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CONSTRUCTION OF UNIVALENT HARMONIC MAPPINGS AND THEIR CONVOLUTIONS

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Abstract. In this article, we make use of convex analytic functions $H_a(z) = [1/(1-a)]\log[(1-az)/(1-z)]$, $a \in \mathbb{R}$, $|a| \leq 1$, $a \neq 1$ and starlike analytic functions $L_b(z) = z/[(1-bz)(1-z)]$, $b \in \mathbb{R}$, $|b| \leq 1$, to construct univalent harmonic functions by means of a transformation on some normalized univalent analytic functions. Besides exploring mapping properties of harmonic functions so constructed, we establish sufficient conditions for their harmonic convolutions or Hadamard products to be locally univalent and sense preserving, univalent and convex in some direction.

1. Introduction

Let $\mathcal{A}(\mathbb{D})$ be the set of analytic functions f defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$ and denote by $S \subset \mathcal{A}(\mathbb{D})$ the usual class of univalent functions in $\mathcal{A}(\mathbb{D})$. Obviously for $f \in S$, $f'(z) \neq 0$ for any $z \in \mathbb{D}$. Let K, S^* and C be subclasses of S consisting of convex, starlike and close to convex functions, respectively. Furthermore, denote by S_2 the subclass of K consisting of all those functions f for which zf' also belongs to K.

We consider two analytic functions

$$H_a(z) = \frac{1}{1-a}\log\frac{1-az}{1-z}, \quad a \in \mathbb{R},\ |a| \leq 1,\ a \neq 1 \tag{1}$$

and

$$L_b(z) = \frac{z}{(1-bz)(1-z)}, \quad b \in \mathbb{R},\ |b| \leq 1. \tag{2}$$

It is known (see [2]) that H_a is a convex univalent function for $|a| \leq 1$, $a \neq 1$ and obviously, the function $H_0(z) = -\log(1-z)$ belongs to the class S_2. Furthermore, $L_b \in S^*$ (see [8]), L_1 is the well known Koebe function and L_0 is the right half plane mapping.

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Our main motive in this article is the construction of locally univalent and sense-preserving harmonic mappings using the properties of H_a and L_b. A complex-valued function $f = u + iv$ is said to be harmonic in $\mathbb{D}$ if u and v are real harmonic in $\mathbb{D}$. A harmonic function f can be represented as $f = h + \overline{g}$ where h and g are analytic in $\mathbb{D}$. Furthermore, a harmonic function $f = h + \overline{g}$ is locally univalent and sense preserving in $\mathbb{D}$, if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second complex dilatation (or simply dilatation) of f, defined by $w(z) := g'(z)/h'(z)$, in $\mathbb{D}$ has modulus less than one. The "only if "part of this condition is a result of Lewy [3] (see also Clunie and Sheil-Small [1]). We denote by S_H the class of univalent harmonic functions $f = h+\overline{g}$ such that $h(0) = g(0) = h'(0) - 1 = 0$ and $|g'(0)| < 1$. If $g'(0) = 0$, we denote the class of functions $f = h + \overline{g} \in S_H$ by S_H^0.

A domain $\Omega \subset \mathbb{C}$ is called convex in the direction α $(0 \leq \alpha < \pi)$ if the intersection of Ω with any line parallel to the line through 0 and the point $e^{i\alpha}$ is either empty or an interval. A function $f \in S_H$ is called convex in the direction of α if it maps $\mathbb{D}$ onto the domain convex in the direction of α. For $f \in S_H$, if $f(\mathbb{D})$ is convex in the direction of the imaginary axis $(\alpha = \pi/2)$, then f is said to be convex in the direction of the imaginary axis. Similarly, $f \in S_H$ is said to be convex in the direction of the real axis if $f(\mathbb{D})$ is convex in the direction of the real axis $(\alpha = 0)$.

For two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n$ we denote their convolution or Hadamard product by $f * g$ and define it as: $(f * F)(z) = \sum_{n=0}^{\infty} a_n A_n z^n$. The convolution (or harmonic convolution) of two harmonic functions $f = h+\overline{g}$ and $F = H+\overline{G}$, is denoted by $f \tilde{*} F$ and is defined as: $f \tilde{*} F = h * H + \overline{g * G}$.

In 2012, Stacey Muir [4] defined a transformation $T_\lambda[f]$, $\lambda > 0$, of functions $f \in S$ given by $T_\lambda[f](z) = \frac{f(z)+\lambda z f'(z)}{1+\lambda} + \overline{\frac{f(z)-\lambda z f'(z)}{1+\lambda}}$, $z \in \mathbb{D}$ and proved that $T_\lambda[f] \in S_H$ and is convex in the direction of the imaginary axis if and only if $f \in K$. The present authors in [9], generalized this transformation as

$$C_{\lambda,h}[f](z) = \frac{f(z) + \lambda(h * f)(z)}{1 + \lambda} + \overline{\frac{f(z) - \lambda(h * f)(z)}{1 + \lambda}}, z \in \mathbb{D}, \tag{3}$$

where $f, h \in S$ and $\lambda > 0$; and established the following result:

THEOREM 1.1. *Let $f, h \in S$ and for $\lambda > 0$, let $C_{\lambda,h}[f]$ be defined as in* (3). *Then $C_{\lambda,h}[f]$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if*

$$Re\left(\frac{(h * zf')(z)}{zf'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

For $f \in K$, they established the univalence of $C_{\lambda,h_i}[f], i = 1, 2, 3, 4$, where h_1, h_2, h_3, h_4 are given by

$$h_1(z) = \sum_{n=1}^{\infty} nz^n,\ h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n,\ h_3(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n,\ h_4(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n.$$

As $(h_1 * f)(z) = zf'(z)$, so $C_{\lambda,h_1}[f](z) = T_\lambda[f](z)$. Moreover, $(h_2 * f)(z) = [f(z) + zf'(z)]/2, (h_3 * f)(z) = \int_0^z [f(\zeta)/\zeta] d\zeta$ and $(h_4 * f)(z) = (2/z) \int_0^z f(\zeta) d\zeta$.

In this paper, we explore the properties of harmonic functions

$$C_{\lambda,H_a}[f](z) = \frac{f(z)+\lambda(H_a*f)(z)}{1+\lambda} + \overline{\frac{f-\lambda(H_a*f)(z)}{1+\lambda}} \tag{4}$$

and

$$C_{\lambda,L_b}[f](z) = \frac{f(z)+\lambda(L_b*f)(z)}{1+\lambda} + \overline{\frac{f-\lambda(L_b*f)(z)}{1+\lambda}}, \tag{5}$$

where $\lambda > 0$, $f \in S$ and where H_a and L_b are given by (1) and (2), respectively. We also find conditions on f and g such that harmonic convolutions of $C_{\lambda,H_a}[f]$ and $C_{\lambda,L_b}[g]$ are locally univalent and sense preserving, univalent and convex in some direction.

2. Main results

We shall need the following two lemmas to prove our main theorems.

LEMMA 2.1 ([6]). *Let ψ and G be analytic in $\mathbb{D}$ with $\psi(0) = G(0) = 0$. If ψ is convex and G is starlike, then for each analytic function F satisfying $\Re(F(z)) > 0$ in $\mathbb{D}$, we have*

$$\Re\left\{\frac{(\psi * FG)(z)}{(\psi * G)(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

LEMMA 2.2 ([1]). *Let a harmonic mapping $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$. Then f is univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of α, $0 \le \alpha < \pi$, if and only if $h - e^{2i\alpha}g$ is a univalent analytic mapping of $\mathbb{D}$ onto a domain convex in the direction of α.*

THEOREM 2.3. *Let $f \in S$ and H_a be as given in (1). If*

$$\Re\left[\frac{f(z)-f(az)}{(1-a)zf'(z)}\right] > 0, \quad z \in \mathbb{D}, \tag{6}$$

then $C_{\lambda,H_a}[f]$ defined by (4) is locally univalent and sense preserving in $\mathbb{D}$. If $f \in K$, then $C_{\lambda,H_a}[f] \in S_H$ and is convex in the directions of both the real and the imaginary axes.

Proof. From Theorem 1.1, $C_{\lambda,H_a}[f]$ is locally univalent and sense preserving in $\mathbb{D}$ if $\Re\left[\frac{(H_a*f)'(z)}{f'(z)}\right] > 0$. Since $z(H_a*f)'(z) = zH_a'(z)*f(z) = \frac{1}{1-a}\left[\frac{-az}{1-az}+\frac{z}{1-z}\right]*f(z) = \left[\frac{f(z)-f(az)}{1-a}\right]$, then $\Re\left[\frac{(H_a*f)'(z)}{f'(z)}\right] = \left[\frac{f(z)-f(az)}{(1-a)zf'(z)}\right] > 0$ in view of (6). Hence the first part of theorem is proved.

Furthermore, it is known that if $f \in K$ then (6) is satisfied. This is, in fact, provided by [6, Lemma 4.2 (with $h = k$ there)] (also see [5, (2.105) on p. 86]). Hence, if $f \in K$, then $C_{\lambda,H_a}[f]$ is locally univalent and sense preserving in $\mathbb{D}$. Now, to complete the proof, let $C_{\lambda,H_a}[f](z) = \frac{f(z)+\lambda(H_a*f)(z)}{1+\lambda} + \overline{\frac{f(z)-\lambda(H_a*f)(z)}{1+\lambda}} = H(z)+\overline{G(z)}$. Then $H(z)-G(z) = \frac{2\lambda}{1+\lambda}(H_a*f)(z)$ and $H(z)+G(z) = \frac{2}{1+\lambda}f(z)$. It is well known that the class K is closed under convolution, so $f \in K$ and $H_a \in K$ imply that $H-G$

and $H+G$ are convex in the direction of the real axis and convex in the direction of the imaginary axis, respectively. Therefore, from Lemma 2.2, we conclude that $C_{\lambda,H_a}[f]$ is convex in the directions of both the real and the imaginary axes. $\square$

If we take $a=-1$, the condition (6) is equivalent to

$$\Re\left[\frac{2zf'(z)}{f(z)-f(-z)}\right]>0, \tag{7}$$

and such functions are known as starlike with respect to symmetric points. If S_S^* denotes the class of all functions sarlike with respect to symmetric points, then it is known that $K\subset S_S^*\subset C$ (see [7] for more details). Also,

$$(H_{-1}*f)(z)=\frac{1}{2}\int_0^z \frac{f(\zeta)-f(-\zeta)}{\zeta}\,d\zeta$$

implies that $H_{-1}*f$ is convex as $f(z)-f(-z)$ is starlike in $\mathbb{D}$ for $f\in S_S^*$. Therefore, from Theorem 2.3 along with Lemma 2.2, we get the following result.

THEOREM 2.4. *Let $f\in S_S^*$ and $H_{-1}(z)=\frac{1}{2}\log\frac{1+z}{1-z}$. Then $C_{\lambda,H_{-1}}[f]\in S_H$ and is convex in the direction of the real axis.*

We present the following example to illustrate our result.

EXAMPLE 2.5. Take $f(z)=f_0(z)=z+z^3/3$ which is univalent as $\Re f_0'(z)>0$ in $\mathbb{D}$. Also it is easy to verify that f_0 satisfies inequality (7), i.e, $f_0\in S_S^*$. Thus by above theorem $C_{\lambda,H_{-1}}[f_0]\in S_H$ and is convex in the direction of the real axis. The image of $\mathbb{D}$ under $C_{3,H_{-1}}[f_0]$ is plotted in Figure 1.

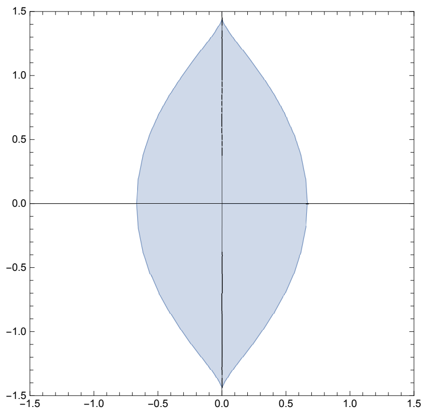

Figure 1: Image of $\mathbb{D}$ under $C_{3,H_{-1}}[f_0]$

Furthermore, in the case $a=0$, the condition (6) reduces to $\Re\left[\frac{f(z)}{zf'(z)}\right]>0$, or equivalently, $\Re\left[\frac{zf'(z)}{f(z)}\right]>0$, which implies that $f\in S^*$. Moreover, $(H_0*f)(z)=\int_0^z[f(\zeta)/\zeta]\,d\zeta$ shows that H_0*f is convex in $\mathbb{D}$ for $f\in S^*$. Therefore, we get the following result.

THEOREM 2.6. *Let $f \in S^*$ and $H_0(z) = -\log(1-z)$. Then $C_{\lambda,H_0}[f] \in S_H$ and is convex in the direction of the real axis.*

We invite the reader to compare Theorem 2.6 with the case $i = 3$ of [9, Theorem 3.3].

EXAMPLE 2.7. Let $F(z) = z/[(1+z/4)(1-z)] = L_{-1/4}(z)$. Then $F \in S^*$ and so, by Theorem 2.6, $C_{\lambda,H_0}[F] \in S_H$ and is convex in the direction of the real axis. In Figure 2 we have plotted the region on which $\mathbb{D}$ is mapped by the function

$$C_{0.5,H_0}[F](z) = \frac{2}{3}\left[\frac{z}{(1+z/4)(1-z)} + \frac{2}{5}\log\left[\frac{1+z/4}{1-z}\right] + \overline{\frac{z}{(1+z/4)(1-z)} - \frac{2}{5}\log\left[\frac{1+z/4}{1-z}\right]}\right].$$

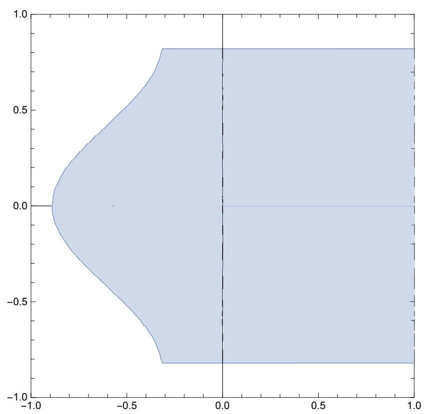

Figure 2: Image of $\mathbb{D}$ under $C_{0.5,H_0}[F]$

THEOREM 2.8. *Let $f \in S$ and L_b be given as in (2). If*

$$\Re\left[\frac{f'(z) - bf'(bz)}{(1-b)f'(z)}\right] > 0, b \neq 1, \tag{8}$$

then $C_{\lambda,L_b}[f]$, defined by (5), is locally univalent and sense preserving in $\mathbb{D}$. Furthermore, if $f \in S_2$, then $C_{\lambda,L_b}[f] \in S_H$ and is convex in the directions of both the real and the imaginary axes.

Proof. In view of Theorem 1.1, $C_{\lambda,L_b}[f]$ is locally univalent and sense preserving in $\mathbb{D}$ if $\Re\left[\frac{(L_b*f)'(z)}{f'(z)}\right] > 0$. On the other hand, $\frac{(L_b*f)'(z)}{f'(z)} = \frac{f'(z)-bf'(bz)}{(1-b)f'(z)}$. Hence, in view of (8), $C_{\lambda,L_b}[f]$ is locally univalent and sense preserving in $\mathbb{D}$. Furthermore, we have

$$\frac{(L_b*f)'(z)}{f'(z)} = \frac{\frac{z}{(1-bz)(1-z)} * zf'(z)}{zf'(z)} = \frac{zf'(z) * \left\{\left(\frac{1}{1-bz}\right)\left(\frac{z}{1-z}\right)\right\}}{zf'(z) * \left(\frac{z}{1-z}\right)}.$$

Now $\Re\,(1/(1-bz)) > 0$ for $|b| \leq 1$, $zf' \in K$ as $f \in S_2$ and $z/(1-z) \in K \subset S^*$. Thus by Lemma 2.1 we get that $\Re[(L_b * f)'(z)/f'(z)] > 0$ for $f \in S_2$, which in turn implies that $C_{\lambda,L_b}[f]$ is locally univalent and sense preserving in $\mathbb{D}$. Furthermore, let

$$C_{\lambda,L_b}[f](z) = \frac{f(z)+\lambda(L_b * f)(z)}{1+\lambda} + \overline{\frac{f(z)-\lambda(L_b * f)(z)}{1+\lambda}} = H(z) + \overline{G(z)}.$$

Then $H(z)-G(z) = [2\lambda/(1+\lambda)](L_b * f)(z)$ and $H(z)+G(z) = [2/(1+\lambda)]f(z)$. Now $f \in S_2$ implies that $zf' \in K$ and so $z(L_b * f)' = L_b * zf' \in S^*$. Thus $L_b * f$ and f both are in K and consequently $H-G$ and $H+G$ are convex in the direction of the real axis and convex in the direction of the imaginary axis, respectively. Therefore, from Lemma 2.2, we have $C_{\lambda,L_b}[f]$ is convex in the directions of both the real and the imaginary axes. □

Remark 2.9. Note that the above result does not hold true for $f \in K$ and this can be easily seen by considering the function

$$C_{1,z/(1-z^2)}[z/(1-z)](z) = \frac{1}{2}\left[\frac{z}{1-z} + \frac{z}{1-z^2} + \overline{\frac{z}{1-z} - \frac{z}{1-z^2}}\right].$$

It is evident from Figure 3 that this function is not even univalent in $\mathbb{D}$.

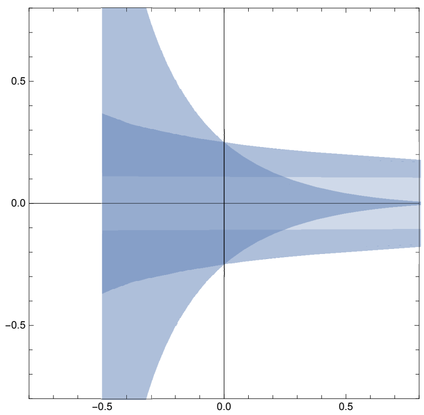

Figure 3: Image of $\mathbb{D}$ under $C_{1,z/(1-z^2)}[z/(1-z)]$

Remark 2.10. If b approaches to 1^-, then bz approaches to z. Therefore we get

$$\Re\left[\frac{f'(z)-bf'(bz)}{(1-b)f'(z)}\right] = \Re\left[\frac{[\frac{z(f(z)-f(bz))}{(1-b)z}]'(z)}{f'(z)}\right] = \Re\left[\frac{(zf'(z))'(z)}{f'(z)}\right] \text{(as } b \to 1^-).$$

However, since $\Re\left[\frac{(zf'(z))'(z)}{f'(z)}\right] = \Re\left[1+\frac{zf''(z)}{f'(z)}\right] > 0$, for $f \in K$, in the limiting case when $b \to 1^-$, we obtain the following: *For $f \in K$, $C_{\lambda,L_1}[f] \in S_H$ and is convex in the direction of the imaginary axis* (compare with [4, Theorem 3.2]).

EXAMPLE 2.11. Taking $f(z) = f_2(z) = -\log(1-z)$ and $b = -1$ in (8), we have
$$\Re\left[\frac{f'(z) - bf'(bz)}{(1-b)f'(z)}\right] = \Re\left[\frac{f_2'(z) + f_2'(-z)}{2f_2'(z)}\right] = \Re\left(\frac{1}{1+z}\right) > 0,$$
for $z \in \mathbb{D}$. Also $f_2 \in S_2$, so Theorem 2.8 implies that $C_{\lambda,L_{-1}}[f_2] \in S_H$ and is convex in the directions of the real as well as the imaginary axes. The region $C_{10,L_{-1}}[f_2](\mathbb{D})$ is depicted in Figure 4.

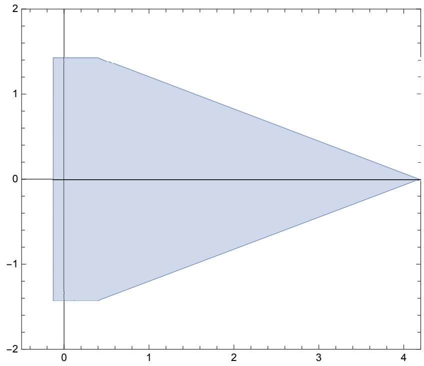

Figure 4: Image of $\mathbb{D}$ under $C_{10,L_{-1}}[f_2]$

The convolutions or Hadamard products of harmonic functions remain an active subject of study nowadays as almost nothing significant in general is known in this area unlike the case of analytic functions which is now a well explored domain of knowledge. We present the following results on harmonic convolutions related to the present investigation.

THEOREM 2.12. *Let* $F_1 \in K$ *and* $F_2 \in S_2$. *Then, for* $\lambda_1, \lambda_2 > 0$, $-1 \le a < 1$, $-1 \le b \le 1$, $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,L_b}[F_2]$ *is locally univalent and sense preserving in* $\mathbb{D}$.

Proof. We write $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,L_b}[F_2](z) = M(z) + \overline{N(z)}$, where
$$M(z)=\frac{(F_1*F_2)(z)+\lambda_1(H_a*F_1*F_2)(z)+\lambda_2(L_b*F_1*F_2)(z)+\lambda_1\lambda_2(H_a*L_b*F_1*F_2)(z)}{(1+\lambda_1)(1+\lambda_2)}$$
$$N(z)=\frac{(F_1*F_2)(z)-\lambda_1(H_a*F_1*F_2)(z)-\lambda_2(L_b*F_1*F_2)(z)+\lambda_1\lambda_2(H_a*L_b*F_1*F_2)(z)}{(1+\lambda_1)(1+\lambda_2)}.$$
First, we note that M and N are analytic in $\mathbb{D}$. Moreover $M'(0) = 1$ implies that M' is not identically zero on $\mathbb{D}$. Therefore, $W(z) = N'(z)/M'(z)$ is analytic, considered as defined in $\mathbb{D}$ and any singularity is isolated. We will show that $|W(z)| < 1$ for all those $z \in \mathbb{D}$ where W is defined. This in turn will show that W has only removable singularities on $\mathbb{D}$. Then, we may define W on all of $\mathbb{D}$ and by using Maximum Principle, conclude that $|W(z)| < 1$ for $z \in \mathbb{D}$ which proves that $M + \overline{N}$ is locally univalent and sense preserving in $\mathbb{D}$.

For all $z \in \mathbb{D}$ where $W(z) = N'(z)/M'(z)$ is defined, we have $|W(z)| < 1$ if and only if $\Re\left[\frac{M'(z)-N'(z)}{M'(z)+N'(z)}\right] > 0$. Now,

$$\begin{aligned}\Re\left[\frac{M'(z)-N'(z)}{M'(z)+N'(z)}\right] =& \Re\left[\frac{\lambda_1(H_a * F_1 * F_2)'(z) + \lambda_2(L_b * F_1 * F_2)'(z)}{(F_1 * F_2)'(z) + \lambda_1\lambda_2(H_a * L_b * F_1 * F_2)'(z)}\right]\\ =&\lambda_1\Re\left[\frac{(H_a * F_1 * F_2)'(z)}{(F_1 * F_2)'(z) + \lambda_1\lambda_2(H_a * L_b * F_1 * F_2)'(z)}\right]\\ &+ \lambda_2\Re\left[\frac{(L_b * F_1 * F_2)'(z)}{(F_1 * F_2)'(z) + \lambda_1\lambda_2(H_a * L_b * F_1 * F_2)'(z)}\right]\end{aligned} \tag{9}$$

Note that

$$\Re\left[\frac{(H_a * F_1 * F_2)'(z)}{(F_1 * F_2)'(z)}\right] = \Re\left[\frac{(F_1 * F_2)(z) * \frac{z}{(1-az)(1-z)}}{(F_1 * F_2)(z) * \frac{z}{(1-z)^2}}\right]$$

$$=\Re\left[\frac{(F_1 * F_2)(z) * \left\{\left(\frac{1-z}{1-az}\right)\left(\frac{z}{(1-z)^2}\right)\right\}}{(F_1 * F_2)(z) * \frac{z}{(1-z)^2}}\right].$$

Now $F_1 \in K$, $F_2 \in S_2 \subset K$ implies $F_1 * F_2 \in K$ and $\Re[(1-z)/(1-az)] > 0$ for all a, $-1 \leq a < 1$. So in view of Lemma 2.1, we get

$$\Re\left[\frac{(H_a * F_1 * F_2)'(z)}{(F_1 * F_2)'(z)}\right] > 0. \tag{10}$$

Furthermore,

$$\Re\left[\frac{(L_b * F_1 * F_2)'(z)}{(F_1 * F_2)'(z)}\right] = \Re\left[\frac{z(F_1 * F_2)'(z) * \frac{z}{(1-bz)(1-z)}}{z(F_1 * F_2)'(z) * \frac{z}{1-z}}\right] > 0, \tag{11}$$

because of Lemma 2.1, as $z(F_1 * F_2)' = F_1 * zF_2' \in K$ $(F_2 \in S_2)$ and $\Re[1/(1-bz)] > 0$ for all b, $-1 \leq b \leq 1$. On the similar lines we can easily show that

$$\Re\left[\frac{(H_a * F_1 * F_2)'(z)}{(H_a * L_b * F_1 * F_2)'(z)}\right] > 0 \tag{12}$$

and

$$\Re\left[\frac{(L_b * F_1 * F_2)'(z)}{(H_a * L_b * F_1 * F_2)'(z)}\right] > 0. \tag{13}$$

Now, since $\lambda_1, \lambda_2 > 0$, from (9), (10), (11), (12) and (13), we get

$$\Re\left[\frac{M'(z)-N'(z)}{M'(z)+N'(z)}\right] > 0. \qquad \square$$

Theorem 2.13. *Let $F_1, F_2 \in K$ and λ_1, λ_2 be positive real numbers. Then, for $-1 \leq a < 1$, $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,H_a}[F_2] \in S_H$ and is convex in the direction of the real axis.*

Proof. Let $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,H_a}[F_2](z) = X(z) + \overline{Y(z)}$. Using the similar argument as in the proof of Theorem 2.12, it is sufficient to prove that for all those $z \in \mathbb{D}$ where $Y'(z)/X'(z)$ is defined, we have $\Re\left[\frac{X'(z)+Y'(z)}{X'(z)-Y'(z)}\right] > 0$. Now,

$$\Re\left[\frac{X'(z)+Y'(z)}{X'(z)-Y'(z)}\right] = \Re\left[\frac{(F_1 * F_2)'(z) + \lambda_1\lambda_2(H_a * H_a * F_1 * F_2)'(z)}{(\lambda_1+\lambda_2)(H_a * F_1 * F_2)'(z)}\right]$$

$$= \frac{1}{\lambda_1+\lambda_2}\Re\left[\frac{(F_1*F_2)'(z)}{(H_a*F_1*F_2)'(z)}\right] + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\Re\left[\frac{(H_a*H_a*F_1*F_2)'(z)}{(H_a*F_1*F_2)'(z)}\right] \tag{14}$$

Note that

$$\Re\left[\frac{(H_a*F_1*F_2)'(z)}{(F_1*F_2)'(z)}\right] = \Re\left[\frac{(F_1*F_2)(z)*\frac{z}{(1-az)(1-z)}}{(F_1*F_2)(z)*\frac{z}{(1-z)^2}}\right]$$

$$= \Re\left[\frac{(F_1*F_2)(z)*\left\{\left(\frac{1-z}{1-az}\right)\left(\frac{z}{(1-z)^2}\right)\right\}}{(F_1*F_2)(z)*\left\{\frac{z}{(1-z)^2}\right\}}\right].$$

Now $F_1, F_2 \in K$ imply that $F_1 * F_2 \in K$ and $\Re[(1-z)/(1-az)] > 0$ for all a, $-1 \le a < 1$. So, in view of Lemma 2.1, we get

$$\Re\left[\frac{(H_a*F_1*F_2)'(z)}{(F_1*F_2)'(z)}\right] > 0$$

or equivalently,

$$\Re\left[\frac{(F_1*F_2)'(z)}{(H_a*F_1*F_2)'(z)}\right] > 0. \tag{15}$$

Following similar steps, we easily get

$$\Re\left[\frac{(H_a*H_a*F_1*F_2)'(z)}{(H_a*F_1*F_2)'(z)}\right] > 0. \tag{16}$$

Since $\lambda_1, \lambda_2 > 0$, from (14), (15) and (16), we get

$$\Re\left[\frac{X'(z)+Y'(z)}{X'(z)-Y'(z)}\right] > 0.$$

This implies that $|Y'(z)/X'(z)| < 1$ and hence $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,H_a}[F_2]$ is locally univalent and sense preserving in $\mathbb{D}$. Moreover, $X(z)-Y(z) = \frac{2(\lambda_1+\lambda_2)}{(1+\lambda_1)(1+\lambda_2)}(H_a*F_1*F_2)(z)$ and $H_a*F_1*F_2 \in K$ implies that $C_{\lambda_1,H_a}[F_1]\tilde{*}C_{\lambda_2,H_a}[F_2] \in S_H$ and is convex in the direction of the real axis (in view of Lemma 2.2). This completes our proof. □

THEOREM 2.14. *Let $F_1, F_2 \in S_2$, and $\lambda_1, \lambda_2 > 0$ be real numbers. Then, for $-1 \le b \le 1$, $C_{\lambda_1,L_b}[F_1]\tilde{*}C_{\lambda_2,L_b}[F_2] \in S_H$ and is convex in the direction of the real axis.*

Proof. If we write $C_{\lambda_1,L_b}[F_1]\tilde{*}C_{\lambda_2,L_b}[F_2](z) = R(z) + \overline{S(z)}$, then it is easy to verify that

$$R(z) = \frac{(F_1*F_2)(z) + (\lambda_1+\lambda_2)(L_b*F_1*F_2)(z) + \lambda_1\lambda_2(L_b*L_b*F_1*F_2)(z)}{(1+\lambda_1)(1+\lambda_2)}$$

$$S(z) = \frac{(F_1*F_2)(z) - (\lambda_1+\lambda_2)(L_b*F_1*F_2)(z) + \lambda_1\lambda_2(L_b*L_b*F_1*F_2)(z)}{(1+\lambda_1)(1+\lambda_2)}.$$

By the similar argument as in the proof of Theorem 2.7, $C_{\lambda_1,L_b}[F_1]\tilde{*}C_{\lambda_2,L_b}[F_2]$ is locally univalent and sense preserving in $\mathbb{D}$ if for all those $z \in \mathbb{D}$, where $S'(z)/R'(z)$ is defined, we have $|S'(z)/R'(z)| < 1$. But this is equivalent to showing that for all such $z \in \mathbb{D}$,

$$\Re\left[\frac{R'(z)+S'(z)}{R'(z)-S'(z)}\right] > 0. \tag{17}$$

Now, following the same steps as in the proof of Theorem 2.13 and noting that $F_1, F_2 \in S_2$ imply that $L_b * F_1 * F_2 \in S_2 \subset K$ and $L_b * L_b * F_1 * F_2 \in K$, we can prove that (17) is true. The remaining part of the proof follows using Lemma 2.2, because $R(z) - S(z) = \frac{2(\lambda_1+\lambda_2)}{(1+\lambda_1)(1+\lambda_2)}(L_b * F_1 * F_2)(z)$ and $L_b * F_1 * F_2 \in K$. □

From Theorem 2.3 we note that $C_{\lambda,H_0}[F_1] \in S_H$ is convex in the direction of the imaginary axis (also in the direction of the real axis) if $F_1 \in K$ and Theorem 2.8 provides us that $C_{\lambda,L_1}[F_2] \in S_H$ and is convex in the direction of the imaginary axis (also in the direction of the real axis) if $F_2 \in S_2$. In the following theorem we show that for particular choices of a and b, we can obtain even better result (in comparison to Theorem 2.12) as we can allow F_2 to be in a larger class K (instead of S_2) to get the following theorem.

Theorem 2.15. *Let $F_1, F_2 \in K$, and λ_1, λ_2 be positive real numbers. Then $C_{\lambda_1,H_0}[F_1]\tilde{*}C_{\lambda_2,L_1}[F_2] \in S_H$ and is convex in the direction of the imaginary axis. Here $H_0 = -\log(1-z)$ and $L_1 = z/(1-z)^2$.*

Proof. If we write $C_{\lambda_1,H_0}[F_1]\tilde{*}C_{\lambda_2,L_1}[F_2](z) = P(z) + \overline{Q(z)}$, then

$$P(z) = \frac{(1+\lambda_1\lambda_2)(F_1 * F_2)(z) + \lambda_1(F_1 * F_2 * H_0)(z) + \lambda_2(F_1 * F_2 * L_1)(z)}{(1+\lambda_1)(1+\lambda_2)}$$

and $$Q(z) = \frac{(1+\lambda_1\lambda_2)(F_1 * F_2)(z) - \lambda_1(F_1 * F_2 * H_0)(z) - \lambda_2(F_1 * F_2 * L_1)(z)}{(1+\lambda_1)(1+\lambda_2)}.$$

Applying the similar argument as in the proof of Theorem 2.12, $C_{\lambda_1,H_0}[F_1]\tilde{*}C_{\lambda_2,L_1}[F_2]$ is locally univalent and sense preserving in $\mathbb{D}$ if for all those $z \in \mathbb{D}$ where $Q'(z)/P'(z)$ is defined, we have $|Q'(z)/P'(z)|<1$, or equivalently, if for all such $z \in \mathbb{D}$, $\Re\left[\frac{P'(z)-Q'(z)}{P'(z)+Q'(z)}\right]>0$. Now, we observe that

$$\begin{aligned}\Re\left[\frac{P'(z)-Q'(z)}{P'(z)+Q'(z)}\right] &= \Re\left[\frac{\lambda_1(F_1 * F_2 * H_0)'(z) + \lambda_2(F_1 * F_2 * L_1)'(z)}{(1+\lambda_1\lambda_2)(F_1 * F_2)'(z)}\right]\\ &= \frac{\lambda_1}{1+\lambda_1\lambda_2}\Re\left[\frac{(F_1 * F_2 * H_0)'(z)}{(F_1 * F_2)'(z)}\right] + \frac{\lambda_2}{1+\lambda_1\lambda_2}\Re\left[\frac{(F_1 * F_2 * L_1)'(z)}{(F_1 * F_2)'(z)}\right].\end{aligned} \tag{18}$$

Note that

$$\Re\left[\frac{(F_1 * F_2 * L_1)'(z)}{(F_1 * F_2)'(z)}\right] = \Re\left[\frac{(F_1 * F_2)(z) * \left\{\left(\frac{1+z}{1-z}\right)\left(\frac{z}{(1-z)^2}\right)\right\}}{(F_1 * F_2)(z) * \frac{z}{(1-z)^2}}\right] > 0, \tag{19}$$

in view of Lemma 2.1, as $F_1 * F_2 \in K$ and $\Re[(1+z)/(1-z)] > 0$ for $z \in \mathbb{D}$. In the same way, we have

$$\Re\left[\frac{(F_1 * F_2 * H_0)'(z)}{(F_1 * F_2)'(z)}\right] > 0. \tag{20}$$

Therefore, from (18), (19) and (20), we get $\Re\left[\frac{P'(z)-Q'(z)}{P'(z)+Q'(z)}\right] > 0$ as $\lambda_1, \lambda_2 > 0$. Now, to complete the proof, we note that $P(z)+Q(z) = \frac{2(1+\lambda_1\lambda_2)}{(1+\lambda_1)(1+\lambda_2)}(F_1 * F_2)(z)$. But $F_1 * F_2$ is convex in the direction of the imaginary axis as $F_1 * F_2 \in K$; so, in view of Lemma 2.2, $C_{\lambda_1,H_0}[F_1]\tilde{*}C_{\lambda_2,L_1}[F_2] \in S_H$ and is convex in the direction of the imaginary axis. □

We conclude this paper by stating the following result which, although a little bit out of context here, is interesting in its own right. Its proof runs on the same lines as that of Theorem 2.15 and hence is omitted.

THEOREM 2.16. *Let $F_1, F_2 \in K$ and $\lambda_1, \lambda_2 > 0$ be positive real numbers. Then $C_{\lambda_1,h_2}[F_1]\tilde{*}C_{\lambda_2,h_4}[F_2] \in S_H$ and is convex in the direction of the imaginary axis. Here $h_2(z) = (1/2)[z/(1-z) + z/(1-z)^2]$ and $h_4(z) = (2/z)\int_0^z \zeta/(1-\zeta)\,d\zeta$.*

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REFERENCES

[1] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I Math., **9** (1984), 3–25.

[2] S.Y. Lee, *Application of convolution operators to some problems in geometric function theory*, J. Korean Math. Soc., **24(2)** (1987), 217–237.

[3] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Am. Math. Soc., **42(10)** (1936), 689–692.

[4] S. Muir, *Harmonic mappings convex in one or every direction*, Comput. Methods Funct. Theory, **12(1)** (2012), 221–239.

[5] St. Ruscheweyh, *Convolutions in geometric function theory, Presses Univ*, Montréal, Montreal, Que, 1982.

[6] S. Ruscheweyh, T. Sheil-Small, *Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv., **48(1)** (1973), 119–135.

[7] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, **11(1)** (1959), 72–75.

[8] S. Singh, R, Singh, *Convex and starlike functions*, Colloq. Math., **2(50)** (1986), 271–279.

[9] C. Singla, S. Gupta, S. Singh, *Some new harmonic mappings convex in one direction and their convolution*, Filomat, **33(19)** (2019), 6131–6139.

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