MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
76, 3 (2024), 159–173
September 2024

research paper
оригинални научни рад
DOI: 10.57016/MV-DMVS3382

ON (α, β, γ)-METRICS

Nasrin Sadeghzadeh and Tahere Rajabi

Abstract. In this paper, we introduce a new class of Finsler metrics that generalize the well-known (α,β)-metrics. These metrics are defined by a Riemannian metric α and two 1-forms $\beta = b_i(x)y^i$ and $\gamma = \gamma_i(x)y^i$. This new class of metrics not only generalizes (α,β)-metrics, but also includes other important Finsler metrics, such as all (generalized) γ-changes of generalized (α,β)-metrics, (α,β)-metrics, and spherically symmetric Finsler metrics in $\mathbb{R}^n$. We find a necessary and sufficient condition for this new class of metrics to be locally projectively flat. Furthermore, we prove the conditions under which these metrics are of Douglas type.

1. Introduction

(α,β)-metrics form a special class of Finsler metrics, in part because they are computationally tractable. An (α,β)-metric on a smooth manifold M is defined by $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\phi = \phi(s)$ is a C^∞ scalar function on $(-b_0, b_0)$ satisfying certain regularity conditions, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

In [7] we have studied a new generalization of the (α,β)-metrics which is defined by a Finsler metric F and a 1-form $\gamma = \gamma_iy^i$ on an n-dimensional manifold M. Then the metric is given by $\bar{F} = F\psi(\tilde{s})$, where $\tilde{s} := \frac{\gamma}{F}$, $\|\gamma\|_F < g_0$ and $\psi(\tilde{s})$ is a positive C^∞ function on $(-g_0, g_0)$. These metrics could be seen as β-change of a Finsler metric.

Suppose $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ is a (α,β)- metric. For every 1-form $\gamma \neq \beta$, $\bar{F} = \alpha\phi(s)\psi(\tilde{s})$ is not necessarily an (α,β)-metric. If $F = \alpha + \beta$ is a Randers metric and $\bar{F} = F + \gamma$ is a Randers change of F, then $\bar{F} = \alpha + \beta + \gamma$ is a Randers metric. With this idea, we have defined a new generalization of the (α,β)-metrics in the form of $\bar{F} = \alpha\Psi(s,\bar{s})$, where $\Psi(s,\bar{s}) = \phi(s)\psi(\frac{\bar{s}}{\phi(s)})$, $\bar{s} = \frac{\gamma}{\alpha}$.

In this paper we intend to generalize the above metric. We consider a new generalization of the (α,β)-metrics which is defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and two 1-forms $\beta = b_iy^i$ and $\gamma = \gamma_iy^i$ on an n-dimensional manifold M. Then the

2020 Mathematics Subject Classification: 53B40, 53C60
Keywords and phrases: Finsler geometry; (α,β,γ)-metrics; projectively flat; Douglas space.

metric is given by $F=\alpha\Psi(s,\bar{s})$, where $s=\frac{\beta}{\alpha}$, $\bar{s}=\frac{\gamma}{\alpha}$, $\|\beta\|_\alpha<g_0$ and $\Psi(s,\bar{s})$ is a positive C^∞ function on $(-b_0,b_0)\times(-g_0,g_0)$ is a Finsler metric, which we call (α,β,γ)-metric.

This class of Finsler metrics generalizes (α,β)-metrics in a natural way. But the main reason for our interest in them is that they include some Finsler metrics such as all (generalized) γ-change of generalized (α,β)-metrics, (α,β)-metrics and spherical symmetric Finsler metrics in R^n [10, 12]. As an example, let us consider the transformed 2nd root metric F: $\bar{F}=\sqrt{F^2+\beta}+\gamma$, where $\beta=b_{ij}(x)y^iy^i$ and $\gamma=c_i(x)y^i$ is a one-form on the manifold M^n.

There are some generalizations of the (α,β)-metrics introduced in the various papers. A generalization of the (α,β)-metric was presented in [5,8,9], which coincides with the (α,β,γ)-metric in the case $p=2$. Another generalization of the (α,β)-metrics are the general (α,β)-metrics, which were first introduced by C. Yu and H. Zhu in [11]. By definition, a general (α,β)-metric F can be expressed in the following form $F=\alpha\phi(b^2,s)$, where $b:=\|\beta\|_\alpha$. In the future, we can similarly define the general (α,β,γ)-metric given by $F=\alpha\phi(b^2,g^2,s,\bar{s})$, where $b:=\|\beta\|_\alpha$ and $g:=\|\gamma\|_\alpha$.

2. Preliminaries

Let M be a smooth manifold and $TM:=\bigcup_{x\in M}T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x\in M$. A Finsler metric on M is a function $F:TM\longrightarrow[0,+\infty)$ with the following properties

- F is C^∞ on $TM\backslash\{0\}$;
- F is positively 1-homogeneous on the fibers of tangent bundle TM;
- for each $x\in M$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\big[F^2(y+su+tv)\big]|_{t,s=0},\quad u,v\in T_xM.$$

Let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y:T_xM\otimes T_xM\otimes T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\big[\mathbf{g}_{y+tw}(u,v)\big]|_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=0$ if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i}.$$

where $G^i(x,y)$ are local functions on TM_0 given by

$$G^i=\frac{1}{4}g^{il}\Big\{\frac{\partial g_{jl}}{\partial x^k}+\frac{\partial g_{lk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^l}\Big\}y^jy^k. \tag{1}$$

G is called the associated spray to (M, F). The projection of an integral curve of the spray **G** is called a geodesic in M.

A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ is said to be projectively flat if all geodesics are straight in $\mathcal{U}$. It is well-known that a Finsler metric F on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ is projectively flat if and only if it satisfies the following system of equations, $F_{x^k y^j} y^k - F_{x^j} = 0$. This fact is due to G. Hamel [4]. In this case, $G^i = P y^i$, where $P = P(x, y)$ is given by $P = \frac{F_{x^k} y^k}{2F}$. The scalar function P is called the projective factor of F.

3. (α, β, γ)-metrics

Definition 3.1. For a Riemannian metric α and two 1-form $\beta = b_i(x) y^i$ and $\gamma = \gamma_i(x) y^i$ on an n-dimensional manifold M, an (α, β, γ)-metric F can be expressed as the form $F = \alpha \Psi(s, \bar{s}), \quad s := \frac{\beta}{\alpha}, \quad \bar{s} := \frac{\gamma}{\alpha}$, where $\|\beta\|_\alpha < b_0$, $\|\gamma\|_\alpha < g_0$ and $\Psi(s, \bar{s})$ is a positive C^∞ function on $(-b_0, b_0) \times (-g_0, g_0)$.

Proposition 3.2. *For an (α, β, γ)-metric $F = \alpha \Psi(s, \bar{s})$, where $s = \frac{\beta}{\alpha}$ and $\bar{s} = \frac{\gamma}{\alpha}$, the fundamental tensor is given by*

$$\begin{aligned} g_{ij} &= \rho a_{ij} + \rho_0 b_i b_j + \bar{\rho}_0 \gamma_i \gamma_j \\ &\quad + \rho_1 (b_i \alpha_j + b_j \alpha_i) + \bar{\rho}_1 (\gamma_i \alpha_j + \gamma_j \alpha_i) + \rho_2 \alpha_i \alpha_j + \rho_3 (b_i \gamma_j + b_j \gamma_i), \end{aligned} \tag{2}$$

where

$$\rho := \Psi(\Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}}), \quad \rho_0 := \Psi\Psi_{ss} + \Psi_s\Psi_s, \quad \rho_1 := \Psi\Psi_s - s\rho_0 - \bar{s}\rho_3, \tag{3}$$

$$\rho_2 := -s\rho_1 - \bar{s}\bar{\rho}_1, \quad \bar{\rho}_0 := \Psi\Psi_{\bar{s}\bar{s}} + \Psi_{\bar{s}}\Psi_{\bar{s}}, \quad \bar{\rho}_1 := \Psi\Psi_{\bar{s}} - \bar{s}\bar{\rho}_0 - s\rho_3,$$

$$\rho_3 := \Psi\Psi_{s\bar{s}} + \Psi_s\Psi_{\bar{s}}. \tag{4}$$

Moreover,

$$\det(g_{ij}) = \Psi^{n+1} \big(\Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}}\big)^{n-2} \Gamma \; det(a_{ij}), \tag{5}$$

where

$$\begin{aligned} \Gamma &:= \Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}} + (b^2 - s^2)\Psi_{ss} + (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}} + 2(\theta - s\bar{s})\Psi_{s\bar{s}} \\ &\quad + \big[(b^2 - s^2)(g^2 - \bar{s}^2) - (\theta - s\bar{s})^2\big] J, \end{aligned} \tag{6}$$

and

$$b^2 := a^{ij} b_i b_j, \quad g^2 := a^{ij} \gamma_i \gamma_j, \quad \theta := a^{ij} b_i \gamma_j, \quad J := \frac{\Psi_{ss}\Psi_{\bar{s}\bar{s}} - \Psi_{s\bar{s}}\Psi_{s\bar{s}}}{\Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}}}.$$

$$\begin{aligned} g^{ij} &= \frac{1}{\rho}\Big\{ a^{ij} - \frac{1}{\Gamma}\Big[\Psi_{ss} + (g^2 - \bar{s}^2)J\Big] b^i b^j - \frac{1}{\Gamma}\Big[\Psi_{\bar{s}\bar{s}} + (b^2 - s^2)J\Big]\gamma^i\gamma^j \\ &\quad - \frac{1}{\Psi\Gamma}\Big[\rho_1 + \pi_2(\theta - s\bar{s}) - \pi_1(g^2 - \bar{s}^2)\Big](b^i\alpha^j + b^j\alpha^i) \\ &\quad - \frac{1}{\Psi\Gamma}\Big[\bar{\rho}_1 - \pi_2(b^2 - s^2) + \pi_1(\theta - s\bar{s})\Big](\gamma^i\alpha^j + \gamma^j\alpha^i) \end{aligned} \tag{7}$$

$$+\frac{1}{\Psi^2\Gamma}\bigg(\Big[s\Psi+(b^2-s^2)\Psi_s+(\theta-s\bar{s})\Psi_{\bar{s}}\Big]\Big[\rho_1+\pi_2(\theta-s\bar{s})-\pi_1(g^2-\bar{s}^2)\Big]$$
$$+\Big[\bar{s}\Psi+(g^2-\bar{s}^2)\Psi_{\bar{s}}+(\theta-s\bar{s})\Psi_s\Big]\Big[\bar{\rho}_1-\pi_2(b^2-s^2)+\pi_1(\theta-s\bar{s})\Big]\bigg)\alpha^i\alpha^j\bigg\},$$

where

$$\pi_1:=\Psi_{\bar{s}}\Psi_{s\bar{s}}-\Psi_s\Psi_{\bar{s}\bar{s}}+s\Psi J,\quad \pi_2:=\Psi_s\Psi_{s\bar{s}}-\Psi_{\bar{s}}\Psi_{ss}+\bar{s}\Psi J. \tag{8}$$

Moreover, the Cartan tensor of F is given by

$$\begin{aligned}C_{ijk}=&\frac{\rho_1}{2}\Big[h_k\alpha_{ij}+h_i\alpha_{jk}+h_j\alpha_{ik}\Big]+\frac{\bar{\rho}_1}{2}\Big[\bar{h}_k\alpha_{ij}+\bar{h}_i\alpha_{jk}+\bar{h}_j\alpha_{ik}\Big]\\&+\frac{(\rho_0)_{\bar{s}}}{2\alpha}\Big[h_ih_j\bar{h}_k+h_jh_k\bar{h}_i+h_ih_k\bar{h}_j\Big]+\frac{(\bar{\rho}_0)_s}{2\alpha}\Big[\bar{h}_i\bar{h}_jh_k+\bar{h}_j\bar{h}_kh_i+\bar{h}_i\bar{h}_kh_j\Big]\\&+\frac{(\rho_0)_s}{2\alpha}h_ih_jh_k+\frac{(\bar{\rho}_0)_{\bar{s}}}{2\alpha}\bar{h}_i\bar{h}_j\bar{h}_k.\end{aligned} \tag{9}$$

REMARK 3.3. One could easily show that the above preposition satisfies for any (α,β)-metric just by putting $\bar{s}=0$, and satisfies for any (α,γ)-metric just by putting $s=0$.

Proof. Recall that the fundamental tensor and Cartan tensor of a Finsler metric F are given by $g_{ij}=\frac{1}{2}[F^2]_{y^iy^j}=FF_{y^iy^j}+F_{y^i}F_{y^j}$ and $C_{ijk}=\frac{1}{2}(g_{ij})_{y^k}$, respectively. Direct computations yield

$$\begin{aligned}s_{y^i}&=\frac{1}{\alpha}h_i,\quad\text{where}\quad h_i:=b_i-s\alpha_i,\quad \alpha_i=\alpha_{y^i},\\ \bar{s}_{y^i}&=\frac{1}{\alpha}\bar{h}_i,\quad\text{where}\quad \bar{h}_i:=\gamma_i-\bar{s}\alpha_i,\\ \Psi_{y^i}&=\frac{1}{\alpha}\big[\Psi_sh_i+\Psi_{\bar{s}}\bar{h}_i\big],\\ (\Psi_s)_{y^i}&=\frac{1}{\alpha}\big[\Psi_{ss}h_i+\Psi_{s\bar{s}}\bar{h}_i\big],\\ (\Psi_{\bar{s}})_{y^i}&=\frac{1}{\alpha}\big[\Psi_{\bar{s}s}h_i+\Psi_{\bar{s}\bar{s}}\bar{h}_i\big],\\ (h_i)_{y^j}&=-\frac{1}{\alpha}h_j\alpha_i-s\alpha_{ij},\quad\text{where}\quad \alpha_{ij}=\alpha_{y^iy^j}=\frac{1}{\alpha}(a_{ij}-\alpha_i\alpha_j).\\ (\bar{h}_i)_{y^j}&=-\frac{1}{\alpha}\bar{h}_j\alpha_i-\bar{s}\alpha_{ij}.\end{aligned}$$

Let $\ell_i=F_{y^i}$ and $\ell_{ij}=F_{y^iy^j}$. By above equations we have

$$\ell_i=\Psi\alpha_i+\Psi_sh_i+\Psi_{\bar{s}}\bar{h}_i, \tag{10}$$

$$\ell_{ij}=\big[\Psi-s\Psi_s-\bar{s}\Psi_{\bar{s}}\big]\alpha_{ij}+\frac{1}{\alpha}\Psi_{ss}h_ih_j+\frac{1}{\alpha}\Psi_{\bar{s}\bar{s}}\bar{h}_i\bar{h}_j+\frac{1}{\alpha}\Psi_{s\bar{s}}\big[h_i\bar{h}_j+h_j\bar{h}_i\big]. \tag{11}$$

Then we get (2). We can rewrite (2) as follows

$$\bar{g}_{ij}=\rho\bigg\{a_{ij}+\delta_1b_ib_j+\delta_2\gamma_i\gamma_j+\delta_0(b_i+\gamma_i)(b_j+\gamma_j)+\frac{\rho_2}{\rho}[\alpha_i+\frac{\rho_1}{\rho_2}b_i+\frac{\bar{\rho}_1}{\rho_2}\gamma_i][\alpha_j+\frac{\rho_1}{\rho_2}b_j+\frac{\bar{\rho}_1}{\rho_2}\gamma_j]\bigg\},$$

where $\delta_0:=\frac{1}{\rho}(\rho_3-\frac{\rho_1\bar{\rho}_1}{\rho_2})$, $\delta_1:=\frac{1}{\rho}(\rho_0-\frac{\rho_1^2}{\rho_2})-\delta_0$, $\delta_2:=\frac{1}{\rho}(\bar{\rho}_0-\frac{\bar{\rho}_1^2}{\rho_2})-\delta_0$.

Using [2, Lemma 1.1.1] four times, we obtain (5) and (7). □

Remark 3.4. Notice that by Cauchy-Schwartz inequality we have $\theta^2 = (a^{ij}b_i\gamma_j)^2 \leq (a^{ij}b_ib_j)(a^{ij}\gamma_i\gamma_j) = b^2g^2$.

We need to prove the following proposition.

Proposition 3.5. *Let M be an n-dimensional manifold. An (α, β, γ)-metric $F = \alpha\Psi(s, \bar{s})$, $s = \frac{\beta}{\alpha}$, $\bar{s} = \frac{\gamma}{\alpha}$ is a Finsler metric for any Riemannian α and 1-forms $\beta = b_iy^i, \gamma = \gamma_iy^i$ where $\|\beta\|_\alpha < b_0$, $\|\gamma\|_\alpha < g_0$, $\theta - s\bar{s} \geq 0$ if and only if the positive C^∞ function $\Psi = \Psi(s, \bar{s})$ satisfying*

$$\Pi := \Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}} > 0, \qquad \Gamma > 0, \tag{12}$$

when $n \geq 3$ or $\Gamma > 0$, when $n = 2$, where Γ is given by (6) and s, $\bar{s}$, b, g are arbitrary numbers with $|s| \leq b < b_0$ and $|\bar{s}| \leq g < g_0$.

Proof. The case $n = 2$ is similar to $n \geq 3$, so we only prove the proposition for $n \geq 3$. It is easy to verify that F is a function with regularity and positive homogeneity. In the following we will consider the strong convexity condition.

Assume that (12) is satisfied, then we could write $\Pi\Gamma$ as a second order equation in Π as follows

$$\Pi\Gamma = \Pi^2 + (a + \bar{a})\Pi + (a\bar{a} - b\bar{b}) > 0, \tag{13}$$

where

$$a := (b^2 - s^2)\Psi_{ss} + (\theta - s\bar{s})\Psi_{s\bar{s}}, \qquad b := (b^2 - s^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{\bar{s}\bar{s}},$$
$$\bar{a} := (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}} + (\theta - s\bar{s})\Psi_{s\bar{s}}, \qquad \bar{b} := (g^2 - \bar{s}^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{ss}.$$

The above inequality holds if and only if one of the following holds:
(i) $\Delta < 0$ where $\Delta = (a + \bar{a})^2 - 4(a\bar{a} - b\bar{b})$;

(ii) $\Delta = 0$, then $\Pi \neq \omega$ and $\Pi\Gamma = (\Pi - \omega)^2$ where $\omega = -\frac{1}{2}(a + \bar{a})$;

(iii) $\Delta > 0$, then $0 < \Pi < \omega_1$ or $\Pi > \omega_2$ where $\omega_1 := -\frac{1}{2}\big[(a + \bar{a}) + \sqrt{\Delta}\big]$ and $\omega_2 := -\frac{1}{2}\big[(a + \bar{a}) - \sqrt{\Delta}\big]$. Note that $\omega_1 < \omega_2$.

Consider a family of functions $\Psi_t(s, \bar{s}) = 1 - t + t\Psi(s, \bar{s})$, $0 \leq t \leq 1$. Put $F_t = \alpha\Psi_t(s, \bar{s})$ and $g^t_{ij} = \frac{1}{2}[F^2_t]_{y^iy^j}$, then $F_0 = \alpha$ and $F_1 = F$. We are going to prove $\Pi_t > 0$ and $\Gamma_t > 0$ for any $0 \leq t \leq 1$, $|s| \leq b < b_0$ and $|\bar{s}| \leq g < g_0$. It is easy to see that $\Pi_t = 1 - t + t\Pi > 0$. Moreover $\Pi_t\Gamma_t = \Pi^2_t + t(a + \bar{a})\Pi_t + t^2(a\bar{a} - b\bar{b})$. Then we have $\Delta_t = t^2\Delta$ where

$$\Delta = (a + \bar{a})^2 - 4(a\bar{a} - b\bar{b}). \tag{14}$$

It is easy to see that for $\Delta_t(s, \bar{s}) < 0$, the equation $\Pi_t\Gamma_t$ is always positive, i.e. $\Gamma_t > 0$.

Now suppose that there are t_0 and $(s_0, \bar{s}_0)$ such that $\Delta_{t_0}(s_0, \bar{s}_0) > 0$. Since $\Delta_t(s, \bar{s})$ is continuous with respect to t and $(s, \bar{s})$, then there is $D \subset (-b_0, b_0) \times (-g_0, g_0)$ such that $\forall(s, \bar{s}) \in D \quad \Delta_t(s, \bar{s}) > 0$ and $\forall(s, \bar{s}) \in \partial D \quad \Delta_t(s, \bar{s}) = 0$, where ∂D is border of D. Then on D we have

$$\Pi_t\Gamma_t = (\Pi_t - t\omega_1)(\Pi_t - t\omega_2). \tag{15}$$

If on D we have $\Gamma_t(s, \bar{s}) > 0$, then there is not anything to prove. Now suppose that there exits $\mathcal{U} \subset D$ such that for $(s, \bar{s}) \in \overline{\mathcal{U}} = \mathcal{U} \bigcup \partial\mathcal{U}$ we have $\Gamma_t(s, \bar{s}) \leq 0$. Since

Γ_0, Γ_1 are both positive, then by continuity Γ_t we get $\exists t_1, t_2 \in (0,1)$ s.t. $\Gamma_{t_1}(s,\bar{s}) = \Gamma_{t_2}(s,\bar{s}) = 0$; $\forall (s,\bar{s}) \in \overline{\mathcal{U}}$. By (15) we have

$$(\Pi_{t_1} - t_1\omega_1)(\Pi_{t_1} - t_1\omega_2) = 0, \quad and \quad (\Pi_{t_2} - t_2\omega_1)(\Pi_{t_2} - t_2\omega_2) = 0. \tag{16}$$

Then for $t_1 \leq t \leq t_2$ we get $\forall (s,\bar{s}) \in \overline{\mathcal{U}} \quad \Gamma_t(s,\bar{s}) \leq 0$, and $\forall (s,\bar{s}) \in D - \overline{\mathcal{U}} \quad \Gamma_t(s,\bar{s}) > 0$. By continuity Γ_t we have $\Gamma_t(s,\bar{s}) = 0, \quad t_1 \leq t \leq t_2, \quad (s,\bar{s}) \in \partial\mathcal{U}$. Then (15) yields $\Pi_t = t\omega_1$ or $\Pi_t = t\omega_2$. In this case by (16) we get $t_1 = t_2$ which is a contradiction. So $\Gamma_t(s,\bar{s}) > 0$ on D.

Now let there is $D_1 \subset (-b_0, b_0) \times (-g_0, g_0)$ such that $\Delta(s,\bar{s}) = 0$ for every $(s,\bar{s}) \in D_1$. Then we see that for every $0 \leqslant t \leqslant 1$ and $(s,\bar{s}) \in D_1$ we have $\Delta_t(s,\bar{s}) = 0$. One could easily get $\Pi_t\Gamma_t - t^2\Pi\Gamma = (1-t)\big(1 - t + 2t(\Pi + \frac{a+\bar{a}}{2})\big)$. If for some $0 < t < 1$ we have $1 - t + 2t(\Pi + \frac{a+\bar{a}}{2}) \geqslant 0$ then $\Pi_t\Gamma_t \geqslant t^2\Pi\Gamma > 0$ and therefore $\Gamma_t > 0$. Now we assume that there are $0 < t < 1$ such that

$$1 - t + 2t(\Pi + \frac{a+\bar{a}}{2}) < 0. \tag{17}$$

which one could easily get $1 - t + t(\Pi - \frac{a+\bar{a}}{2}) < \frac{1}{2}(1-t) \neq 0$. Thus

$$\Pi_t\Gamma_t = (\Pi_t - \omega_t)^2 = \big(1 - t + t(\Pi - \omega)\big)^2 = \big(1 - t + t(\Pi - \frac{a+\bar{a}}{2})\big)^2 > 0. \tag{18}$$

Then for this $0 < t < 1$ we get $\Gamma_t > 0$, too.

All above arguments yield $\Gamma_t > 0$ for any $0 \leq t \leq 1$. Then $\det(g^t_{ij}) > 0$ for all $0 \leq t \leq 1$. Since (g^0_{ij}) is positive definite, we conclude that (g^t_{ij}) is positive definite for any $t \in [0,1]$. Therefore, F_t is a Finsler metric for any $t \in [0,1]$.

Conversely, assume that $F = \alpha\Psi(s,\bar{s})$ is a Finsler metric for any Riemannian metric α and 1-forms β and γ with $b < b_0$ and $g < g_0$. Then $\Psi = \Psi(s,\bar{s})$ and $\det(g_{ij})$ are positive. By Proposition 3.2, $\det(g_{ij}) > 0$ is equivalent to $\Pi^{n-2}\Gamma > 0$, which implies $\Pi \neq 0$ when $n \geq 3$. Noting that $\Psi(0,0) > 0$, one could get the inequality $\Pi > 0$. $\Gamma > 0$ also holds because of $\det(g_{ij}) > 0$. □

Example 3.6. In [7], a new class of Finsler metrics called (F,γ)-metrics was introduced. A Finsler metric $\bar{F}$ is called (F,γ)-metric if it has the following form $\bar{F} = F\psi(\tilde{s}), \quad \tilde{s} = \frac{\gamma}{F}$, where F is a Finsler metric and $\gamma = \gamma_i y^i$ is a 1-form on an n-dimensional manifold M, $\psi(\tilde{s})$ is a positive C^∞ function on $(-g_0, g_0)$ and $\|\gamma\|_F < g_0$. It has been shown that $\bar{F}$ is a Finsler metric if and only if the positive C^∞ function $\psi(\tilde{s})$ satisfying

$$\psi - \tilde{s}\psi' > 0, \qquad \psi - \tilde{s}\psi' + (p^2 - \tilde{s}^2)\psi'' > 0, \tag{19}$$

when $n \geq 3$ or $\psi - \tilde{s}\psi' + (p^2 - \tilde{s}^2)\psi'' > 0$, when $n = 2$, where $p^2 := g^{ij}\gamma_i\gamma_j$. Now suppose that F is an (α,β)-metric, i.e. $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$. Then

$$\bar{F} = \alpha\phi(s)\psi(\tilde{s}). \tag{20}$$

Let $\bar{s} = \frac{\gamma}{\alpha}$ and $\Psi := \phi(s)\psi(\frac{\bar{s}}{\phi(s)})$. Then (20) is an (α,β,γ)-metric. A direct computation gives $\Pi = (\phi - s\phi')(\psi - \tilde{s}\psi')$, $\Gamma = \big[\phi - s\phi' + (b^2 - s^2)\phi''\big]\big[\psi - \tilde{s}\psi' + (p^2 - \tilde{s}^2)\psi''\big]$. By these relations we can conclude that if F be an (α,β)-metric, then $\bar{F}$ is Finsler metric iff $\Pi > 0$ and $\Gamma > 0$.

For 1-form $\beta = b_i(x)y^i$ and $\gamma = \gamma_i(x)y^i$, we have

$$ {}^{\beta}r_{ij} := \frac{1}{2}\big(b_{i|j} + b_{j|i}\big), \qquad {}^{\beta}s_{ij} := \frac{1}{2}\big(b_{i|j} - b_{j|i}\big). \tag{21} $$

$$ {}^{\gamma}r_{ij} := \frac{1}{2}\big(\gamma_{i|j} + \gamma_{j|i}\big), \qquad {}^{\gamma}s_{ij} := \frac{1}{2}\big(\gamma_{i|j} - \gamma_{j|i}\big). \tag{22} $$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. Moreover, we define

$$ {}^{\beta}r_{i0} := {}^{\beta}r_{ij}y^j, \quad {}^{\beta}r_j := b^i\, {}^{\beta}r_{ij}, \quad {}^{\beta}r_0 := {}^{\beta}r_j y^j, \quad {}^{\beta}r_{00} = {}^{\beta}r_{ij}y^iy^j, $$

$$ {}^{\beta}s_{i0} := {}^{\beta}s_{ij}y^j, \quad {}^{\beta}s_j := b^i\, {}^{\beta}s_{ij}, \quad {}^{\beta}s_0 := {}^{\beta}s_j y^j, \quad {}^{\beta}s^i_0 = a^{ij}\, {}^{\beta}s_{j0}, \quad {}^{\beta}\bar{s}_0 := {}^{\beta}s^i_0\gamma_i, $$

and

$$ {}^{\gamma}r_{i0} := {}^{\gamma}r_{ij}y^j, \quad {}^{\gamma}r_j := b^i\, {}^{\gamma}r_{ij}, \quad {}^{\gamma}r_0 := {}^{\gamma}r_j y^j, \quad {}^{\gamma}r_{00} = {}^{\gamma}r_{ij}y^iy^j, $$

$$ {}^{\gamma}s_{i0} := {}^{\gamma}s_{ij}y^j, \quad {}^{\gamma}s_j := b^i\, {}^{\gamma}s_{ij}, \quad {}^{\gamma}s_0 := {}^{\gamma}s_j y^j, \quad {}^{\gamma}s^i_0 = a^{ij}\, {}^{\gamma}s_{j0}, \quad {}^{\gamma}\bar{s}_0 := {}^{\gamma}s^i_0 b_i. $$

4. Spray coefficients of F

In this section, to compute G^i, we use a technique used by Matsumoto in [6].

For $F = \alpha\Psi(s,\bar{s})$ we can get

$$ \beta_{x^j} = b_{0|j} + b_r G^r_j, \qquad \gamma_{x^j} = \gamma_{0|j} + \gamma_r G^r_j, $$

$$ s_{x^j} = \frac{1}{\alpha}(b_{0|j} + h_r G^r_j), \qquad \bar{s}_{x^j} = \frac{1}{\alpha}(\gamma_{0|j} + \bar{h}_r G^r_j), \tag{23} $$

where $G^i_j = {}^{\alpha}G^i_{y^j}$. Moreover, by $\alpha_{|i} = 0$ and $\alpha_{i|j} = 0$ we have

$$ \alpha_{x^j} = \alpha_r G^r_j, \quad (\alpha_i)_{x^j} = \alpha_{ir}G^r_j + \alpha_r G^r_{ij}, \tag{24} $$

where $G^r_{ij} = {}^{\alpha}G^r_{y^iy^j}$. Then

$$ (h_i)_{x^j} = b_{i|j} - \frac{1}{\alpha}b_{0|j}\alpha_i - \frac{1}{\alpha}h_r G^r_j\alpha_i + h_r G^r_{ij} - s\alpha_{ir}G^r_j, $$

$$ (\bar{h}_i)_{x^j} = \gamma_{i|j} - \frac{1}{\alpha}\gamma_{0|j}\alpha_i - \frac{1}{\alpha}\bar{h}_r G^r_j\alpha_i + \bar{h}_r G^r_{ij} - \bar{s}\alpha_{ir}G^r_j. \tag{25} $$

Differentiating (10) with respect to x^j and using (23), (24) and (25) yield

$$ \begin{aligned} \frac{\partial \ell_i}{\partial x^j} &= \Psi_s b_{i|j} + \Psi_{\bar{s}}\gamma_{i|j} + \frac{1}{\alpha}\Big[\Psi_{ss}b_{0|j} + \Psi_{s\bar{s}}\gamma_{0|j}\Big]h_i + \frac{1}{\alpha}\Big[\Psi_{s\bar{s}}b_{0|j} + \Psi_{\bar{s}\bar{s}}\gamma_{0|j}\Big]\bar{h}_i \\ &+ \Big[\Psi\alpha_r + \Psi_s h_r + \Psi_{\bar{s}}\bar{h}_r\Big]G^r_{ij} + (\Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}})\alpha_{ir}G^r_j \\ &+ \frac{1}{\alpha}\Big[\Psi_{ss}h_ih_r + \Psi_{\bar{s}\bar{s}}\bar{h}_i\bar{h}_j + \Psi_{s\bar{s}}(h_i\bar{h}_j + \bar{h}_ih_j)\Big]G^r_j. \end{aligned} \tag{26} $$

Let ";" denotes the horizontal covariant derivative with respect to Cartan connection of F. Next, we deal with $\ell_{i;j} = 0$, that is $\frac{\partial \ell_i}{\partial x^j} = \ell_{ir}N^r_j + \ell_r\Gamma^r_{ij}$. Let us define

$$ D^i_{jk} := \Gamma^i_{jk} - G^i_{jk}, \quad D^i_j := D^i_{jk}y^k = N^i_j - G^i_j, \quad D^i := D^i_j y^j = 2G^i - 2\,{}^{\alpha}G^i. \tag{27} $$

Then $\frac{\partial \ell_i}{\partial x^j} = \ell_{ir}(D^r_j + G^r_j) + \ell_r(D^r_{ij} + G^r_{ij})$. Putting (10) and (11) in above equation yields

$$\frac{\partial \ell_i}{\partial x^j} = \ell_{ir}D^r_j + \ell_r D^r_{ij} + \Big[\Psi\alpha_r + \Psi_s h_r + \Psi_{\bar{s}}\bar{h}_r\Big]G^r_{ij}$$
$$+\Big[(\Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}})\alpha_{ir} + \frac{1}{\alpha}\Psi_{ss}h_i h_r + \frac{1}{\alpha}\Psi_{\bar{s}\bar{s}}\bar{h}_i\bar{h}_r + \frac{1}{\alpha}\Psi_{s\bar{s}}(h_i\bar{h}_r + \bar{h}_r h_i)\Big]G^r_j. \tag{28}$$

By comparing (26) and (28) we get the following

$$\Psi_s b_{i|j} + \Psi_{\bar{s}}\gamma_{i|j} = \ell_{ir}D^r_j + \ell_r D^r_{ij} - \frac{1}{\alpha}\big[\Psi_{ss}b_{0|j} + \Psi_{s\bar{s}}\gamma_{0|j}\big]h_i - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}b_{0|j} + \Psi_{\bar{s}\bar{s}}\gamma_{0|j}\big]\bar{h}_i. \tag{29}$$

Thus by (21) and (22) we have

$$\begin{aligned} 2\Psi_s\ {}^{\beta}r_{ij} + 2\Psi_{\bar{s}}\ {}^{\gamma}r_{ij} &= \ell_{ir}D^r_j + \ell_{jr}D^r_i + 2\ell_r D^r_{ij} \\ &\quad - \frac{1}{\alpha}\big[\Psi_{ss}b_{0|j} + \Psi_{s\bar{s}}\gamma_{0|j}\big]h_i - \frac{1}{\alpha}\big[\Psi_{ss}b_{0|i} + \Psi_{s\bar{s}}\gamma_{0|i}\big]h_j \\ &\quad - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}b_{0|j} + \Psi_{\bar{s}\bar{s}}\gamma_{0|j}\big]\bar{h}_i - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}b_{0|i} + \Psi_{\bar{s}\bar{s}}\gamma_{0|i}\big]\bar{h}_j, \end{aligned} \tag{30}$$

$$\begin{aligned} 2\Psi_s\ {}^{\beta}s_{ij} + 2\Psi_{\bar{s}}\ {}^{\gamma}s_{ij} &= \ell_{ir}D^r_j - \ell_{jr}D^r_i \\ &\quad - \frac{1}{\alpha}\big[\Psi_{ss}b_{0|j} + \Psi_{s\bar{s}}\gamma_{0|j}\big]h_i + \frac{1}{\alpha}\big[\Psi_{ss}b_{0|i} + \Psi_{s\bar{s}}\gamma_{0|i}\big]h_j \\ &\quad - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}b_{0|j} + \Psi_{\bar{s}\bar{s}}\gamma_{0|j}\big]\bar{h}_i + \frac{1}{\alpha}\big[\Psi_{s\bar{s}}b_{0|i} + \Psi_{\bar{s}\bar{s}}\gamma_{0|i}\big]\bar{h}_j. \end{aligned} \tag{31}$$

Contracting (30) and (31) with y^j implies that

$$\begin{aligned} 2\Psi_s\ {}^{\beta}r_{i0} + 2\Psi_{\bar{s}}\ {}^{\gamma}r_{i0} &= \ell_{ir}D^r + 2\ell_r D^r_i - \frac{1}{\alpha}\big[\Psi_{ss}\ {}^{\beta}r_{00} + \Psi_{s\bar{s}}\ {}^{\gamma}r_{00}\big]h_i \\ &\quad - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}\ {}^{\beta}r_{00} + \Psi_{\bar{s}\bar{s}}\ {}^{\gamma}r_{00}\big]\bar{h}_i. \end{aligned} \tag{32}$$

$$\begin{aligned} 2\Psi_s\ {}^{\beta}s_{i0} + 2\Psi_{\bar{s}}\ {}^{\gamma}s_{i0} &= \ell_{ir}D^r - \frac{1}{\alpha}\big[\Psi_{ss}\ {}^{\beta}r_{00} + \Psi_{s\bar{s}}\ {}^{\gamma}r_{00}\big]h_i \\ &\quad - \frac{1}{\alpha}\big[\Psi_{s\bar{s}}\ {}^{\beta}r_{00} + \Psi_{\bar{s}\bar{s}}\ {}^{\gamma}r_{00}\big]\bar{h}_i. \end{aligned} \tag{33}$$

If you subtract (33) from (32), you get

$$\Psi_s(\ {}^{\beta}r_{i0} -\ {}^{\beta}s_{i0}) + \Psi_{\bar{s}}(\ {}^{\gamma}r_{i0} -\ {}^{\gamma}s_{i0}) = \ell_r D^r_i. \tag{34}$$

The contraction of (34) with y^i leads to

$$\Psi_s\ {}^{\beta}r_{00} + \Psi_{\bar{s}}\ {}^{\gamma}r_{00} = \ell_r D^r. \tag{35}$$

To obtain the spray coefficients of F, we first propose the following lemma.

Lemma 4.1. *The system of algebraic equations (i) $\ell_{ir}A^r = B_i$, (ii) $\ell_r A^r = B$, has unique solution A^r for given B and B_i such that $B_i y^i = 0$. The solution is given by*

$$A^i = (\alpha_r A^r)\alpha^i + \frac{\alpha}{\Pi}B^i - \frac{\alpha}{\Pi\Gamma}(\mu_1 h^i + \mu_2 \bar{h}^i), \tag{36}$$

where $B^i = a^{il}B_l$, $h^i = a^{il}h_l$, $\bar{h}^i = a^{il}\bar{h}_l$ and

$$\Pi := \Psi - s\Psi_s - \bar{s}\Psi_{\bar{s}},$$
$$\mu_1 := \big[\Psi_{ss} + (g^2 - \bar{s}^2)J\big]B_r b^r + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]B_r\gamma^r,$$

$$\mu_2 := \big[\Psi_{\bar{s}\bar{s}} + (b^2 - s^2)J\big]B_r\gamma^r + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]B_r b^r.$$

Proof. By contracting (11) with b^i and γ^i we have

$$\ell_{ij}b^i = \frac{1}{\alpha}\big[\Pi + (b^2 - s^2)\Psi_{ss} + (\theta - s\bar{s})\Psi_{s\bar{s}}\big]h_j + \frac{1}{\alpha}\big[(b^2 - s^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{\bar{s}\bar{s}}\big]\bar{h}_j, \tag{37}$$

$$\ell_{ij}\gamma^i = \frac{1}{\alpha}\big[(\theta - s\bar{s})\Psi_{ss} + (g^2 - \bar{s}^2)\Psi_{s\bar{s}}\big]h_j + \frac{1}{\alpha}\big[\Pi + (\theta - s\bar{s})\Psi_{s\bar{s}} + (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}}\big]\bar{h}_j. \tag{38}$$

Next contracting equation (i) with b^i and γ^i and using (37) and (38) we get the following

$$\begin{cases} \big[\Pi + (b^2 - s^2)\Psi_{ss} + (\theta - s\bar{s})\Psi_{s\bar{s}}\big]h_jA^j + \big[(b^2 - s^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{\bar{s}\bar{s}}\big]\bar{h}_jA^j = \alpha B_j b^j \\ \big[(\theta - s\bar{s})\Psi_{ss} + (g^2 - \bar{s}^2)\Psi_{s\bar{s}}\big]h_jA^j + \big[\Pi + (\theta - s\bar{s})\Psi_{s\bar{s}} + (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}}\big]\bar{h}_jA^j = \alpha B_j\gamma^j. \end{cases}$$

By solving the above system we obtain

$$\begin{aligned} h_jA^j &= \frac{\alpha}{\Pi\Gamma}\Big\{\big[\Pi + (\theta - s\bar{s})\Psi_{s\bar{s}} + (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}}\big]B_jb^j \\ &\quad - \big[(b^2 - s^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{\bar{s}\bar{s}}\big]B_j\gamma^j\Big\}, \end{aligned} \tag{39}$$

$$\begin{aligned} \bar{h}_jA^j &= \frac{\alpha}{\Pi\Gamma}\Big\{\big[\Pi + (b^2 - s^2)\Psi_{ss} + (\theta - s\bar{s})\Psi_{s\bar{s}}\big]B_j\gamma^j \\ &\quad - \big[(\theta - s\bar{s})\Psi_{ss} + (g^2 - \bar{s}^2)\Psi_{s\bar{s}}\big]B_jb^j\Big\}. \end{aligned} \tag{40}$$

Substituting (10) in equation (ii) yields $\Psi\alpha_jA^j + \Psi_s h_jA^j + \Psi_{\bar{s}}\bar{h}_jA^j = B$. By (39) and (40) we get

$$\begin{aligned} \alpha_jA^j = \frac{1}{\Psi}\Big\{B - \frac{\alpha}{\Pi\Gamma}\Big(\Psi_s\big[\Pi + (\theta - s\bar{s})\Psi_{s\bar{s}} + (g^2 - \bar{s}^2)\Psi_{\bar{s}\bar{s}}\big] - \Psi_{\bar{s}}\big[(\theta - s\bar{s})\Psi_{ss} + (g^2 - \bar{s}^2)\Psi_{s\bar{s}}\big]\Big)B_jb^j \\ - \frac{\alpha}{\Pi\Gamma}\Big(\Psi_{\bar{s}}\big[\Pi + (b^2 - s^2)\Psi_{ss} + (\theta - s\bar{s})\Psi_{s\bar{s}}\big] - \Psi_s\big[(b^2 - s^2)\Psi_{s\bar{s}} + (\theta - s\bar{s})\Psi_{\bar{s}\bar{s}}\big]\Big)B_j\gamma^j\Big\}. \end{aligned}$$

Applying (11) in equation (i) yields

$$\frac{\Pi}{\alpha}\big[a_{ij}A^j - (\alpha_jA^j)\alpha_i\big] + \frac{1}{\alpha}\big[(\Psi_{ss}h_i + \Psi_{s\bar{s}}\bar{h}_i)h_jA^j + (\Psi_{s\bar{s}}h_i + \Psi_{\bar{s}\bar{s}}\bar{h}_i)\bar{h}_jA^j\big] = B_i.$$

Contracting this equation with a^{ij} and using (39) and (40) one could get (36). □

Now, we are able to obtain the spray coefficients of F.

The equations (33) and (35) constitute the system of algebraic equations whose solution from Lemma 4.1 is given by $D^i = (\alpha_rD^r)\alpha^i + \frac{\alpha}{\Pi}B^i - \frac{\alpha}{\Pi\Gamma}(\mu_1h^i + \mu_2\bar{h}^i)$, where

$$B_i = 2\Psi_s\ {}^{\beta}s_{i0} + 2\Psi_{\bar{s}}\ {}^{\gamma}s_{i0} + \frac{1}{\alpha}\big[\Psi_{ss}\ {}^{\beta}r_{00} + \Psi_{s\bar{s}}\ {}^{\gamma}r_{00}\big]h_i + \frac{1}{\alpha}\big[\Psi_{s\bar{s}}\ {}^{\beta}r_{00} + \Psi_{\bar{s}\bar{s}}\ {}^{\gamma}r_{00}\big]\bar{h}_i,$$

$$B = \Psi_s\ {}^{\beta}r_{00} + \Psi_{\bar{s}}\ {}^{\gamma}r_{00},$$

$$B_ib^i = 2\Psi_s\ {}^{\beta}s_0 + 2\Psi_{\bar{s}}\ {}^{\gamma}\bar{s}_0 + \frac{1}{\alpha}\big[\Psi_{ss}\ {}^{\beta}r_{00} + \Psi_{s\bar{s}}\ {}^{\gamma}r_{00}\big](b^2 - s^2) + \frac{1}{\alpha}\big[\Psi_{s\bar{s}}\ {}^{\beta}r_{00} + \Psi_{\bar{s}\bar{s}}\ {}^{\gamma}r_{00}\big](\theta - s\bar{s}),$$

$$B_i\gamma^i = 2\Psi_s\ {}^{\beta}\bar{s}_0 + 2\Psi_{\bar{s}}\ {}^{\gamma}s_0 + \frac{1}{\alpha}\big[\Psi_{ss}\ {}^{\beta}r_{00} + \Psi_{s\bar{s}}\ {}^{\gamma}r_{00}\big](\theta - s\bar{s}) + \frac{1}{\alpha}\big[\Psi_{s\bar{s}}\ {}^{\beta}r_{00} + \Psi_{\bar{s}\bar{s}}\ {}^{\gamma}r_{00}\big](g^2 - \bar{s}^2).$$

Now put $D^i = 2\bar{G}^i - 2G^i$ and then we get the followin.

Proposition 4.2. *The spray coefficients G^i are related to ${}^{\alpha}G^i$ by*

$$G^i = {}^{\alpha}G^i + \frac{\alpha}{A}\big[\Psi_s\ {}^{\beta}s^i_0 + \Psi_{\bar{s}}\ {}^{\gamma}s^i_0\big] + \frac{1}{2\Gamma}\big[\Gamma_1 b^i + \Gamma_2\gamma^i + \frac{1}{\Psi}\Gamma_3\alpha^i\big], \tag{41}$$

where

$$\Gamma_1 := \big[\Psi_{ss} + (g^2 - \bar{s}^2)J\big]\mathcal{R}^{\beta} + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]\ \mathcal{R}^{\gamma}, \tag{42}$$

$$\Gamma_2 := \big[\Psi_{\bar{s}\bar{s}} + (b^2 - s^2)J\big]\mathcal{R}^{\gamma} + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]\mathcal{R}^{\beta},$$

$$\Gamma_3 := \big[\rho_1 + \pi_2(\theta - s\bar{s}) - \pi_1(g^2 - \bar{s}^2)\big]\mathcal{R}^{\beta} + \big[\bar{\rho}_1 - \pi_2(b^2 - s^2) + \pi_1(\theta - s\bar{s})\big]\mathcal{R}^{\gamma}, \tag{43}$$

and

$$\mathcal{R}^{\beta} := {}^{\beta}r_{00} - \frac{2\alpha}{\Pi}\big[\Psi_s\ {}^{\beta}s_0 + \Psi_{\bar{s}}\ {}^{\gamma}\bar{s}_0\big], \qquad \mathcal{R}^{\gamma} := {}^{\gamma}r_{00} - \frac{2\alpha}{\Pi}\big[\Psi_s\ {}^{\beta}\bar{s}_0 + \Psi_{\bar{s}}\ {}^{\gamma}s_0\big].$$

5. Projectively flat (α,β,γ)-metrics

Lemma 5.1. *An (α,β,γ)-metric $F = \alpha\Psi(s,\bar{s})$, where $s = \frac{\beta}{\alpha}$ and $\bar{s} = \frac{\gamma}{\alpha}$, is projectively flat on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ if and only if*

$${}^{\alpha}h_{ij}\ {}^{\alpha}G^i + \frac{\alpha}{\Pi}\big[\Psi_s\ {}^{\beta}s_{j0} + \Psi_{\bar{s}}\ {}^{\gamma}s_{j0}\big] + \frac{1}{2\Gamma}\big[\Gamma_1 h_j + \Gamma_2\bar{h}_j\big] = 0, \tag{44}$$

where Γ_1 and Γ_2 are given by (43) and ${}^{\alpha}h_{ij} = a_{ij} - \alpha_i\alpha_j$.

Proof. Let $F = \alpha\Psi(s,\bar{s})$ be a projectively flat metric on $\mathcal{U}$. Therefore, we have

$$G^i = Py^i \tag{45}$$

Contracting (45) with ${}^{\alpha}h_{ij}$ and using (41) we get (44).

Conversely, suppose that (44) holds. Contracting (44) by a^{ij} yields

$$\frac{\alpha}{\Pi}\big[\Psi_s\ {}^{\beta}s^j_0 + \Psi_{\bar{s}}\ {}^{\gamma}s^j_0\big] = -\frac{1}{2\Gamma}\big[\Gamma_1 h^j + \Gamma_2\bar{h}^j\big] - \big[{}^{\alpha}G^i - {}^{\alpha}G^r\alpha_r\alpha^i\big].$$

Applying it to (41) leads to

$$G^i = \Big\{\ {}^{\alpha}G^r\alpha_r + \frac{1}{2\Gamma}\big[s\Gamma_1 + \bar{s}\Gamma_2 + \frac{1}{\Psi}\Gamma_3\big]\Big\}\alpha^i.$$

This implies that F is projectively flat. □

Example 5.2. We consider an (α,β,γ)-metric in the following form $F = \alpha e^{\frac{\beta}{\alpha}} + \gamma$, $\Psi(s,\bar{s}) = e^s + \bar{s}$. Let $b_0 > 0$ and $g_0 > 0$ be the largest numbers such that

$$\Pi = (1-s)e^s > 0, \quad \Gamma = (1 - s + b^2 - s^2)e^s > 0, \quad |s| < b < b_0,\ |\bar{s}| < g < g_0. \tag{46}$$

Note that F is a Finsler metric if and only if β and γ satisfy that $b := \|\beta\|_{\alpha} < b_0$ and $g := \|\gamma\|_{\alpha} < g_0$.

For this metric we can prove the following lemma.

Lemma 5.3. *The (α,β,γ)-metric $F = \alpha e^{\frac{\beta}{\alpha}} + \gamma$ is locally projectively flat if and only if β is parallel with respect to α and γ is closed.*

Recall that 1-form β is closed ($d\beta = 0$) if and only if ${}^{\beta}s_{ij} = 0$, and β is parallel with respect to α if and only if $b_{i|j} = 0$, i.e. ${}^{\beta}s_{ij} = 0$ and ${}^{\beta}r_{ij} = 0$.

Proof. let $F = \alpha e^{\frac{\beta}{\alpha}} + \gamma$ be locally projectively flat. Putting (46) into (44) yields

$$h_{ij}\ {}^{\alpha}G^i + \frac{\alpha^2}{(\alpha-\beta)e^{\frac{\beta}{\alpha}}}\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0} + {}^{\gamma}s_{j0}\big] + \frac{\alpha^2}{2\big[\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2\big]}\Big\{\ {}^{\beta}r_{00} - \frac{2\alpha^2}{(\alpha-\beta)e^{\frac{\beta}{\alpha}}}\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 + {}^{\gamma}\bar{s}_0\big]\Big\}h_j = 0.$$

By multiplying this equation by $2\alpha^2(\alpha-\beta)\big[\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2\big]e^{\frac{\beta}{\alpha}}$, we get

$$(\alpha-\beta)\big[\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2\big]e^{\frac{\beta}{\alpha}}(a_{ij}\alpha^2 - y_iy_j)\ {}^{\alpha}G^i + 2\alpha^4\big[\alpha^2-\alpha\beta+b^2\alpha^2-\beta^2\big]\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0} + {}^{\gamma}s_{j0}\big] + \alpha^2(\alpha-\beta)e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j - \beta y_j) - 2\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 + {}^{\gamma}\bar{s}_0\big](\alpha^2b_j - \beta y_j) = 0.$$

We can rewrite this equation as a polynomial in y^i and α. This gives

$$0=\Big\{-2\beta\big[2\alpha^2+b^2\alpha^2-\beta^2\big]e^{\frac{\beta}{\alpha}}(a_{ij}\alpha^2-y_iy_j)\ {}^{\alpha}G^i+2\alpha^4\big[\alpha^2+b^2\alpha^2-\beta^2\big]\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big] -\alpha^2\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j)-2\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0\big](\alpha^2b_j-\beta y_j)\Big\} +\alpha\Big\{2\big[\alpha^2+b^2\alpha^2\big]e^{\frac{\beta}{\alpha}}(a_{ij}\alpha^2-y_iy_j)\ {}^{\alpha}G^i-2\beta\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big]+\alpha^2e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j)\Big\}.$$

α^{even} is rational in y^i and α is irrational. Then we have two following equations:

$$-2\beta\big[2\alpha^2+b^2\alpha^2-\beta^2\big]e^{\frac{\beta}{\alpha}}(a_{ij}\alpha^2-y_iy_j)\ {}^{\alpha}G^i + 2\alpha^4\big[\alpha^2+b^2\alpha^2-\beta^2\big]\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0} + {}^{\gamma}s_{j0}\big] - \alpha^2\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j) - 2\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 + {}^{\gamma}\bar{s}_0\big](\alpha^2b_j-\beta y_j) = 0, \tag{47}$$

and

$$2\big[\alpha^2+b^2\alpha^2\big]e^{\frac{\beta}{\alpha}}(a_{ij}\alpha^2-y_iy_j)\ {}^{\alpha}G^i - 2\beta\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0} + {}^{\gamma}s_{j0}\big] + \alpha^2e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j) = 0. \tag{48}$$

Then we have

$$(\alpha^2+b^2\alpha^2)\Big\{2\alpha^4\big[\alpha^2+b^2\alpha^2-\beta^2\big]\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big] - \alpha^2\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j) - 2\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0\big](\alpha^2b_j-\beta y_j)\Big\} = -\beta\big[2\alpha^2+b^2\alpha^2-\beta^2\big]\Big\{-2\beta\alpha^4\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big]+\alpha^2e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(\alpha^2b_j-\beta y_j)\Big\}.$$

Therefore

$$2\alpha^2\Big\{(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\Big\}\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big] +\Big\{\beta(\alpha^2-\beta^2)e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00} - 2\alpha^2(\alpha^2+b^2\alpha^2)\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0\big]\Big\}(\alpha^2b_j-\beta y_j) = 0. \tag{49}$$

Contracting (49) with b^j leads to

$$2\alpha^2(\alpha^2-\beta^2)(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)+\beta(\alpha^2-\beta^2)(b^2\alpha^2-\beta^2)e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}=0. \tag{50}$$

Since $\alpha^2 \not\equiv 0 \pmod{\beta}$ Then $\alpha^2-\beta^2 \neq 0$. The term of (50) which does not contain α^2 is $-\beta^3 e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}$. Notice $-\beta^3 e^{\frac{\beta}{\alpha}}$ is not divisible by α^2, then ${}^{\beta}r_{00}=k(x)\alpha^2$ where we can consider two cases.

Case 1: $k(x)=0$. Substituting ${}^{\beta}r_{00}=0$ into (50) implies that $(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)=0$. If $\alpha^2+b^2\alpha^2-\beta^2=0$, then the term which does not contain α^2 is β^2, which implies that $\beta^2=0$ and is a contradiction. Hence

$$e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0=0. \tag{51}$$

Putting ${}^{\beta}r_{00}=0$ and (51) into (49) leads to $\Big[(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\Big]\big[e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{j0}+{}^{\gamma}s_{j0}\big]=0$. If $(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2=0$, then by a similar argument, we get $\beta^4=0$ which is a contradiction. Therefore

$$e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{i0}+{}^{\gamma}s_{i0}=0. \tag{52}$$

Differentiating (52) with respect to y^j and y^k imply that

$$-(\alpha_j h_k+\alpha_k h_j-s\alpha\alpha_{jk})\ {}^{\beta}s_{i0}+h_j h_k\ {}^{\beta}s_{i0}+\alpha h_j\ {}^{\beta}s_{ik}+\alpha h_k\ {}^{\beta}s_{ij}=0.$$

Contracting it with $b^j b^k$ yields

$$(b^2-s^2)\Big[(-3s+b^2-s^2)\ {}^{\beta}s_{i0}-2\alpha\ {}^{\beta}s_i\Big]=0. \tag{53}$$

Contracting (53) with b^i leads to $(-3s+b^2-s^2)\ {}^{\beta}s_0=0$.

If $-3s+b^2-s^2=0$, then $-3\alpha\beta+b^2\alpha^2-\beta^2=0$. By separating it in the rational and irrational terms of y^i, we get $\beta=0$. But this leads to a contradiction. Then ${}^{\beta}s_0=0$, that is ${}^{\beta}s_i=0$. Putting ${}^{\beta}s_i=0$ in (53) yields ${}^{\beta}s_{i0}=0$. Substituting it into (52) implies that ${}^{\gamma}s_{i0}=0$. From ${}^{\beta}s_{i0}=0$ and ${}^{\gamma}s_{i0}=0$, we get ${}^{\beta}s_{ij}=0$, ${}^{\gamma}s_{ij}=0$.

Case 2: $k(x)\neq 0$. Let ${}^{\beta}r_{00}=k(x)\alpha^2$. Substituting ${}^{\beta}r_{00}=k(x)\alpha^2$ into (50) implies that

$$(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)+\beta(b^2\alpha^2-\beta^2)e^{\frac{\beta}{\alpha}}k(x)=0. \tag{54}$$

The term of (54) which does not contain α^2 is $-\beta^2(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)-\beta^3 e^{\frac{\beta}{\alpha}}k(x)$. Then we have $(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)=-\beta e^{\frac{\beta}{\alpha}}k(x)$. Putting it into (54) yields $-\alpha^2\beta e^{\frac{\beta}{\alpha}}k(x)=0$. This implies that $k(x)=0$, then ${}^{\beta}r_{00}=0$. Similar to **Case 1**, we can conclude that ${}^{\beta}s_{ij}={}^{\gamma}s_{ij}=0$. □

6. Douglas spaces by (α,β,γ)-metrics

In [3], Douglas introduced the local functions $D^i_{j\,kl}$ on TM_0 defined by

$$D^i_{j\,kl}:=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\Big(G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big).$$

It is easy to verify that $D := D^i_{j\,kl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_0. D is called the Douglas tensor. The Finsler space (M, F) is called a Douglas space if and only if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [1].

By (41) one can gets $G^i y^j - G^j y^i = ({}^{\alpha}G^i y^j - {}^{\alpha}G^j y^i) + B^{ij}$, where

$$
\begin{aligned}
B^{ij} :=& \frac{\alpha}{\Pi}\big[\Psi_s\ ({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) + \Psi_{\bar{s}}\ ({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i)\big] \\
&+ \frac{1}{2\Gamma}\Big\{\big[\Psi_{ss} + (g^2 - \bar{s}^2)J\big]\mathcal{R}^{\beta} + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]\ \mathcal{R}^{\gamma}\Big\}(b^i y^j - b^j y^i) \\
&+ \frac{1}{2\Gamma}\Big\{\big[\Psi_{\bar{s}\bar{s}} + (b^2 - s^2)J\big]\mathcal{R}^{\gamma} + \big[\Psi_{s\bar{s}} - (\theta - s\bar{s})J\big]\mathcal{R}^{\beta}\Big\}(\gamma^i y^j - \gamma^j y^i). \qquad (55)
\end{aligned}
$$

Example 6.1. Let F be the metric that introduced in Example 5.2. We can prove (α, β, γ)-metric $F = \alpha e^{\frac{\beta}{\alpha}} + \gamma$ is Doaglus if and only if β is parallel with respect to α and γ is closed.

Proof. Substituting (46) into (55) implies that

$$
\begin{aligned}
B^{ij} =& \frac{\alpha^2}{(\alpha-\beta)e^{\frac{\beta}{\alpha}}}\Big[({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i)e^{\frac{\beta}{\alpha}} + ({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i)\Big] \\
&+ \frac{\alpha^2}{2\big[\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2\big]}\Big[\ {}^{\beta}r_{00} - \frac{2\alpha^2}{(\alpha-\beta)e^{\frac{\beta}{\alpha}}}(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 + {}^{\gamma}\bar{s}_0)\Big](b^i y^j - b^j y^i).
\end{aligned}
$$

Suppose that F is a Douglas space, that is B^{ij} are $hp(3)$. Multiplying this equation by $2(\alpha-\beta)\big[\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2\big]e^{\frac{\beta}{\alpha}}$ yields

$$
\begin{aligned}
&2(\alpha-\beta)\big[\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2\big]e^{\frac{\beta}{\alpha}}B^{ij} = \\
&2\alpha^2\big[\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2\big]e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) + 2\alpha^2\big[\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2\big]({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i) \\
&+ \Big[\alpha^2(\alpha-\beta)e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00} - 2\alpha^4 e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 - 2\alpha^4\ {}^{\gamma}\bar{s}_0\Big](b^i y^j - b^j y^i).
\end{aligned}
$$

By separating it in rational and irrational terms of y^i, we obtain two equations as follows:

$$
\begin{aligned}
2(\alpha^2 + b^2\alpha^2)e^{\frac{\beta}{\alpha}}B^{ij} =& -2\alpha^2\beta e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) - 2\alpha^2\beta({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i) \\
&+ \alpha^2 e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(b^i y^j - b^j y^i). \qquad (56)
\end{aligned}
$$

and

$$
\begin{aligned}
-2\beta(2\alpha^2 + b^2\alpha^2 - \beta^2)e^{\frac{\beta}{\alpha}}B^{ij} =& 2\alpha^2(\alpha^2 + b^2\alpha^2 - \beta^2)e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) \\
&+ 2\alpha^2(\alpha^2 + b^2\alpha^2 - \beta^2)({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i) \\
&+ \Big[-\alpha^2\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00} - 2\alpha^4 e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 - 2\alpha^4\ {}^{\gamma}\bar{s}_0\Big](b^i y^j - b^j y^i). \qquad (57)
\end{aligned}
$$

Eliminating B^{ij} from (56) and (57) yields

$$
\begin{aligned}
&(\alpha^2 - b^2\alpha^2)\Big\{2\alpha^2(\alpha^2 + b^2\alpha^2 - \beta^2)e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) + 2\alpha^2(\alpha^2 + b^2\alpha^2 - \beta^2)({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i) \\
&+ \Big[-\alpha^2\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00} - 2\alpha^4 e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0 - 2\alpha^4\ {}^{\gamma}\bar{s}_0\Big](b^i y^j - b^j y^i)\Big\} = \\
&-\beta(2\alpha^2 - b^2\alpha^2 - \beta^2)\Big\{-2\alpha^2\beta e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0 y^j - {}^{\beta}s^j_0 y^i) - 2\alpha^2\beta({}^{\gamma}s^i_0 y^j - {}^{\gamma}s^j_0 y^i) + \alpha^2 e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(b^i y^j - b^j y^i)\Big\}.
\end{aligned}
$$

By simplifying this equation one implies that

$$2\Big[(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\Big]e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0y^j-{}^{\beta}s^j_0y^i)+2\Big[(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\Big]({}^{\gamma}s^i_0y^j-{}^{\gamma}s^j_0y^i) \quad (58)$$
$$+\Big[-(\alpha^2+b^2\alpha^2)\big(\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}+2\alpha^2e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+2\alpha^2\ {}^{\gamma}\bar{s}_0\big)+\beta e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}(2\alpha^2+b^2\alpha^2-\beta^2)\Big](b^iy^j-b^jy^i)=0.$$

By contracting it with b_iy_j, we get

$$2\alpha^2(\alpha^2-\beta^2)(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)+\beta(\alpha^2-\beta^2)(b^2\alpha^2-\beta^2)e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}=0. \quad (59)$$

The term of (59) which does not contain α^2 is $-\beta^3e^{\frac{\beta}{\alpha}}\ {}^{\beta}r_{00}$. Notice that $-\beta^3e^{\frac{\beta}{\alpha}}$ is not divisible by α^2, then ${}^{\beta}r_{00}=k(x)\alpha^2$ and we can consider two cases.

Case 1: $k(x)=0$. Substituting ${}^{\beta}r_{00}=0$ into (59) implies that

$$2\alpha^2(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)=0$$

If $\alpha^2+b^2\alpha^2-\beta^2=0$, then the term which does not contain α^2 is β^2. This implies that $\beta^2=0$ which leads to a contradiction. Hence

$$e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0=0. \quad (60)$$

Putting ${}^{\beta}r_{00}=0$ and (60) into (58) leads to

$$\Big[(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\Big]\Big[e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0y^j-{}^{\beta}s^j_0y^i)+({}^{\gamma}s^i_0y^j-{}^{\gamma}s^j_0y^i)\Big]=0.$$

By a similar argument, we get $(\alpha^2+b^2\alpha^2-\beta^2)^2-\alpha^2\beta^2\neq 0$. Therefore

$$e^{\frac{\beta}{\alpha}}({}^{\beta}s^i_0y^j-{}^{\beta}s^j_0y^i)+({}^{\gamma}s^i_0y^j-{}^{\gamma}s^j_0y^i)=0. \quad (61)$$

Contracting (61) with y_j yields

$$e^{\frac{\beta}{\alpha}}\ {}^{\beta}s^i_0+{}^{\gamma}s^i_0=0\Longrightarrow e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_{i0}+{}^{\gamma}s_{i0}=0. \quad (62)$$

Differentiating (62) with respect to y^j and y^k and multiplying it by α^2 imply that

$$-(\alpha_jh_k+\alpha_kh_j-s\alpha\alpha_{jk})\ {}^{\beta}s_{i0}+h_jh_k\ {}^{\beta}s_{i0}+\alpha h_j\ {}^{\beta}s_{ik}+\alpha h_k\ {}^{\beta}s_{ij}=0.$$

Contracting it with b^jb^k yields

$$(b^2-s^2)\Big[(-3s+b^2-s^2)\ {}^{\beta}s_{i0}-2\alpha\ {}^{\beta}s_i\Big]=0. \quad (63)$$

Contracting (63) with b^i leads to $(-3s+b^2-s^2)\ {}^{\beta}s_0=0$. If $-3s+b^2-s^2=0$, then $-3\alpha\beta+b^2\alpha^2-\beta^2=0$. By separating it in rational and irrational terms of y^i, we get $\beta=0$. But this leads to a contradiction. Then ${}^{\beta}s_0=0$, that is ${}^{\beta}s_i=0$. Putting ${}^{\beta}s_i=0$ in (63) yields ${}^{\beta}s_{i0}=0$. Substituting it into (62) implies that ${}^{\gamma}s_{i0}=0$. From ${}^{\beta}s_{i0}=0$ and ${}^{\gamma}s_{i0}=0$, we get ${}^{\beta}s_{ij}=0$, ${}^{\gamma}s_{ij}=0$.

Case 2: $k(x)\neq 0$. Let ${}^{\beta}r_{00}=k(x)\alpha^2$. Putting ${}^{\beta}r_{00}=k(x)\alpha^2$ into (59) implies that

$$2(\alpha^2+b^2\alpha^2-\beta^2)(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)+\beta(b^2\alpha^2-\beta^2)e^{\frac{\beta}{\alpha}}k(x)=0. \quad (64)$$

The term of (64) which does not contain α^2 is $-2\beta^2(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)-\beta^3e^{\frac{\beta}{\alpha}}k(x)$. Then we have $2(e^{\frac{\beta}{\alpha}}\ {}^{\beta}s_0+{}^{\gamma}\bar{s}_0)=-\beta e^{\frac{\beta}{\alpha}}k(x)$. Putting it into (64) yields $-\alpha^2\beta e^{\frac{\beta}{\alpha}}k(x)=0$. This implies that $k(x)=0$, then ${}^{\beta}r_{00}=0$. Therefore similar to **Case 1**, we can conclude that ${}^{\beta}s_{ij}={}^{\gamma}s_{ij}=0$. □

References

[1] S. Bacso, M. Matsumoto, *On the Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen, **51** (1997), 385–406.

[2] S. S. Chern, Z. Shen, *Riemann-Finsler Geometry*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[3] J. Douglas, *The general geometry of paths*, Ann. Math. **29** (1927-1928), 143–168.

[4] G. Hamel, *Über die Geometrieen in denen die Geraden die Kürzesten sind*, Math. Ann. **57** (1903), 231–264.

[5] M. A. Javaloyes and M. Sánchez, *On the definition and examples of Finsler metrics*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **13(3)** (2014), 813–858.

[6] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ., **14(3)** (1974), 477-498.

[7] T. Rajabi, N. Sadeghzadeh, *A new class of Finsler metrics*, Mat. Vesn, **73(1)** (2021), 1–13.

[8] V. Rovenski, *The new Minkowski norm and integral formulae for a manifold with a set of one-forms*, Balkan J. Geom. Appl., **23(1)** (2018), 75–99.

[9] V. Rovenski, P. Walczak, *Deforming convex bodies in Minkowski geometry*, Int. J. Math., **33(1)** (2022), 2250003.

[10] A. Tayebi, T. Tabatabaeifar, E. Peyghan, *On Kropina change for mth root Finsler metrics*, Ukr. Math. J., **66(1)** (2014), 160–164.

[11] C. Yu, H. Zhu,*On a new class of Finsler metrics*, Differential Geom. Appl. **29**-2 (2011), 244-254.

[12] N. L. Youssef, S. H. Abed, S. G. Elgendi, *Generalized β-conformal change and special Finsler spaces*, International Journal of Geometric Methods in Modern Physics, **9(3)**(2012), 1250016.

(received 25.07.2021; in revised form 30.09.2022; available online 02.07.2024)

Department of Mathematics, Faculty of Science, University of Qom, Iran
E-mail: nsadeghzadeh@qom.ac.ir
ORCID iD: https://orcid.org/0000-0002-1058-384X

Department of Mathematics, Faculty of Science, University of Qom, Iran
E-mail: t.rajabi.j@gmail.com
ORCID iD: https://orcid.org/0009-0003-8246-2005