a gr-C-2^A-secondary submodule of $\Im'$.

(b) If S' is a gr-C-2^A-secondary submodule of $\psi(\Im)$, then $\psi^{-1}(S')$ is a gr-C-2^A-secondary submodule of $\Im$.

Proof. (a) As $S \neq 0$, and ψ is a graded monomorphism, we have $\psi(S) \neq 0$, let $r, s \in h(\Gamma)$, $L' \leq_\Omega \Im'$, and $rs\,\psi(S) \subseteq L'$. Then $rs\,S \subseteq \psi^{-1}(L')$. Since S is *gr*-C-2^A-secondary submodule of $\Im$, $r\,S \subseteq \psi^{-1}(L')$ or $s\,S \subseteq \psi^{-1}(L')$ or $rs \in Gr(Ann_\Gamma(S))$. Therefore, $r\,\psi(S) \subseteq \psi(\psi^{-1}(L')) = \psi(\Im) \cap L' \subseteq L'$ or $s\,\psi(S) \subseteq \psi(\psi^{-1}(L')) = \psi(\Im) \cap L' \subseteq L'$ or $rs \in Gr(Ann_\Gamma(\psi(S)))$, as desired.

(b) If $\psi^{-1}(S') = 0$, then $\psi(\Im) \cap S' = \psi\,\psi^{-1}(S') = \psi(0) = 0$. So $S' = 0$, which is a contradiction. Therefore $\psi^{-1}(S') \neq 0$. Let $r, s \in h(\Gamma)$, $L \leq_\Omega \Im$, and $rs\,\psi^{-1}(S') \subseteq L$. Then $rs\,S' = rs(\psi(\Im) \cap S') = rs\,\psi\,\psi^{-1}(S') \subseteq \psi(L)$. As S' is *gr*-C-2^A-secondary submodule of $\psi(\Im)$, $r\,S' \subseteq \psi(L)$ or $s\,S' \subseteq \psi(L)$ or $rs \in Gr(Ann_\Gamma(S'))$. Thus $r\,\psi^{-1}(S') \subseteq \psi^{-1}\psi(L) = L$ or $s\,\psi^{-1}(S') \subseteq \psi^{-1}\psi(L) = L$ or $rs \in Gr(Ann_\Gamma(\psi^{-1}(S')))$, as needed. □

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