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ALMOST YAMABE SOLITON AND ALMOST RICCI-BOURGUIGNON SOLITON WITH GEODESIC VECTOR FIELDS

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Abstract. The aim of this paper is to prove some results about almost Yamabe soliton and almost Ricci-Bourguignon soliton with special soliton vector field. In fact, we prove that every compact non-trivial almost Ricci-Bourguignon soliton with constant scalar curvature is isometric to a Euclidean sphere. Then we show that every compact almost Ricci-Bourguignon soliton whose soliton vector field is divergence-free is Einstein and its soliton vector field is Killing. Finally, we prove that a complete almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) has V as the contact vector field of a contact manifold M with metric g and its Reeb vector field is geodesic, then it becomes a Ricci-Bourguignon soliton and g has constant scalar curvature. In particular, if V is strict, then g is a compact Sasakian Einstein.

1. Introduction

On an n-dimensional smooth manifold M, a Riemannian metric g and a non-vanishing vector field V define an almost Yamabe soliton [8] if there exists a smooth function λ on M such that

$$\mathcal{L}_V g = (\lambda - R)g, \tag{1}$$

respectively, an almost Ricci-Bourguignon soliton if there exists a smooth function λ on M such that

$$2Ric + \mathcal{L}_V g = 2(\lambda + \rho R)g, \tag{2}$$

where $\mathcal{L}_V$ denotes the Lie derivative operator in the direction of the vector field V, Ric denotes the Ricci curvature tensor field of g, R is the scalar curvature of g, and ρ is a real constant. For λ constant, they reduce to a Yamabe soliton and a Ricci-Bourguignon soliton respectively. As in the case of almost Ricci solitons, almost

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Ricci-Bourguignon solitons give rise to a self-similar solution of the Ricci-Bourguignon flow

$$\frac{\partial}{\partial t}g = -2(Ric - \rho R g), \quad g(0) = g_0.$$

This flow was first considered by Bourguignon [3] and then the short-time existence and uniqueness of the solution of the Ricci-Bourguignon flow on $[0, T)$ was shown by Catino et al. [5] for $\rho < \frac{1}{2(n-1)}$.

An almost Yamabe soliton (M, g, V, λ) (or an almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) is said to be shrinking, steady or expanding if λ is positive, zero or negative, respectively. If the potential vector field V is of gradient type, $V = \nabla f$, for a smooth function $f : M \to \mathbb{R}$, then an almost Yamabe soliton (resp. an almost Ricci-Bourguignon soliton) is called a gradient almost Yamabe soliton (or a gradient almost Ricci-Bourguignon soliton). An almost Ricci-Bourguignon soliton (M, g, V, λ, ρ) is called trivial if V is a Killing vector field, i.e. $\mathcal{L}_V g = 0$.

If $\rho = 0$, then the almost Ricci-Bourguignon soliton reduces to the almost Ricci soliton, which was first introduced by Pigola et al. [12], and after that many authors have obtained some properties of the almost Ricci soliton. Some characterization results for compact almost Ricci solitons were obtained in [1, 13, 14]. Gradient Ricci-Bourguignon solitons were studied in detail, for example in [6, 7], where the authors called them gradient ρ-Einstein solitons. Then Dwivedi [9] obtained some results on the almost Ricci-Bourguignon soliton.

We recall an operator $\square$ acting on a smooth vector field V such that $\square V$ is a vector field and in a local coordinate system $\{x^i\}$ it has components $-(g^{jk}\nabla_j\nabla_k V^i + R^i_j V^j$ where $R^i_j = g^{ik}R_{jk}$ and R_{jk} are the components of the Ricci tensor. From [16] we have the following definition.

Definition 1.1. A vector field V on a Riemannian manifold (M, g) is called a geodesic vector field if $\square V = 0$.

Note that the condition for a geodesic vector field is also equivalent to the condition $g^{jk}\mathcal{L}_V\Gamma^i_{jk} = 0$ and this shows that a Killing vector field and an affine Killing vector field are special examples of a geodesic vector field. Moreover, this definition of a geodesic vector field is different from a vector field whose integral curves are geodesics (see [15,16]). In this paper, we first give the following rigidity result for almost Yamabe soliton.

Proposition 1.2. *If an almost Yamabe soliton (M, g, V, λ) has a divergence-free soliton vector field V, then V is a Killing vector field.*

Motivated by the almost Ricci soliton case, we will prove the following theorem and generalize results of [1].

Theorem 1.3. *Let $(M^n, g, V, \lambda, \rho)$ be a compact oriented almost Ricci-Bourguignon soliton. If $d\mu$ denotes the volume form with respect to g, then*

$$\int_M \left|Ric - \frac{R}{n}g\right|^2 d\mu = \frac{n-2}{2n}\int_M g(\nabla R, V)d\mu. \tag{3}$$

Furthermore, if $n > 2$, the almost-Ricci-Bourguignon soliton is non-trivial and the scalar curvature is constant, then (M, g) is isometric to a Euclidean sphere and almost-Ricci-Bourguignon soliton is gradient.

Consequently, we have the following rigidity result for the almost Ricci-Bourguignon soliton.

Corollary 1.4. *If a compact Ricci-Bourguignon soliton (M, g, V, λ, ρ) has a divergence-free soliton vector field V, then V is a Killing vector field and g is an Einstein metric.*

Remark 1.5. Every Killing vector field is divergence-free, but the converse need not be true in general. Therefore, Theorems 1.2 and 1.4 provide a condition under which the converse holds.

We now note that the above result holds for a Ricci-Bourguignon soliton without the compactness condition, as follows.

Theorem 1.6. *If a Ricci-Bourguignon soliton (M, g, V, λ, ρ) has a divergence-free soliton vector field V, then for $\rho \neq \frac{1}{n}$ the vector field V is Killing and g is Einstein metric.*

We study almost Yamabe solitons and almost Ricci-Bourguignon solitons when the vector field of the soliton is a geodesic vector field. In fact, we show the following.

Theorem 1.7. *In an almost Yamabe soliton (M, g, V, λ), V is Killing if and only if V is a geodesic vector field in the sense of the Definition 1.1.*

We also prove that following is true.

Theorem 1.8. *In an almost Ricci-Bourguignon soliton (M, g, V, λ, ρ), $\lambda + \rho R$ is constant if and only if V is a geodesic vector field in the sense of the Definition 1.1.*

In the following we give some well-known definitions and basic formulas for the contact geometry of [2]. A $(2m+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a 1-form η on M such that $\eta \wedge (d\eta)^m \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ (it is called a Reeb vector field) such that $d\eta(\xi, \dot{}) = 0$ and $\eta(\xi) = 1$. If we polarize $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a $(1,1)$ tensor field ϕ such that $d\eta(X, Y) = g(x, \phi Y)$, $\eta(X) = g(\xi, X)$, $\phi^2 = -I + \eta \otimes \xi$, $\phi(\xi) = 0$, for any vector fields X, Y on M. If the vector field ξ is Killing, then the contact manifold is called K-contact manifold and in this case we have $Ric(\xi, X) = 2mg(\xi, X)$ for any vector field X on M. A vector field X on a contact manifold is called a contact vector field if $\mathcal{L}_X \eta = f\eta$ for a smooth function f on M, and it is called strict if $f = 0$. A contact metric g on M is called Sasakian if the almost-Kähler structure induced on the cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ is Kähler.

Finally, we prove the following results for an almost Ricci-Bourguignon soliton on contact manifolds.

Theorem 1.9. *If a complete* $(2m+1)$*-dimensional almost Ricci-Bourguignon soliton* (M,g,V,λ,ρ) *has* V *as the contact vector field of a* K*-contact manifold* M *with metric* g*, then it becomes a Ricci-Bourguignon soliton and* g *has constant scalar curvature. In particular, if* V *is strict, then* g *is a compact Sasakian Einstein.*

Remark 1.10. All Sasakian manifolds are K-contact but the converse need not be true except in dimension three. Therefore, Theorem 1.9 gives a condition under which the converse holds.

Corollary 1.11. *If a complete* $(2m+1)$*-dimensional almost Ricci-Bourguignon soliton* (M,g,V,λ,ρ) *has* V *as contact vector field of a contact manifold* M *with metric* g *and its Reeb vector field is geodesic in the sense of the Definition 1.1, then it becomes a Ricci-Bourguignon soliton and* g *has constant scalar curvature. In particular, if* V *is strict, then* g *is a compact Sasakian Einstein.*

2. Proofs of the results

In this section, we prove our results.

Proof (Proposition 1.2). Taking trace of the almost Yamabe soliton equation (1) and the condition $\mathrm{div}V=0$, we get $R=\lambda$. Using this in (1) shows that V is Killing. □

Below we give a lemma that will be used throughout the paper. For a proof of the lemma we need the following formula from [15, pp. 23],

$$\mathcal{L}_V\Gamma^k_{ij}=\frac{1}{2}g^{kl}\left\{\nabla_i(\mathcal{L}_Vg_{jl})+\nabla_j(\mathcal{L}_Vg_{il})-\nabla_l(\mathcal{L}_Vg_{ij})\right\},\tag{4}$$

and

$$\nabla_l(\mathcal{L}_V\Gamma^k_{ij})-\nabla_j(\mathcal{L}_V\Gamma^k_{il})=\mathcal{L}_VR^k_{lji},\tag{5}$$

where Γ^k_{ij} and R^k_{lji} are the Christoffel symbols and the components of the curvature tensor of the metric g in local coordinates, respectively.

Lemma 2.1. *Let* (M^n,g,V,λ,ρ) *be an almost Ricci-Bourguignon soliton. Then we have*

$$\mathcal{L}_VR=\big(1-2(n-1)\rho\big)\Delta R-2\rho R^2-2\lambda R+2|Ric|^2-2(n-1)\Delta\lambda.\tag{6}$$

Proof. If we take the Lie derivative of the relation $g_{lk}g^{kj}=\delta^j_l$ along the vector field V, use the equation (2) and then multiply the equation obtained by g^{il}, we obtain

$$\mathcal{L}_Vg^{ij}=2R^{ij}-2(\lambda+\rho R)g^{ij}.\tag{7}$$

If you substitute (7) into (4), you obtain

$$\mathcal{L}_V\Gamma^k_{ij}=\nabla^kR_{ij}-\nabla_jR^k_i-\nabla_iR^k_j-\nabla^k(\lambda+\rho R)g_{ij}+\nabla_j(\lambda+\rho R)\delta^k_i+\nabla_i(\lambda+\rho R)\delta^k_j.\tag{8}$$

By replacing (8) with (5) and using the Ricci identity, you get

$$\begin{aligned}\mathcal{L}_VR^k_{lji}&=\nabla_j\nabla_lR^k_i-\nabla_l\nabla_jR^k_i+\nabla_j\nabla_iR^k_l-\nabla_l\nabla_iR^k_j+\nabla_l\nabla^kR_{ij}-\nabla_j\nabla^kR_{il}\\&+(\nabla_l\nabla_i)(\lambda+\rho R)\delta^k_j-(\nabla_l\nabla^k(\lambda+\rho R))g_{ij}-(\nabla_i\nabla_j(\lambda+\rho R))\delta^k_l+(\nabla_j\nabla^k(\lambda+\rho R))g_{il}.\end{aligned}$$

Contracting this equation with g^{lk}, you get

$$\begin{aligned}\mathcal{L}_V R_{ij} &= \nabla_j\nabla_i R - \nabla_k\nabla_j R_i^k - \nabla_k\nabla_i R_j^k \\ &\quad + \Delta R_{ij} - (\Delta(\lambda+\rho R))g_{ij} - (n-2)\nabla_i\nabla_j(\lambda+\rho R).\end{aligned} \tag{9}$$

Taking the Lie derivative of $R = g^{ij}R_{ij}$ along the vector field V and using the equations (7) and (9), we obtain (6). □

Proof (Theorem 1.3). We can write (6) as

$$g(\nabla R, V) = \big(1 - 2(n-1)\rho\big)\Delta R - 2\rho R^2 - 2\lambda R + 2|Ric|^2 - 2(n-1)\Delta\lambda.$$

By integrating both sides of the last equation and applying the divergence theorem, we obtain that

$$\begin{aligned}\frac{1}{2}\int_M g(\nabla R, V)d\mu &= \int_M \Big[|Ric|^2 - \rho R^2 - \lambda R\Big]d\mu \\ &= \int_M \Big[\Big|Ric - \frac{R}{n}g\Big|^2 - \frac{(n\rho-1)R^2 + n\lambda R}{n}\Big]d\mu.\end{aligned} \tag{10}$$

The contraction of (2) leads to $\mathrm{divV} = n\lambda + (n\rho - 1)R$, then

$$\int_M R\mathrm{div}V d\mu = \int_M \big(n\lambda R + (n\rho-1)R^2\big)d\mu.$$

Since

$$\mathrm{div}(RV) = g(\nabla R, V) + R\mathrm{div}V, \tag{11}$$

we conclude

$$\int_M g(\nabla R, V)d\mu = -\int_M R\mathrm{div}V d\mu. \tag{12}$$

Inserting (12) into (11) results in

$$\int_M g(\nabla R, V)d\mu = -\int_M \big(n\lambda R + (n\rho-1)R^2\big)d\mu. \tag{13}$$

If you insert (13) into (10), you get (3). If the scalar curvature R is constant, then (3) implies that g is Einstein. Consequently, the equation (2) reduces to $\mathcal{L}_V g = 2(\lambda + \rho R - \frac{R}{n})g$. Assuming that λ is not constant, V is a non-homothetic conformal vector field on M. We set $h := \lambda + \rho R - \frac{R}{n}$. We have $\mathcal{L}_V g = 2hg$ and from [15, pp. 26], $\mathcal{L}_V R_{ij} = (n-2)\nabla_i\nabla_j h - (\Delta h)g_{ij}$. If we take the Lie derivative of the relation $R_{ij} = \frac{R}{n}g_{ij}$ along V, we get

$$\Big(\Delta h + \frac{2R}{n}h\Big)g_{ij} = (2-n)\nabla_i\nabla_j h. \tag{14}$$

If you take the trace of the above equation, you get $\Delta h = -\frac{R}{n-1}h$ and this shows that

$$\Delta h^2 = 2|\nabla h|^2 + 2h\Delta h = 2|\nabla h|^2 - \frac{2R}{n-1}h^2.$$

If you integrate the above equation over the compact M and using the divergence theorem, you get

$$\int_M |\nabla h|^2 d\mu = \frac{R}{n-1}\int_M h^2 d\mu.$$

This implies that R is positive. From (14) and $\Delta h = -\frac{R}{n-1}h$ we conclude

$$\nabla_i\nabla_j h = -\frac{R}{n(n-1)}hg_{ij}. \qquad (15)$$

Obata's Theorem [11] now implies that (M, g) is isomorphic to a Euclidean sphere of radius $\sqrt{\frac{n(n-1)}{R}}$. We can write (15) as $\mathcal{L}_{\nabla h}g = -\frac{2R}{n(1-n)}hg$ or equivalently $\mathcal{L}_{-\frac{R}{n(1-n)}\nabla h}g = 2hg$. Since V is also conformal and satisfies $\mathcal{L}_V g = 2hg$, we can use the Hoge-de-Rham decomposition to conclude that $V = -\frac{R}{n(1-n)}\nabla h + Z$, where Z is a Killing vector field, so V is the gradient of a smooth function and thus the proof of the theorem is complete. □

Proof (Corollary 1.4). If we replace $\mathrm{div}V = 0$ in equation (3), then $\int_M |Ric - \frac{R}{n}g|^2 d\mu = 0$, which implies that $Ric = \frac{R}{n}g$, i.e. g is Einstein metric. Substituting $Ric = \frac{R}{n}g$ and $(1-n\rho)R = n\lambda$ into (2) gives $\mathcal{L}_V g = 0$, which shows that the vector field V is Killing. □

Proof (Theorem 1.6). In this case, the equation $(1-n\rho)R = n\lambda$ applies. Since λ is constant, we conclude that R is also constant. Thus, the formulas $(1-n\rho)R = n\lambda$ and (6) imply that $|Ric|^2 = \frac{R^2}{n}$. Substituting this into $|Ric|^2 - \frac{R^2}{n} = |Ric - \frac{R}{n}g|^2$ gives $Ric = \frac{R}{n}g$, i.e. g is Einstein metric. Finally, (2) implies that the vector field V is Killing. □

Proof (Theorem 1.7). If you take the trace of the equation (1) and then take the covariant derivative of it in an orthonormal frame, you get

$$\nabla_j\nabla_i V^i = n\nabla_j(\lambda - R). \qquad (16)$$

In addition, the differentiation of (1) results in

$$\nabla_i\nabla_j V^i + \nabla_i\nabla^i V_j = 2\nabla_j(\lambda - R). \qquad (17)$$

If you subtract equation (16) from (17), you get $R_{kj}V^k + \nabla_i\nabla^i V_j = (2-n)\nabla_j(\lambda - R)$. This shows that $\Box V = (2-n)\nabla_j(\lambda - R)$. Therefore, equation (1) implies that V is a geodesic vector field if and only if V is Killing. □

Proof (Theorem 1.8). First we take the trace of the equation (1) and then we take the covariant derivative of it in an orthonormal frame and get

$$\nabla_j\nabla_i V^i + \nabla_j R = n\nabla_j(\lambda + \rho R). \qquad (18)$$

By differentiating (1) and using the twice contracted second Bianchi identity $2\mathrm{div}Ric = 2\nabla R$, we also arrive at the conclusion

$$\nabla_i\nabla_j V^i + \nabla_i\nabla^i V_j + \nabla_j R = 2\nabla_j(\lambda + \rho R). \qquad (19)$$

If you subtract equation (18) from (19), you obtain $R_{kj}V^k + \nabla_i\nabla^i V_j = (2-n)\nabla_j(\lambda + \rho R)$. This results in $\Box V = (2-n)\nabla_j(\lambda + \rho R)$. Therefore, V is a geodesic vector field if and only if $\lambda + \rho R$ is constant. □

Proof (Theorem 1.9). According to the definition of the contact manifold, we have $\omega = \eta \wedge (d\eta)^m \neq 0$, then ω is a volume element and

$$\mathcal{L}_V \omega = (\text{div}V)\omega. \tag{20}$$

Equation $\mathcal{L}_V \eta = f\eta$ implies that $\mathcal{L}_V d\eta = d\mathcal{L}_V \eta = df \wedge \eta + f d\eta$. It follows from the equation (20) that $\text{div}V = (m+1)f$.

On the other hand, from (2) we have $\text{div}V = (2m+1)\lambda + ((2m+1)\rho - 1)R$. Then

$$[1-(2m+1)\rho]\, R = (2m+1)\lambda - (m+1)f. \tag{21}$$

With the help of the formula $\eta(X) = g(\xi, X)$ we derive

$$(\mathcal{L}_V \eta)(X) = (\mathcal{L}_V g(\xi, X) + g(\mathcal{L}_V \xi, X), \tag{22}$$

for any vector field X on M. From (2) and $Ric(\xi, X) = 2mg(\xi, X)$ we get

$$(\mathcal{L}_V g)(\xi, X) = 2(\lambda + \rho R - 2m)g(\xi, X). \tag{23}$$

Inserting (23) and $\mathcal{L}_X \eta = f\eta$ into (22) results in $g(\mathcal{L}_V \xi, X) = f\eta(X) - 2(\lambda + \rho R - 2m)g(\xi, X)$ for any vector field X on M and this shows that

$$\mathcal{L}_V \xi = (f - 2\lambda - 2\rho R + 4m)\xi. \tag{24}$$

The inner product of (24) with ξ yields $g(\mathcal{L}_V \xi, \xi) = (f - 2\lambda - 2\rho R + 4m)$. With the Lie derivative of $g(\xi, \xi) = 1$ along V and using equation (2) and $Ric(\xi, \xi) = 2m$ we also obtain $g(\mathcal{L}_V \xi, \xi) = 2m - \lambda - \rho R$.

If we compare the two values of $g(\mathcal{L}_V \xi, \xi)$, the result is $f = \lambda + \rho R - 2m$, then

$$\mathcal{L}_V \eta = (\lambda + \rho R - 2m)\eta, \quad \mathcal{L}_V \xi = (2m - \lambda - \rho R)\xi. \tag{25}$$

Let Q denote the Ricci operator defined by $g(QX, Y) = Ric(X, Y)$ for arbitrary vector fields X, Y on M. If we take the Lie derivative of $d\eta(X, Y) = g(X, \phi Y)$ along the vector field V and use equation (2) and $\mathcal{L}_V \eta = f\eta$, we obtain

$$\eta(Y)\nabla f - (Yf)\xi + 2(f - 2\lambda - 2\rho R)\phi Y = -4Q(\phi Y) + 2(\mathcal{L}_X \phi)Y. \tag{26}$$

Substituting ξ for Y in (26) results in

$$\nabla f - (\xi f)\xi = 2(\mathcal{L}_X \phi)\xi. \tag{27}$$

If we now take the Lie derivative of $\phi(\xi) = 0$ along the vector field V again and use the second equation of (25), we obtain $(\mathcal{L}_X \phi)\xi = 0$ and insert it into (27), we obtain $\nabla f = (\xi f)\xi$, i.e. $df = (\xi f)\eta$. If you take the exterior derivative of this and then take the wedge product with η, you get $(\xi f)\eta \wedge d\eta = 0$. Since $\eta \wedge d\eta$ is nonzero everywhere then $\xi f = 0$, i.e. $df = 0$ and this shows that f is constant on M. Therefore, (26) reduces to $\mathcal{L}_X \phi = 2Q\phi - (2m + \lambda + \rho R)\phi$. Since $f = \lambda + \rho R - 2m$ and f is constant, we conclude that $\lambda + \rho R$ is constant and (21) shows that R is also constant. Thus λ is constant and the almost Ricci-Bourguignon soliton is simply a Ricci-Bourguignon soliton. This completes the proof of the first part of the theorem.

To prove the second part, we choose $f = 0$, then $\lambda + \rho R = 2m$ and thus we have from (21) that $R = 2m(2m+1)$. Since λ, R are constant, (6) implies that $|Ric - 2mg|^2 = 0$, i.e. $Ric = 2mg$. This shows that g is an Einstein metric. Since (M, g) is complete, (M, g) is compact according to Myers' theorem. From [4,10] every compact K-contact manifold with an Einstein constant greater than -2 is Sasakian. This completes the proof of the theorem. □

Proof (Corollary 1.11). Since the Reeb vector field is geodesic, we have from [13, Theorem 3] that the metric contact manifold is a K-contact manifold. Therefore, the Theorem 1.9 completes the proof. □

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