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NEW MULTIPLE FIXED POINT THEOREMS FOR SUM OF TWO OPERATORS AND APPLICATION TO A SINGULAR GENERALIZED STURM-LIOUVILLE MULTIPOINT BVP

Lydia Bouchal and Karima Mebarki

Abstract. In this paper, we develop some new multiple fixed point theorems for the sum of two operators $T + S$ where $I - T$ is Lipschitz invertible and S is a k-set contraction on translate of a cone in a Banach space. New existence criteria for multiple positive solutions of a singular generalized Sturm-Liouville multipoint boundary value problem are established. The article ends with an illustrative example.

1. Introduction

Starting from the Krasnosel'skii fixed point theorem [11], the fixed point theory for the sum of operators has developed rapidly and has been extended both in theory and in application to many problems of various kinds of nonlinear sciences. On the other hand, Krasnoselskii's fixed point theorem in cones, which appeared in 1960 (see, e.g., [7, 8]), is one of the most useful principles for proving the existence, localization, and multiplicity of nonnegative solutions of various nonlinear problems. According to this theorem, a solution is localized in a conical shell of a Banach space.

Recently, in 2019, the authors in [4] opened a new line of research in the theory of fixed points in ordered Banach spaces for the sum of operators. Several fixed point theorems, including Krasnosel'skii type theorems in cones, have been established for a sum of an expansive operator and a set contraction. Recent developments of positive fixed point theorems in this direction and their applications can be found in [1, 2, 5]. These works are motivated by the fact that many problems from different fields of science (chemical reactors, neutron transport, population biology, infectious diseases, epidemiology, economics, applied mechanics, fluid mechanics, . . .) can be formulated for a sum of two operators.

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Positivity of solutions of nonlinear equations, especially ordinary, fractional, partial differential equations and integral equations, is a very important issue in applications where a positive solution can represent a density, temperature, velocity, gravity, etc. The positivity condition can be described mathematically by introducing a cone $\mathcal{P}$ in a Banach space E which is a closed convex subset such that $\lambda\mathcal{P} \subset \mathcal{P}$ for all positive real numbers λ and $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. Note that a cone $\mathcal{P}$ induces a partial order $\leq$ in E defined by $x \leq y$ if and only if $y - x \in \mathcal{P}$. We denote $\mathcal{P} \setminus \{0\}$ by $\mathcal{P}^*$.

The first part of this paper contains generalizations of some fixed point theorems for the sum of operators. More precisely, it is devoted to the study of existence, multiplicity, positivity and localization of solutions for abstract equations of the form: $Tx + Sx = x$, $x \in D$, where $(I - T)$ is a Lipschitz invertible mapping, S is a k-set contraction, and D is a translation of a cone of a Banach space.

In the second part of the paper, we discuss the existence of multiple positive solutions of the following singular Sturm-Liouville multipoint boundary value problem (BVP for short):

$$\begin{cases} -u''(t) = h(t)f(t, u(t), u'(t)), \quad 0 < t < 1, \\ au(0) - bu'(0) = \displaystyle\sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \displaystyle\sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1}$$

where $a, b, c, d \in [0, \infty)$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ $(m \geq 3)$, $a_i, b_i \in [0, \infty)$ are constants for $i = 1, 2, \ldots, m-2$ and $\rho = ac + ad + bc > 0$.

By a positive solution, we mean a function $u \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2((0,1))$ such that $u(t) \geq 0$ on $[0,1]$, it is not identically zero and satisfies (1).

Multipoint boundary value problem theory has developed rapidly in the last twenty years. Since the original work of Il'in and Moiseev [9] on the existence of solutions for a linear multipoint BVP, special attention has been given to the study of multipoint BVP for nonlinear ordinary differential equations. Various approaches have been used to deal with this type of problem: Leray-Schauder continuation theorem, fixed point theorems in cones, coincidence degree theory, and the method of upper and lower solutions.

In [14], Zhang and Sun, using the fixed point theorem of Avery and Peterson, discussed the existence of three positive solutions of the problem (1) for the case where $h \in \mathcal{C}([0,1], [0,\infty))$.

In [15], Zhang used the same approach to obtain a multiplicity result for this problem in the singular case (where h can be singular at $t = 0$ and/or $t = 1$).

In this work, new existence criteria for at least three positive solutions of the problem (1) is established using one of our results from Section 3. This study is carried out under much weaker conditions than those established in [14] and [15]. Moreover, the possibility of the existence of countable many positive solutions is also discussed in this paper.

The paper is organized as follows. In Section 2, we give some useful preliminary results. In Section 3 we develop some new multiple fixed point theorems for the sum of two operators on translates of cones in Banach space. New general criteria for the existence of multiple nontrivial positive solutions of the generalized Sturm-Liouville multipoint BVP (1) are established in Section 4. To illustrate the application of our main criteria, an example is presented in Section 5. Finally, in Section 6 we compare the obtained results with some existing ones.

2. Preliminary

Let E be a real Banach space. Recall the following

Definition 2.1. A mapping $K : E \to E$ is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The notion of set contraction is related to the Kuratowski measure of noncompactness. Recall that the Kuratowski measure of noncompactness $\alpha(V)$ of a bounded subset V of a Banach space E is the infimum of positive numbers δ such that there are finitely many sets of diameter at most δ covering V. For the main properties of the measure of noncompactness we refer the reader to [8].

Definition 2.2. A mapping $A : E \to E$ is called k-set contraction if it is continuous and bounded and there is a constant $k \geq 0$ such that $\alpha(A(Y)) \leq k\alpha(Y)$, for any bounded set $Y \subset E$. The mapping A is called a strict k-set contraction if $k < 1$.

Obviously, if $A : E \to E$ is a completely continuous mapping, then A is a 0-set contraction.

In the following, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|\cdot\|)$. For a given $\omega \in E$, we consider the translation of the cone $\mathcal{P}$, namely $\mathcal{K}_\omega = \mathcal{P} + \omega = \{x + \omega \ : x \in \mathcal{P}\}$. Then the set $\mathcal{K}_\omega$ is closed and convex, that is, a retract of E. Let Ω be a subset of $\mathcal{K}_\omega$ and U be a bounded open subset of $\mathcal{K}_\omega$. The fixed point index $i_*(T + S, U \cap \Omega, \mathcal{K}_\omega)$ is defined by

$$i_*\left(T + S, U \cap \Omega, \mathcal{K}_\omega\right) = \begin{cases} i\left((I-T)^{-1}S, U, \mathcal{K}_\omega\right), & \text{if } U \cap \Omega \neq \emptyset \\ 0, & \text{if } U \cap \Omega = \emptyset. \end{cases} \tag{2}$$

It is well-defined if $T : \Omega \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $S : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I-T)(\Omega)$. For more details see [4, 5].

3. New multiple fixed point theorems for sum of two operators

It is known that if D is a bounded open subset of a Banach space E and A is a strict set contraction mapping defined on the closure of D and taking values in E, then the Leray-Schauder boundary condition $Ax \neq \lambda x$ for all $x \in \partial D$, $\lambda > 1$, is sufficient to guarantee the existence of a fixed point for A. For the importance of this condition and its extensions in the study of nonlinear problems, we refer the reader to [6, 10]. In this paper, we develop an extension of the Leray-Schauder boundary condition by considering a translation of the cone $\mathcal{P}$ defined above. First, we present our result for the class of strict set contractions. Then we extend it for a class of k-set contractions perturbed by an operator T such that $(I-T)$ is Lipschitz invertible.

Lemma 3.1. *Let $\mathcal{K}_\omega$ be a translate of a cone $\mathcal{P}$ and $U \subset \mathcal{K}_\omega$ a bounded open subset with $\omega \in U$. Assume that $A : \overline{U} \to \mathcal{K}_\omega$ is a strict k-set contraction without fixed point on ∂U and there exists $\varepsilon > 0$ small enough such that*

$$Ax - \omega \neq \lambda(x-\omega) \ \text{ for all } x \in \partial U \text{ and } \lambda \geq 1+\varepsilon. \tag{3}$$

Then the fixed point index $i\,(A, U, \mathcal{K}_\omega) = 1$.

Proof. Consider the homotopic deformation $H : [0,1] \times \overline{U} \to \mathcal{K}_\omega$ defined by $H(t,x) = \frac{t}{\varepsilon+1}\,(Ax-\omega)+\omega$. The operator H is continuous and uniformly continuous in t for each x, and the mapping $H(t,.)$ is a strict set contraction for each $t \in [0,1]$. In addition, $H(t,.)$ has no fixed point on ∂U. Otherwise, there would exist some $x_0 \in \partial U$ and $t_0 \in [0,1]$ such that $\frac{t_0}{\varepsilon+1}\,(Ax_0-\omega)+\omega = x_0$; then

- If $t_0 = 0$, we get $x_0 = \omega$, contradicting $\omega \in U$.

- If $t_0 \in (0,1]$, we get $Ax_0 - \omega = \frac{1+\varepsilon}{t_0}(x_0-\omega)$ with $\frac{1+\varepsilon}{t_0} \geq 1+\varepsilon$, contradicting the assumption (3). From the invariance under homotopy and the normalization properties of the index, we deduce

$$i\,(\frac{1}{\varepsilon+1}\,A + \frac{\varepsilon}{\varepsilon+1}\,\omega, U, \mathcal{K}_\omega) = i\,(\omega, U, \mathcal{K}_\omega) = 1.$$

Now, we show that $i\,(A, U, \mathcal{K}_\omega) = i\,(\frac{1}{\varepsilon+1}\,A + \frac{\varepsilon}{\varepsilon+1}\,\omega\,, U, \mathcal{K}_\omega)$. Since A has no fixed point in ∂U and $(I-A)(\partial U)$ is a closed set (see [13, Lemma 1]), we get $0 \notin \overline{(I-A)(\partial U)}$. Hence, $\gamma := dist(0, (I-A)(\partial U)) = \inf_{x\in\partial U} \|x - Ax\| > 0$. Let ε be sufficiently small so that $\|\frac{\varepsilon}{\varepsilon+1}(Ax-\omega)\| < \frac{\gamma}{2}$ and $\frac{\varepsilon+2}{\varepsilon+1}k < 1$. Hence

$$\left\| Ax - \left(\frac{1}{\varepsilon+1}Ax + \frac{\varepsilon}{\varepsilon+1}\omega\right)\right\| = \left\|\frac{\varepsilon}{\varepsilon+1}\left(Ax-\omega\right)\right\|,\ \forall\, x \in \partial U.$$

Define the convex deformation $G : [0,1] \times \overline{U} \to \mathcal{K}_\omega$ by

$$G(t,x) = tAx + (1-t)\left(\frac{1}{\varepsilon+1}\,Ax + \frac{\varepsilon}{\varepsilon+1}\,\omega\right).$$

The operator G is continuous and uniformly continuous in t for each x, and the mapping $G(t,.)$ is a strict set contraction, with constant $\frac{\varepsilon+2}{\varepsilon+1}k$, for each $t \in [0,1]$. In addition, $G(t,.)$ has no fixed point on ∂U. In fact, for all $x \in \partial U$, we have

$$\begin{aligned}\|x - G(t,x)\| &= \left\| x - tAx - (1-t)\left(\frac{1}{\varepsilon+1}\,Ax + \frac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\| \\ &\geq \left\| x - \left(\frac{1}{\varepsilon+1}\,Ax + \frac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\| - t\left\| Ax - \left(\frac{1}{\varepsilon+1}\,Ax + \frac{\varepsilon}{\varepsilon+1}\,\omega\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&= \left\| x - Ax + \frac{\varepsilon}{\varepsilon+1} Ax - \frac{\varepsilon}{\varepsilon+1}\omega \right\| - t \left\| \frac{\varepsilon}{\varepsilon+1}\Big(Ax - \omega\Big) \right\| \\
&\geq \|x - Ax\| - \left\| \frac{\varepsilon}{\varepsilon+1}\Big(Ax - \omega\Big) \right\| - t \left\| \frac{\varepsilon}{\varepsilon+1}\Big(Ax - \omega\Big) \right\| \\
&> \gamma - \frac{\gamma}{2} - \frac{\gamma}{2} = 0.
\end{aligned}
$$

Then our claim follows from the invariance by homotopy property of the index. □

Now, we extend the previous result to the case of a k-set contraction perturbed by an operator T such that $(I - T)$ is Lipschitz invertible.

LEMMA 3.2. *Let $\mathcal{K}_\omega$ be a translate of a cone $\mathcal{P}$. Let Ω be a subset of $\mathcal{K}_\omega$ and U a bounded open subset of $\mathcal{K}_\omega$. Assume that $T : \Omega \to E$ is such that $(I - T)$ is a Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I - T)(\Omega)$. Suppose that $T + S$ has no fixed point on $\partial U \cap \Omega$. Then we have the following results:*

(i) If $\omega \in U$ and there exists $\varepsilon > 0$ small enough such that

$$Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega) \ \text{ for all } \ \lambda \geq 1 + \varepsilon,\ x \in \partial U \ \text{ and } \ \lambda x + (1 - \lambda)\omega \in \Omega,$$

then the fixed point index $i_*\left(T + S, U \cap \Omega, \mathcal{K}_\omega\right) = 1$.

(ii) If there exists $u_0 \in \mathcal{P}^*$ such that

$$Sx \neq (I - T)(x - \lambda u_0), \ \text{ for all } \lambda > 0 \ \text{ and } \ x \in \partial U \cap (\Omega + \lambda u_0),$$

then the fixed point index $i_*\left(T + S, U \cap \Omega, \mathcal{K}_\omega\right) = 0$.

Proof. (i) The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{K}_\omega$ is a strict γk-set contraction without fixed point on $\partial U \cap \Omega$, and our hypothesis implies

$$(I - T)^{-1}Sx - \omega \neq \lambda(x - \omega) \quad \text{for all} \quad x \in \partial U \quad \text{and} \quad \lambda \geq 1 + \varepsilon.$$

Then, our claim follows from (2) and Lemma 3.1.

(ii) The mapping $(I - T)^{-1}S : \overline{U} \to \mathcal{K}_\omega$ is a strict γk-set contraction. Assume to the contrary that $i_*\left(T + S, U \cap \Omega, \mathcal{K}_\omega\right) \neq 0$; then $i((I - T)^{-1}S, U, \mathcal{K}_\omega) \neq 0$. For each $r > 0$, define the homotopy $H(t, x) = (I - T)^{-1}Sx + tru_0$, for $x \in \overline{U}$ and $t \in [0, 1]$. The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t, \cdot)$ is a strict γk-set contraction mapping for each t, and $H([0, 1] \times \overline{U}) = ((I - T)^{-1}S(\overline{U}) + tru_0) \subset \mathcal{K}_\omega$. In addition, $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial U$. Otherwise, there would exist some $(t_0, x_0) \in [0, 1] \times \partial U$ such that $H(t_0, x_0) = x_0$; then $(I - T)^{-1}Sx_0 = x_0 - t_0 r u_0$, and so $x_0 - t_0 r u_0 \in \Omega$. Hence $Sx_0 = (I - T)(x_0 - t_0 r u_0)$, for some $x_0 \in \partial U \cap (\Omega + t_0 r u_0)$, which contradicts our assumption. By homotopy invariance property of the fixed point index, we deduce that $i((I - T)^{-1}S + ru_0, U, \mathcal{K}_\omega) = i((I - T)^{-1}S, U, \mathcal{K}_\omega) \neq 0$. Thus, from the existence property of the fixed point index, for each $r > 0$, there exists $x_r \in U$ such that

$$x_r - (I - T)^{-1}Sx_r = ru_0. \tag{4}$$

Letting $r \to +\infty$ the left-hand side of (4) is bounded, while the right-hand side is not, which is a contradiction. Therefore $i_*(T + S, U \cap \Omega, \mathcal{K}_\omega) = 0$. □

Remark 3.3. (a) The result (i) in Lemma 3.2 is an extension of [4, Proposition 2.11], [5, Proposition 4.1] and [3, Proposition 4].

(b) The result (ii) in Lemma 3.2 and additional results concerning the computation of the fixed point index for the sum $T + S$ on translates of cones, are given in [5].

In the following result the existence is proved of at least three fixed points for the operator $T + S$ on translates of cones.

Theorem 3.4. *Let U_1, U_2 and U_3 be three open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $\omega \in U_1$, and let Ω be a subset of K_ω. Assume that $T : \Omega \to E$ be such that $(I - T)$ is a Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_3 \to E$ a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:*

(i) $Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_1$ and $\lambda x + (1 - \lambda)\omega \in \Omega$,

(ii) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda \geq 0$ and $x \in \partial U_2 \cap (\Omega + \lambda u_0)$,

(iii) $Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_3$ and $\lambda x + (1 - \lambda)\omega \in \Omega$,
Then $T + S$ has at least three nontrivial fixed points $x_1, x_2, x_3 \in \mathcal{K}_\omega$ such that

$$x_1 \in \partial U_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega,$$

or

$$x_1 \in U_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Proof. If $Sx = (I - T)x$ for $x \in \partial U_1 \cap \Omega$, then we get a fixed point $x_1 \in \partial U_1 \cap \Omega$ of the operator $T + S$.

Suppose that $Tx + Sx \neq x$ on $\partial U_1 \cap \Omega$. Without loss of generality, assume that $Tx + Sx \neq x$ on $\partial U_3 \cap \Omega$. By Lemma 3.2, we have $i_* (T + S, U_1 \cap \Omega, \mathcal{K}_\omega) = i_* (T+S, U_3 \cap \Omega, \mathcal{K}_\omega) = 1$ and $i_* (T+S, U_2 \cap \Omega, \mathcal{K}_\omega) = 0$. From the additivity property of the index i_*, we get $i_* (T+S, (U_2 \setminus \overline{U}_1) \cap \Omega, \mathcal{K}_\omega) = -1$, $i_* (T+S, (U_3 \setminus \overline{U}_2) \cap \Omega, \mathcal{K}_\omega) = 1$. Consequently, by the existence property of the index i_*, $T + S$ has at least three fixed points such that $x_1 \in U_1 \cap \Omega$, $x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega$ and $x_3 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$. □

Similarly, we can prove the following results, which are extensions of Theorem 3.4.

Theorem 3.5. *Let $U_1, U_2, \ldots, U_n$ be n $(n \geq 3)$ open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset \cdots \subset U_n$ and $\omega \in U_1$ and Ω be a subset of K_ω. Assume that $T : \Omega \to E$ be such that $(I - T)$ is a Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_n \to E$ is an ℓ-set contraction with $0 \leq \ell < \gamma^{-1}$ such that $T + S$ has no fixed point in $\partial U_{2k+1} \cap \Omega$ for $2k + 1 \in \{1, \ldots, n\}$ and $S(\overline{U}_n) \subset (I - T)(\Omega)$.*
Suppose that $(U_{i+1} \setminus \overline{U}_i) \cap \Omega \neq \emptyset$, for $i \in \{1, \ldots, n - 1\}$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:

(a) $Sx \neq (I - T)(\lambda x + (1 - \lambda)\omega)$, for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_{2k+1}$, for $2k + 1 \in \{1, \ldots, n\}$ and $(\lambda x + (1 - \lambda)\omega) \in \Omega$.

(b) $Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda \geq 0$ and $x \in \partial U_{2k} \cap (\Omega + \lambda u_0)$, for $2k \in \{2, \ldots, n\}$.

Then $T+S$ has n nontrivial fixed points $x_1, x_2, \dots, x_n \in \mathcal{K}_\omega$ satisfying

$$x_1 \in U_1 \cap \Omega, \quad \text{and} \quad x_i \in (U_i \setminus \overline{U}_{i-1}) \cap \Omega, \ \text{for } i \in \{2, \dots, n\}.$$

THEOREM 3.6. *Let $U_1, U_2, \dots, U_{n+1}$ be $n+1$ ($n \geq 3$) open bounded subsets of $\mathcal{K}_\omega$ such that $\overline{U}_1 \subset \overline{U}_2 \subset \cdots \subset U_{n+1}$ and $\omega \in U_1$. Let $T : \Omega \to E$ be such that $(I-T)$ is a Lipschitz invertible mapping with constant $\gamma > 0$, $S : \overline{U}_{n+1} \to E$ is a I-set contraction with $0 \leq \ell < \gamma^{-1}$ such that $T+S$ has no fixed point in $\partial U_{2k} \cap \Omega$ for $2k \in \{2, \dots, n+1\}$ and $S(\overline{U}_{n+1}) \subset (I-T)(\Omega)$.*

Suppose that $(U_{i+1} \setminus \overline{U}_i) \cap \Omega \neq \emptyset$, for $i \in \{1, \dots, n-1\}$, and there exist $u_0 \in \mathcal{P}^*$ and $\varepsilon > 0$ small enough such that the following conditions hold:
(a) $Sx \neq (I-T)(x - \lambda u_0)$, for all $\lambda \geq 0$ and $x \in \partial U_{2k+1} \cap (\Omega + \lambda u_0)$, for $2k+1 \in \{1, \dots, n+1\}$.

(b) $Sx \neq (I-T)(\lambda x + (1-\lambda)\omega)$, for all $\lambda \geq 1+\varepsilon$, $x \in \partial U_{2k}$, for $2k \in \{2, \dots, n+1\}$ and $(\lambda x + (1-\lambda)\omega) \in \Omega$.
Then $T+S$ has n nontrivial fixed points $x_1, x_2, \dots, x_n \in \mathcal{K}_\omega$ satisfying

$$x_i \in (U_{i+1} \setminus \overline{U}_i) \cap \Omega, \quad \text{for} \quad i \in \{1, \dots, n\}.$$

4. Application to a singular generalized Sturm-Liouville multipoint BVP

The aim of this section is to investigate the singular generalized Sturm-Liouville multipoint BVP (1) for existence of multiple positive solutions. For this goal, our new topological approach for the sum of two operators, developed in Section 3, is used.

4.1 General assumptions

We first enunciate the common assumptions that we will use in order to prove our main results. The following assumptions will be assumed in each existence criteria.
$(\mathcal{H}_1)$ $f \in \mathcal{C}([0,1] \times [0,\infty) \times (-\infty,\infty), (-\infty,\infty))$, $|f(t,u,v)| \leq k_1|u|^{p_1} + k_2|v|^{p_2} + k_3$, $t \in [0,1]$, $u, v \in \mathbb{R}$, k_1, k_2, k_3, p_1, p_2 are positive constants.

$(\mathcal{H}_2)$ $h \in \mathcal{C}((0,1), \mathbb{R})$ may be singular at $t = 0$ and/or $t = 1$ and $\int_0^1 G(s,s)h(s)\,ds < \infty$, where G is given by (6).

$(\mathcal{H}_3)$ $\Delta < 0$, $\rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) > 0$, $\rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) > 0$.

4.2 Integral representation of the solutions

Let $x(t) = at + b$ and $y(t) = d + c(1-t)$ for $t \in [0,1]$ and denote

$$\Delta := \begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i x(\xi_i) & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & -\sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix}.$$

In [12], it is proved that, if $\Delta \neq 0$, then the problem (1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s),u'(s))\,ds + \mathcal{A}(hf)\,x(t) + \mathcal{B}(hf)\,y(t), \quad t \in [0,1], \tag{5}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases}(d+c(1-t))(as+b), & 0 \le s \le t \le 1,\\ (at+b)(d+c(1-s)), & 0 \le t \le s \le 1,\end{cases} \tag{6}$$

$$\mathcal{A}(v) := \frac{1}{\Delta}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)v(s)\,ds & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)v(s)\,ds & -\sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix},$$

$$\mathcal{B}(v) := \frac{1}{\Delta}\begin{vmatrix} -\sum\limits_{i=1}^{m-2} a_i x(\xi_i) & \sum\limits_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)v(s)\,ds \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)v(s)\,ds \end{vmatrix}.$$

Set

$$\mathcal{M} := \int_0^1 G(s,s)|h(s)|\,ds,$$

$$\mathbb{A} := \frac{1}{|\Delta|}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i & \rho - \sum\limits_{i=1}^{m-2} a_i y(\xi_i) \\ -\sum\limits_{i=1}^{m-2} b_i & \sum\limits_{i=1}^{m-2} b_i y(\xi_i) \end{vmatrix},$$

$$\mathbb{B} := \frac{1}{|\Delta|}\begin{vmatrix} \sum\limits_{i=1}^{m-2} a_i x(\xi_i) & -\sum\limits_{i=1}^{m-2} a_i \\ \rho - \sum\limits_{i=1}^{m-2} b_i x(\xi_i) & \sum\limits_{i=1}^{m-2} b_i \end{vmatrix}.$$

Let E be the Banach space $\mathcal{C}^1([0,1])$ endowed with the norm

$$\|u\| = \max\left(\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}|u'(t)|\right).$$

For $u \in E$, we define the operators

$$\begin{aligned} Fu(t) &= \int_0^1 G(t,s)h(s)f(s,u(s),u'(s))\,ds + \mathcal{A}(hf)x(t) + \mathcal{B}(hf)y(t),\\ S_1u(t) &= Fu(t) - u(t),\\ S_2u(t) &= \int_0^t (t-s)^2\, g(s)S_1u(s)\, ds, \quad t \in [0,1], \end{aligned}$$

where $g \in \mathcal{C}([0,1],(0,\infty))$.

By (5), it follows that if $u \in E$ satisfy the equation $S_1u = 0$, then it is a solution of the problem (1).

LEMMA 4.1. *Suppose that* ($\mathcal{H}_1$)-($\mathcal{H}_3$) *hold. Let* L *be a real constant and* $u \in E$ *satisfies the equation*

$$S_2u(t) + 2L = 0, \quad t \in [0,1]. \tag{7}$$

Then u *is a solution of the problem* (1).

Proof. We differentiate the integral equation (7) three times with respect to t and we get $g(t)\,S_1u(t) = 0$, $t \in [0,1]$, whereupon $S_1\,u(t) = 0$, $t \in [0,1]$. □

4.3 A *priori* estimates

Suppose
($\mathcal{H}_4$) $g \in \mathcal{C}([0,1],(0,\infty))$ be such that $\int_0^1((1-s)^2+2(1-s)+2)\,g(s)\,ds \le A_1$, for some constant $A_1 > 0$.

Fix $B > 0$ arbitrarily.

LEMMA 4.2. *Suppose that* ($\mathcal{H}_1$)-($\mathcal{H}_3$) *hold. For any* $u \in E$ *with* $\|u\| \le B$, *we have*

$$|Fu(t)| \le \mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right),\ t \in [0,1].$$

Proof. We have

$$\begin{aligned}
|\mathcal{A}(hf)| \le \frac{1}{|\Delta|}\Bigg(&\left(\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)|h(s)||f(s,u(s),u'(s))|\,ds\right)\left(\sum_{i=1}^{m-2} b_iy(\xi_i)\right)\\
&+\left(\sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)|h(s)||f(s,u(s),u'(s))|\,ds\right)\left(\rho-\sum_{i=1}^{m-2} a_iy(\xi_i)\right)\Bigg)\\
\le \Bigg(\frac{1}{|\Delta|}\Bigg(&\left(\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)|h(s)|\,ds\right)\left(\sum_{i=1}^{m-2} b_iy(\xi_i)\right)\\
&+\left(\sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)|h(s)|\,ds\right)\left(\rho-\sum_{i=1}^{m-2} a_iy(\xi_i)\right)\Bigg)\Bigg)\left(k_1\|u\|^{p_1}+k_2\|u\|^{p_2}+k_3\right)\\
\le \Bigg(\frac{1}{|\Delta|}\Bigg(&\left(\sum_{i=1}^{m-2} a_i\int_0^1 G(s,s)|h(s)|\,ds\right)\left(\sum_{i=1}^{m-2} b_iy(\xi_i)\right)\\
&+\left(\sum_{i=1}^{m-2} b_i\int_0^1 G(s,s)|h(s)|\,ds\right)\left(\rho-\sum_{i=1}^{m-2} a_iy(\xi_i)\right)\Bigg)\Bigg)\left(k_1\|u\|^{p_1}+k_2\|u\|^{p_2}+k_3\right)\\
\le \mathcal{M}\mathbb{A}&\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right).
\end{aligned}$$

Similarly, we obtain $|\mathcal{B}(hf)| \le \mathcal{M}\mathbb{B}\left(k_1B^{p_1}+k_2B^{p_2}+k_3\right)$. Then

$$\begin{aligned}
|Fu(t)| &\le \int_0^1 G(t,s)|h(s)||f(s,u(s),u'(s))|\,ds + |\mathcal{A}(hf)|\,x(t) + |\mathcal{B}(hf)|\,y(t)\\
&\le \int_0^1 G(t,s)|h(s)|\left(k_1|u(s)|^{p_1}+k_2|u'(s)|^{p_2}+k_3\right)\,ds
\end{aligned}$$

$$+\mathcal{M}\mathbb{A}\,(k_1\|u\|^{p_1}+k_2\|u\|^{p_2}+k_3)\,x(t)+\mathcal{M}\mathbb{B}\,(k_1\|u\|^{p_1}+k_2\|u\|^{p_2}+k_3)\,y(t)$$
$$\leq \mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)\,,\quad t\in[0,1].$$

This completes the proof. □

LEMMA 4.3. *Suppose that* $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ *hold. Let* $u\in E$ *be such that* $\|u\|\leq B$. *Then*

$$\|S_2u\|\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),$$
$$|(S_2u)''(t)|\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),\;\; t\in[0,1].$$

Proof. Using Lemma 4.2, we arrive at

$$\begin{aligned}|S_2u(t)| &= \left|\int_0^t (t-s)^2 g(s)S_1u(s)\,ds\right| \leq \int_0^t (t-s)^2 g(s)|S_1u(s)|\,ds\\
&\leq \int_0^t (t-s)^2 g(s)|Fu(s)-u(s)|\,ds \leq \int_0^1 (1-s)^2 g(s)(|Fu(s)|+|u(s)|)\,ds\\
&\leq (\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B)\int_0^1 (1-s)^2 g(s)\,ds\\
&\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),\quad t\in[0,1],\end{aligned}$$

and

$$\begin{aligned}|(S_2u)'(t)| &= 2\left|\int_0^t (t-s)g(s)S_1u(s)\,ds\right|\\
&\leq 2\int_0^t (t-s)g(s)|S_1u(s)|\,ds \leq 2\int_0^t (t-s)g(s)|Fu(s)-u(s)|\,ds\\
&\leq 2(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B)\int_0^1 (t-s)g(s)\,ds\\
&\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),\quad t\in[0,1],\end{aligned}$$

Hence,

$$\|S_2u\|\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),$$

and

$$\begin{aligned}|(S_2u)''(t)| &= \left|2\int_0^t g(s)S_1u(s)\,ds\right| \leq 2\int_0^t g(s)|S_1u(s)|\,ds\\
&\leq 2(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B)\int_0^1 g(s)\,ds\\
&\leq A_1(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1B^{p_1}+k_2B^{p_2}+k_3)+B),\quad t\in[0,1].\end{aligned}$$

This completes the proof. □

4.4 Main results

In the sequel, suppose that the constant A_1 which appears in $(\mathcal{H}_4)$ satisfies the following inequality:

$$A_1\,(\mathcal{M}\,(1+(a+b)\mathbb{A}+(c+d)\mathbb{B})\,(k_1R_1^{p_1}+k_2R_1^{p_2}+k_3)+R_1)<2L_1, \tag{8}$$

where L_1, R_1 are such that $r_1 < L_1 < R_1$ with r_1 a positive constant.

The first main existence criteria is the following:

THEOREM 4.4. *If the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_4$) and the inequality* (8) *are satisfied, the problem* (1) *has at least three positive solutions* u_1, u_2, $u_3 \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2((0,1))$ *that satisfy*

$$0 \le \max\{\max_{t\in[0,1]} |u_1(t)|, \max_{t\in[0,1]} |u_1'(t)|\} \le r_1,$$
$$r_1 < \max\{\max_{t\in[0,1]} |u_2(t)|, \max_{t\in[0,1]} |u_2'(t)|\} < L_1,$$
$$L_1 < \max\{\max_{t\in[0,1]} |u_3(t)|, \max_{t\in[0,1]} |u_3'(t)|\} \le R_1.$$

Proof. Let $\mathcal{P} = \{u \in E : u \ge 0 \text{ on } [0,1]\}$. For $u \in \mathcal{P}$ let us define the operators T and S as follows:

$$Tu(t) = (1+\mu\varepsilon)u(t) - \varepsilon L_1, \quad Su(t) = -\varepsilon S_2 u(t) - \mu\varepsilon u(t) - \varepsilon L_1,\ t \in [0,1],$$

where μ is a large enough positive constant and $\varepsilon \ge \frac{4}{\mu}\frac{L_1}{r_1}$. Note that any fixed point $u \in \mathcal{P}$ of the operator $T+S$ is a solution to the problem (1). Define

$$U_1 = \mathcal{P}_{r_1} = \{u \in \mathcal{P} : \|u\| < r_1\}, \quad U_2 = \mathcal{P}_{L_1} = \{u \in \mathcal{P} : \|u\| < L_1\},$$
$$U_3 = \mathcal{P}_{R_1} = \{u \in \mathcal{P} : \|u\| < R_1\},$$
$$\varrho = \frac{1}{\mu}\left(A_1(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)(k_1R_1^{p_1}+k_2R_1^{p_2}+k_3)+R_1\right]+2L_2+\mu R_1)$$
$$\Omega = \overline{\mathcal{P}_\varrho} = \{v \in \mathcal{P} : \|v\| \le \varrho\}.$$

1. For $u_1, u_2 \in \Omega$, we have $\|Tu_1 - Tu_2\| = (1+\mu\varepsilon)\|u_1 - u_2\|$; then T is an expansive operator with constant $1+\mu\varepsilon$. So, $(I-T) : E \to E$ is Lipschitz invertible with constant $\frac{1}{\mu\varepsilon}$.

2. As in [15, Lemma 2.5], by applying Ascoli-Arzelà compactness criterion, we can prove that the operator S is completely continuous; then S is 0-set contraction.

3. We prove that $S(\overline{P_{R_1}}) \subset (I-T)(\Omega)$. Let $u \in \overline{P_{R_1}}$ be arbitrarily chosen. Set $v = \frac{S_2u+2L_1+\mu u}{\mu}$. It is clear that $v \ge 0$ and

$$\|v\| = \|\frac{1}{\mu}(S_2u + 2L_1 + \mu u)\| \le \frac{1}{\mu}(\|S_2u\| + 2L_1 + \mu\|u\|)$$
$$\le \frac{1}{\mu}\left(A_1(\mathcal{M}\left(1+(a+b)\mathbb{A}+(c+d)\mathbb{B}\right)(k_1R_1^{p_1}+k_2R_1^{p_2}+k_3)+R_1\right)+2L_1+\mu R_1) = \varrho.$$

Therefore $v \in \Omega$ and

$$(I-T)v = -\epsilon(\mu v - L_1) = -\epsilon(\mu(\frac{S_2u+2L_1+\mu u}{\mu}) - L_1) = -\epsilon(S_2u+\mu u+L_1) = Su.$$

Thus, $S(\overline{P_{R_1}}) \subset (I-T)(\Omega)$.

4. Assume that there exist $\frac{\varrho}{r_1} \ge \lambda_1 \ge \varepsilon+1$ and $x_1 \in \partial\mathcal{P}_{r_1}$ ($\lambda_1 x_1 \in \Omega$ leads to $\lambda_1 \le \frac{\varrho}{\|x_1\|}$) such that $Sx_1 = (I-T)(\lambda_1 x_1)$. Then $-\varepsilon S_2 x_1 - \mu\varepsilon x_1 - \varepsilon L_1 = -\varepsilon\mu\lambda_1 x_1 + \varepsilon L_1$,

or equivalently $S_2x_1 = \mu(\lambda_1 - 1)x_1 - 2L_1$. So

$$\|S_2x_1\| = \|\mu(\lambda_1 - 1)x_1 - 2L_1\| \geq \mu(\lambda_1 - 1)\|x_1\| - 2L_1$$
$$\geq \mu(\lambda_1 - 1)r_1 - 2L_1 \geq \mu\varepsilon r_1 - 2L_1 \geq 2L_1.$$

Hence, a contradiction with one of the results of Lemma 4.3 and (8).

5. Assume that for any $u_0 \in \mathcal{P}^*$, there exist $\lambda_0 \geq 0$ and $x_0 \in \partial P_{L_1} \cap (\Omega + \lambda_0 u_0)$ such that $Sx_0 = (I - T)(x_0 - \lambda_0 u_0)$. Then $-\varepsilon(S_2x_0 + \mu x_0 + L_1) = -\varepsilon(\mu(x_0 - \lambda_0 u_0) - L_1)$, or equivalently $S_2x_0 = -(\lambda_0 \mu u_0 + 2L_1)$. So $\|S_2x_0\| = \|\lambda_0\mu u_0 + 2L_1\| \geq 2L_1$, which is a contradiction.

6. Assume that there exist $\frac{\varrho}{R_1} \geq \lambda_2 \geq \varepsilon + 1$ and $x_2 \in \partial\mathcal{P}_{R_1}$ such that $Sx_2 = (I - T)(\lambda_2 x_2)$. Then $S_2x_2 = (\lambda_2 - 1)\mu x_2 - 2L_1$. So

$$\|S_2x_2\| = \|\mu(\lambda_1 - 1)x_2 - 2L_1\| \geq \mu(\lambda_2 - 1)R_1 - 2L_1 \geq \mu\varepsilon R_1 - 2L_1 \geq \mu\varepsilon r_1 - 2L_1 \geq 2L_1,$$

which is a contradiction.

Therefore all conditions of Theorem 3.4 hold for $w = 0$. Hence, the problem (1) has at least three solutions u_1, u_2 and u_3 in $\mathcal{P}$ so that $0 \leq \|u_1\| < r_1 < \|u_2\| < L_1 < \|u_3\| \leq R_1$. □

Now, we discuss the existence of countable many positive solutions for the problem (1). If we replace the inequality (8) by the following one

$$A_1\left(\mathcal{M}\left(1 + (a + b)\mathbb{A} + (c + d)\mathbb{B}\right)(k_1 R_n^{p_1} + k_2 R_n^{p_2} + k_3) + R_n\right) < 2L_1, \tag{9}$$

where $n \in \mathbb{N}^*$ fixed and for $i \in \{1, \ldots, n\}$, $L_i, R_i \in (0, \infty)$ are such that $r_i < L_i < R_i$ with $r_i > R_{i-1}, i \geq 2$, and by using similar arguments as in the proof of Theorem 4.4, we can prove the following generalized existence criteria.

Theorem 4.5. *If the assumptions* $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ *and the inequality (9) are satisfied, the problem* (1) *has at least* $2n + 1$ *positive solutions* $u_k \in \mathcal{C}^1([0, 1]) \cap \mathcal{C}^2((0, 1))$, $k \in \{1, \ldots, 2n + 1\}$ *that satisfy* $0 \leq \|u_1\| < r_1 < \|u_2\| < L_1 < \|u_3\| \leq R_1$, $r_k \leq \|u_{2k}\| < L_k < \|u_{2k+1}\| \leq R_k$, *for* $k \in \{2, \ldots, n\}$.

Proof. In this case for $i \in \{1, \ldots, n\}$, we consider

$$U_1^{(i)} = \mathcal{P}_{r_i} = \{u \in \mathcal{P} : \|u\| < r_i\}, \quad U_2^{(i)} = \mathcal{P}_{L_i} = \{u \in \mathcal{P} : \|u\| < L_i\},$$
$$U_3^{(i)} = \mathcal{P}_{R_i} = \{u \in \mathcal{P} : \|u\| < R_i\},$$
$$\varrho = \frac{1}{\mu}(A_1(\mathcal{M}\left(1 + (a + b)\mathbb{A} + (c + d)\mathbb{B}\right)(k_1 R_n^{p_1} + k_2 R_n^{p_2} + k_3) + R_n) + 2L_1 + \mu R_n)$$
$$\Omega = \overline{\mathcal{P}_\varrho} = \{u \in \mathcal{P} : \|u\| \leq \varrho\}.$$
□

5. Example

Let $m = 4$, $a = 4$, $b = 2$, $c = 4$, $d = 2$, $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{2}$, $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{2}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$. We consider the following BVP

$$-u''(t) = h(t)f(t, u(t), u'(t)), \quad 0 < t < 1,$$

$$4u(0) - 2u'(0) = \frac{1}{4}u(\frac{1}{4}) + \frac{1}{2}u(\frac{1}{2}), \qquad (10)$$
$$4u(1) + 2u'(1) = \frac{1}{3}u(\frac{1}{4}) + \frac{1}{2}u(\frac{1}{2}).$$

Where $h(t) = \frac{1}{\sqrt{t}} + \frac{1}{\sqrt{1-t}}$, $t \in (0,1)$, $f(t,y,z) = \frac{1}{10^2}\, t + \frac{1}{10^4}\, y + \frac{1}{10^4}\, z^{\frac{1}{5}}$, $t \in [0,1]$, $y \in [0,\infty)$, $z \in (-\infty,\infty)$.

Let also, $r_1 = 1$, $L_1 = 15$, $R_1 = 20$, $r_2 = 25$, $L_2 = 35$, $R_2 = 40$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$, $A_1 = \frac{1}{2}$, $k_1 = \frac{1}{3}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{3}$. By some calculations, we have $\rho = 32$, $\Delta = -823.66$, and the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold. We have, $G(s,s) = -\frac{1}{32}(4s+2)(4s-6)$. Let $\mathcal{M} = \int_0^1 G(s,s)\, h(s)\, ds = \frac{53}{30}$. Then

$$A_1((\mathcal{M} + (a+b)\mathbb{A} + (c+d)\mathbb{B})\,(k_1 R_2^{p_1} + k_2 R_2^{p_2} + k_3) + R_2) = 25,4559 < 2L_1.$$

Let $g(s) = \frac{s+1}{10}$, $s \in [0,1]$. Then

$$\int_0^1 ((1-s)^2+2(1-s)+2)g(s)\, ds = \frac{1}{10}\int_0^1 ((1-s)^2+2(1-s)+2)s\, ds = \frac{19}{40} = 0,475 \le A_1.$$

Then all assumptions of Theorem 4.5 hold for $n = 2$. Hence, the problem (10) has at least five positive solutions u_1, u_2, u_3, u_4, u_5 such that

$$0 \le \|u_1\| \le 1, \quad 1 < \|u_2\| < 10, \quad 10 < \|u_3\| \le 20. \quad 25 \le \|u_4\| < 35, \quad 35 < \|u_5\| \le 40.$$

6. Comparison and concluding remarks

In this section we compare the results obtained in this paper with those obtained by Zhang-Sun [14] and Zhang [15].

(a) In this work, the nonlinear term f takes values on $\mathbb{R}$ and is associated with the first-order derivative; moreover, f satisfies a general growth condition. The nonlinearity considered in [14] and [15] takes values in $[0,\infty)$ and is said to be piecewise bounded.

(b) The problem studied here is provided with a singular term given by h which takes values on $\mathbb{R}$, and the integral of h on $(0,1)$ need not be finite as in [15], it suffices that $\int_0^1 G(s,s)h(s)\, ds < \infty$.

(c) The conditions $a > \sum_{i=1}^{m-2} a_i$, $c > \sum_{i=1}^{m-2} b_i$ in both [14] and [15] are not of interest in our work.

(d) In this work, we establish sufficient conditions for the existence of countable many positive solutions to the problem (1). However, in [14] and [15] the authors have discussed the existence of only three positive solutions.

(e) Our approach has been applied to prove the existence of finite multiple positive solutions as well as the existence of countable many positive solutions for the problem (1), and it can be used to study the existence of multiple solutions for other classes of differential equations covered by various types of boundary value problems.

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Laboratory of Applied Mathematics, Faculty of Exact Sciences, Bejaia University, 6000 Bejaia, Algeria
E-mail: lydia.bouchal@univ-bejaia.dz, Bouchal_Lydia@outlook.com
ORCID iD: https://orcid.org/0000-0002-2728-0298

Laboratory of Applied Mathematics, Faculty of Exact Sciences, Bejaia University, 6000 Bejaia, Algeria
E-mail: mebarqi_karima@hotmail.fr, karima.mebarki@univ-bejaia.dz
ORCID iD: https://orcid.org/0000-0002-6679-5059