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UNCERTAINTY PRINCIPLES ASSOCIATED WITH THE SHORT TIME QUATERNION COUPLED FRACTIONAL FOURIER TRANSFORM

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Abstract. In this paper, we extend the coupled fractional Fourier transform of complex valued functions to that of the quaternion valued functions on $\mathbb{R}^4$ and call it the quaternion coupled fractional Fourier transform (QCFrFT). We obtain the sharp Hausdorff-Young inequality for QCFrFT and obtain the associated Rènyi uncertainty principle. We also define the short time quaternion coupled fractional Fourier transform (STQCFrFT) and explore its important properties followed by the Lieb's and entropy uncertainty principles.

1. Introduction

Based on the knowledge that the Hermite functions are the eigenfunctions of the Fourier transform (FT) with eigenvalues $e^{in\frac{\pi}{2}}$, Namias [27] introduced in 1980 the fractional Fourier transform (FrFT) with angle θ as an integral transform whose eigenfunctions are the Hermite functions but with eigenvalues $e^{in\theta}$, and which is reduced to FT when $\theta = \frac{\pi}{2}$. This was later refined by McBride and Kerr [22,23]. The extension of FrFT to higher dimensions can be seen in [30], where the kernel of the transform has been obtained by taking the tensor product of n copies of the kernel of the one-dimensional transform. Following the ideas of Namias and the fact that the Hermite functions of two complex variables are eigenfunctions of the two-dimensional FT, Zayed introduced in [33,34] a new definition of the two-dimensional FrFT $\mathcal{F}^{\theta_1,\theta_2}$, which is not a tensor product of two copies of the one-dimensional transform and is given as follows

$$(\mathcal{F}^{\theta_1,\theta_2} f)(\boldsymbol{y}) = \tilde{d}(\mu) \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-i\{\tilde{a}(\mu)(|\boldsymbol{x}|^2+|\boldsymbol{y}|^2)-\boldsymbol{x}\cdot M\boldsymbol{y}\}} d\boldsymbol{x}, \tag{1}$$

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where $\theta_1, \theta_2 \in \mathbb{R}$ and are such that $\theta_1 + \theta_2 \notin 2\pi\mathbb{Z}$ and $\mu = \frac{\theta_1+\theta_2}{2}$, $\nu = \frac{\theta_1-\theta_2}{2}$, $\tilde{a}(\mu) = \frac{\cos\mu}{2}$, $\tilde{b}(\mu,\nu) = \frac{\cos\nu}{\sin\mu}$, $\tilde{c}(\mu,\nu) = \frac{\sin\nu}{\sin\mu}$, $\tilde{d}(\mu) = \frac{ie^{-i\mu}}{2\pi\sin\mu}$, $M = \begin{pmatrix} \tilde{b}(\mu,\nu) & \tilde{c}(\mu,\nu) \\ -\tilde{c}(\mu,\nu) & \tilde{b}(\mu,\nu) \end{pmatrix}$. Furthermore, the transform depends on the angles θ_1 and θ_2, which are coupled, so that the transform parameters are $\mu = \frac{\theta_1+\theta_2}{2}$ and $\nu = \frac{\theta_1-\theta_2}{2}$. Kamalakkannan et al. [19] proved the Parseval identity, the inversion theorem and that the class $\{\mathcal{F}^{\theta_1,\theta_2} : \theta_1, \theta_2 \in \mathbb{R},\ \theta_1 + \theta_2 \notin 2\pi\mathbb{Z}\}$ is a family of unitary operators on $L^2(\mathbb{R}^2)$ which has the additive property $\mathcal{F}^{\theta_1',\theta_2'}(\mathcal{F}^{\theta_1,\theta_2}f) = \mathcal{F}^{\theta_1+\theta_1',\theta_2+\theta_2'}f$ for $\theta_1+\theta_2,\ \theta_1'+\theta_2',\ \theta_1+\theta_2+\theta_1'+\theta_2' \notin 2\pi\mathbb{Z}$. Kamalakkannan et al. [17] extended the FrFT to the n-dimensional FrFT, which is more general than that in [30], and introduced a corresponding convolution structure followed by the convolution theorem. Recently, Shah et al. [28] obtained the Heisenberg uncertainty principle (UP), followed by the local and logarithmic UPs. They also established some concentration-based UP including Amrein-Berthier-Benedicks, Donoho-Stark's UPs, etc.

Although CFrFT generalizes FrFT to two dimensions, it cannot reproduce the local information of non-transient signals due to the presence of a global kernel. Therefore, Kamalakkannan et al. [18] developed a short-time coupled fractional Fourier transform (STCFrFT) and obtained the associated Parseval and inversion formulas, followed by some associated UPs.

The quaternion Fourier transform (QFT) introduced by Ell [11] is useful in analyzing $\mathbb{H}$-valued, i.e. quaternion-valued functions. Due to the non-commutativity of the quaternion product, the QFT can be categorized into different types, namely left-sided, right-sided and two-sided [4, 5, 11]. For the right-sided QFT, Cheng et al. [9] discussed the Plancherel theorem and also obtained its relation to the other two QFTs. Lian [24], proved Pitt's inequality, logarithmic UP (see also [8]), entropy UP for the two-sided QFT with optimal constants. Also in [25], the author gave the sharp Hausdorff-Young (H-Y) inequality together with the Hirschman's entropy UP for two-sided QFT using the standard differential approach. Recently, the QFT has been generalized to the quaternion FrFT (QFrFT) and also to the quaternion quadratic phase Fourier transform (QQPFT) [2, 14]. If the kernels in the definition of the two-sided QFT [26] of the function defined on $\mathbb{R}^2$ are replaced by those of the kernels of the FrFT [1, 27, 30], the two-sided QFrFT is obtained. Similarly, the other variant of the QFrFT can be found in [31].

The generalization of the classical windowed Fourier transform to $\mathbb{H}$-valued functions on $\mathbb{R}^2$ can be found in [6]. The authors obtained some important properties using the properties of right-sided QFT [5]. They also obtained the Heisenberg UP for the quaternion windowed Fourier transform (QWFT) by using the same technique as Wilczok [32]. In [3], the authors also obtained Pitt's inequality and Lieb's inequality for the right-sided QWFT introduced in [6]. In addition to the orthogonality property for the two-sided QWFT, the authors in [20, 21] studied several UPs, including Beckner's UP in terms of entropy and Lieb's UP. Substituting the Fourier kernel in the left, right or two-sided QWFT with the fractional Fourier kernels, results in the corresponding QWFrFT.

As already mentioned, some important properties as well as the UPs of the CFrFT and the STCFrFT for the complex-valued function have been investigated. It is natural to extend these transforms to quaternionic setting. As far as we know, these transforms have not yet been introduced for $\mathbb{H}$-valued functions. This paper deals with the two-sided QCFrFT. We derive the sharp H-Y inequality for QCFrFT, followed by the Rènyi entropy UP for QCFrFT. We also introduce the STQCFrFT and obtain its inner product relation and reconstruction formula in addition to its basic properties such as linearity, translation, etc. The Lieb's and entropy UPs for the proposed STQCFrFT is obtained with the aid of QCFrFT's sharp H-Y inequality.

The paper is organised as follows: In Section 2, we recall some basics of quaternion algebra. In Section 3, we define the two sided QCFrFT and establish its various important properties. In Section 4, we define the two sided STQCFrFT and study its properties together with the Lieb's and entropy UPs, followed by an example of the STQCFrFT. Finally, in Section 5, we conclude this paper.

2. Preliminaries

Let $\mathbb{H} = \{s = s_0 + is_1 + js_2 + ks_3 : s_0, s_1, s_2, s_3 \in \mathbb{R}\}$ be the quaternion algebra, where i, j and k are the imaginary units satisfying the Hamilton's multiplication rules $i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$. For a quaternion $s = s_0 + is_1 + js_2 + ks_3$, we define the following terms

- $\mathrm{Sc}(s) = s_0$, called the scalar part of s which satisfies the cyclic multiplicative symmetry [15], i.e., $\mathrm{Sc}(qrs) = \mathrm{Sc}(sqr) = \mathrm{Sc}(rsq),\ \forall\, q, r, s \in \mathbb{H}$.
- the quaternion conjugate $\bar{s} = s_0 - is_1 - js_2 - ks_3$, which satisfies $\overline{rs} = \bar{s}\bar{r},\ \overline{r+s} = \bar{r} + \bar{s},\ \bar{\bar{s}} = s,\ \forall\, r, s \in \mathbb{H}$.
- the modulus $|s| = \sqrt{s\bar{s}} = \left(\sum_{l=0}^{3} s_l^2\right)^{\frac{1}{2}}$, which satisfies $|rs| = |r||s|,\ \forall\, r, s \in \mathbb{H}$.

A $\mathbb{H}$-valued function g defined on $\mathbb{R}^n$ is of the form $g(\boldsymbol{t}) = g_0(\boldsymbol{t}) + ig_1(\boldsymbol{t}) + jg_2(\boldsymbol{t}) + kg_3(\boldsymbol{t}),\ \boldsymbol{t} \in \mathbb{R}^n$, where g_0, g_1, g_2 and g_3 are real valued. The L^q-norm, $1 \le q < \infty$, of g is defined by

$$\|g\|_{L^q_{\mathbb{H}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g(\boldsymbol{t})|^q \, d\boldsymbol{t}\right)^{\frac{1}{q}} \tag{2}$$

and the collection of all measurable $\mathbb{H}$-valued functions having finite L^q-norm is a Banach space denoted by $L^q_{\mathbb{H}}(\mathbb{R}^n)$. $L^\infty_{\mathbb{H}}(\mathbb{R}^n)$ is the set of all essentially bounded measurable $\mathbb{H}$-valued functions with norm

$$\|g\|_{L^\infty_{\mathbb{H}}(\mathbb{R}^n)} = \text{ess sup}_{\boldsymbol{t}\in\mathbb{R}^n} |g(\boldsymbol{t})|. \tag{3}$$

Moreover, the $\mathbb{H}$-valued inner product

$$(g, h) = \int_{\mathbb{R}^n} g(\boldsymbol{t})\overline{h(\boldsymbol{t})}\, d\boldsymbol{t}, \tag{4}$$

with symmetric real scalar part $\langle g,h\rangle = \mathrm{Sc}\left(\int_{\mathbb{R}^n} g(\boldsymbol{t})\overline{h(\boldsymbol{t})}\,d\boldsymbol{t}\right)$ makes $L^2_{\mathbb{H}}(\mathbb{R}^n)$ a Hilbert space, with the norm given in (2) expressed as

$$\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^n)} = \sqrt{\langle g,g\rangle} = \sqrt{(g,g)} = \left(\int_{\mathbb{R}^n} |g(\boldsymbol{t})|^2\,d\boldsymbol{t}\right)^{\frac{1}{2}}. \tag{5}$$

3. Quaternion Coupled Fractional Fourier transform (QCFrFT)

In this section, we define two-sided QCFrFT and look at some of its key characteristics.

Definition 3.1. Let $\boldsymbol{\alpha} = (\theta_1,\theta_2), \boldsymbol{\beta} = (\theta_1',\theta_2') \in \mathbb{R}^2$ such that $\theta_1+\theta_1',\ \theta_2+\theta_2' \notin 2\pi\mathbb{Z}$. The QCFrFT of $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$, $\boldsymbol{x} = (\boldsymbol{x}_1,\boldsymbol{x}_2) \in \mathbb{R}^2\times\mathbb{R}^2$, is defined by

$$(\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}} f)(\boldsymbol{\omega}) = \int_{\mathbb{R}^4} \mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1) f(\boldsymbol{x}) \mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}, \tag{6}$$

$$\boldsymbol{\omega} = (\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) \in \mathbb{R}^2\times\mathbb{R}^2$$

where
$$\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1) = \tilde{d}(\gamma_1) e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}} \tag{7}$$
and
$$\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2) = \tilde{d}(\gamma_2) e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}, \tag{8}$$

with $\gamma_1=\frac{\theta_1+\theta_1'}{2}$, $\delta_1=\frac{\theta_1-\theta_1'}{2}$, $\tilde{a}(\gamma_1)=\frac{\cos\gamma_1}{2}$, $\tilde{b}(\gamma_1,\delta_1)=\frac{\cos\delta_1}{\sin\gamma_1}$, $\tilde{c}(\gamma_1,\delta_1)=\frac{\sin\delta_1}{\sin\gamma_1}$, $\tilde{d}(\gamma_1)=\frac{ie^{-i\gamma_1}}{2\pi\sin\gamma_1}$, $M_1=\begin{pmatrix}\tilde{b}(\gamma_1,\delta_1) & \tilde{c}(\gamma_1,\delta_1)\\ -\tilde{c}(\gamma_1,\delta_1) & \tilde{b}(\gamma_1,\delta_1)\end{pmatrix}$ and $\gamma_2=\frac{\theta_2+\theta_2'}{2}$, $\delta_2=\frac{\theta_2-\theta_2'}{2}$, $\tilde{a}(\gamma_2)=\frac{\cos\gamma_2}{2}$, $\tilde{b}(\gamma_2,\delta_2)=\frac{\cos\delta_2}{\sin\gamma_2}$, $\tilde{c}(\gamma_2,\delta_2)=\frac{\sin\delta_2}{\sin\gamma_2}$, $\tilde{d}(\gamma_2)=\frac{je^{-j\gamma_2}}{2\pi\sin\gamma_2}$, $M_2=\begin{pmatrix}\tilde{b}(\gamma_2,\delta_2) & \tilde{c}(\gamma_2,\delta_2)\\ -\tilde{c}(\gamma_2,\delta_2) & \tilde{b}(\gamma_2,\delta_2)\end{pmatrix}$.

The corresponding inversion formula is given by

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^4} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)} (\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}} f)(\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)}\,d\boldsymbol{\omega}. \tag{9}$$

Remark 3.2. For $\boldsymbol{\alpha} = \boldsymbol{\beta}$, the kernels $\mathcal{K}^i_{\theta_1,\theta_1}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)$ and $\mathcal{K}^j_{\theta_2,\theta_2}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)$ are the tensor products of two one-dimensional FrFT kernels. Thus, the QCFrFT reduces to the two sided QFrFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$. Moreover, if $\boldsymbol{\alpha} = \boldsymbol{\beta} = (\frac{\pi}{2},\frac{\pi}{2})$, the QCFrFT reduces to the two sided QFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

3.1 QCFrFT in terms of QFT

We now obtain an important relation between QCFrFT and QFT.

$$(\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}} f)(\boldsymbol{\omega}) = \int_{\mathbb{R}^4} \mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1) f(\boldsymbol{x}) \mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}$$
$$= \tilde{d}_0(\gamma_1) e^{-i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2} \left\{\int_{\mathbb{R}^4} \frac{1}{2\pi} e^{i\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1} \tilde{f}(\boldsymbol{x}) \frac{1}{2\pi} e^{j\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2}\,d\boldsymbol{x}\right\} \tilde{d}_0(\gamma_2) e^{-j\tilde{a}(\gamma_2)|\boldsymbol{\omega}_2|^2},$$

where $\tilde{d}_0(\gamma_1) = \frac{ie^{-i\gamma_1}}{\sin\gamma_1}$, $\tilde{d}_0(\gamma_2) = \frac{je^{-j\gamma_2}}{\sin\gamma_2}$ and

$$\tilde{f}(\boldsymbol{x}) = e^{-i\tilde{a}(\gamma_1)|\boldsymbol{x}_1|^2} f(\boldsymbol{x}) e^{-j\tilde{a}(\gamma_2)|\boldsymbol{x}_2|^2}. \tag{10}$$

Thus,

$$(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f)(\boldsymbol{\omega}) = \tilde{d}_0(\gamma_1) e^{-i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2} \left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(-M_1\boldsymbol{\omega}_1, -M_2\boldsymbol{\omega}_2) \tilde{d}_0(\gamma_2) e^{-j\tilde{a}(\gamma_2)|\boldsymbol{\omega}_2|^2}, \tag{11}$$

where
$$\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(\boldsymbol{\omega}) = \int_{\mathbb{R}^4} \frac{1}{2\pi} e^{-i\boldsymbol{x}_1\cdot\boldsymbol{\omega}_1} \tilde{f}(\boldsymbol{x}) \frac{1}{2\pi} e^{-j\boldsymbol{x}_2\cdot\boldsymbol{\omega}_2}\, d\boldsymbol{x} \tag{12}$$

is the quaternion Fourier transform of the function $\tilde{f} \in \mathbb{R}^4$.

Example 3.3 (of QCFrFT). With the assumption that $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^2$ satisfy the conditions in Definition 3.1, we see that the QCFrFT $\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ of the function $f(\boldsymbol{x}) = e^{i\tilde{a}(\gamma_1)|\boldsymbol{x}_1|^2} e^{-A|\boldsymbol{x}_1|^2} e^{-B|\boldsymbol{x}_2|^2} e^{j\tilde{a}(\gamma_2)|\boldsymbol{x}_2|^2}$, $A, B > 0$ is given as

$$\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega}) = \frac{1}{4AB} \tilde{d}_0(\gamma_1) e^{-i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2} e^{-\frac{1}{4}\left(\frac{|M_1\boldsymbol{\omega}_1|^2}{A} + \frac{|M_2\boldsymbol{\omega}_2|^2}{B}\right)} \tilde{d}_0(\gamma_2) e^{-j\tilde{a}(\gamma_2)|\boldsymbol{\omega}_2|^2},$$

where M_1 and M_2 are the matrices given in Definition 3.1. This can be proved using the relation (11) and the fact that

$$\begin{aligned}\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(\boldsymbol{\omega}) &= \left\{\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\boldsymbol{x}_1\cdot\boldsymbol{\omega}_1} e^{-|\sqrt{A}\boldsymbol{x}_1|^2}\, d\boldsymbol{x}_1\right\}\left\{\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-|\sqrt{B}\boldsymbol{x}_2|^2} e^{-j\boldsymbol{x}_2\cdot\boldsymbol{\omega}_2}\, d\boldsymbol{x}_2\right\} \\ &= \left(\frac{1}{2\pi A}\pi e^{-\frac{|\boldsymbol{\omega}_1|^2}{4A}}\right)\left(\frac{1}{2\pi B}\pi e^{-\frac{|\boldsymbol{\omega}_2|^2}{4B}}\right) = \frac{1}{4AB} e^{-\frac{1}{4}\left(\frac{|\boldsymbol{\omega}_1|^2}{A} + \frac{|\boldsymbol{\omega}_2|^2}{B}\right)},\end{aligned}$$

where $\tilde{f}$ is given by (10).

We now obtain the following important inequality, called the H-Y inequality, based on the relation (11) among QCFrFT and the QFT.

Theorem 3.4. *If $f \in L^p_{\mathbb{H}}(\mathbb{R}^4)$ and $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\|\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^4)} \le \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1} \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1} A_p^4 \|f\|_{L^p_{\mathbb{H}}(\mathbb{R}^4)}, \tag{13}$$

where $A_p = \left(\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\right)^{\frac{1}{2}}$.

Proof. Using relation (11), we get

$$\begin{aligned}\|\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^4)} &= |\tilde{d}_0(\gamma_1)||\tilde{d}_0(\gamma_2)| \left(\int_{\mathbb{R}^4} \left|\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(-M_1\boldsymbol{\omega}_1, -M_2\boldsymbol{\omega}_2)\right|^q d\boldsymbol{\omega}\right)^{\frac{1}{q}} \\ &= \frac{|\tilde{d}_0(\gamma_1)||\tilde{d}_0(\gamma_2)|}{(|\det(-M_1)||\det(-M_2)|)^{\frac{1}{q}}} \|\mathcal{F}_{\mathbb{H}}\tilde{f}\|_{L^q_{\mathbb{H}}(\mathbb{R}^4)}.\end{aligned}$$

Applying the sharp Hausdorff-Young inequality [25] for the QFT, yields

$$\|\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^4)} \le \frac{|\tilde{d}_0(\gamma_1)||\tilde{d}_0(\gamma_2)|A_p^4}{(|\det(-M_1)||\det(-M_2)|)^{\frac{1}{q}}} \|\tilde{f}\|_{L^p_{\mathbb{H}}(\mathbb{R}^4)}$$

$$= \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1} A_p^4\|\tilde{f}\|_{L^p_{\mathbb{H}}(\mathbb{R}^4)}.$$

By virtue of (10) we obtain (13). This finishes the proof. □

Remark 3.5.
- For $\boldsymbol{\alpha}=\boldsymbol{\beta}$, equation (13) reduces to the sharp Hausdorff-Young inequality for the QFrFT of the function $f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$.
- For $\boldsymbol{\alpha}=\boldsymbol{\beta}=(\frac{\pi}{2},\frac{\pi}{2})$, equation (13) reduces to the sharp Hausdorff-Young inequality for the QFT of the function $f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

In what follows, we obtain the Parseval's formula associated with the QCFrFT.

Theorem 3.6. *If $f,g\in L^2_{\mathbb{H}}(\mathbb{R}^4)$, then*

$$\langle \mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f, \mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} g\rangle = \langle f,g\rangle. \tag{14}$$

In particular,

$$\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)} = \|\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}. \tag{15}$$

Proof. Applying Parseval's formula for the QFT, we get

$$\begin{aligned}\langle \tilde{f},\tilde{g}\rangle &= \langle \mathcal{F}_{\mathbb{H}}\tilde{f}, \mathcal{F}_{\mathbb{H}}\tilde{g}\rangle\\ &= \mathrm{Sc}\left[|\det(-M_1)||\det(-M_2)|\int_{\mathbb{R}^4}\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(-M_1\boldsymbol{\omega}_1,-M_2\boldsymbol{\omega}_2)\overline{(\mathcal{F}_{\mathbb{H}}\tilde{g})\,(-M_1\boldsymbol{\omega}_1,-M_2\boldsymbol{\omega}_2)}\,d\boldsymbol{\omega}\right].\end{aligned}$$

Using relation (11), we have

$$\begin{aligned}\langle \tilde{f},\tilde{g}\rangle &= \frac{|\det(-M_1)||\det(-M_2)|}{|\tilde{d}_0(\gamma_2)|^2}\int_{\mathbb{R}^4}\mathrm{Sc}\left[\frac{e^{i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2}}{\tilde{d}_0(\gamma_1)}\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{\omega})\overline{\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}g\right)(\boldsymbol{\omega})}\frac{e^{-i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2}}{\overline{\tilde{d}_0(\gamma_1)}}\right]d\boldsymbol{\omega}\\ &= \frac{|\det(-M_1)||\det(-M_2)|}{|\tilde{d}_0(\gamma_2)|^2|\tilde{d}_0(\gamma_1)|^2}\int_{\mathbb{R}^4}\mathrm{Sc}\left[\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{\omega})\overline{\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}g\right)(\boldsymbol{\omega})}\right]d\boldsymbol{\omega} = \langle \mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f,\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}g\rangle.\end{aligned}$$

Again, using equation (10), it can be shown that $\langle\tilde{f},\tilde{g}\rangle=\langle f,g\rangle$. Hence (14) follows. With $f=g$ in (14), we have (15). □

3.2 Rènyi entropy uncertainty principle

The Rènyi entropy UPs for the proposed QCFrFT is obtained in this subsection. Similar findings for the complex FrFT can be seen in [13]. These UPs for the QPFT and two sided quaternion QPFT have recently been discovered in [7, 14, 29], respectively. We recall the following.

Definition 3.7 ([10,13]). If P is a probability density function on $\mathbb{R}^n$, then the Rènyi entropy of P is defined by

$$H_s(P) = \frac{1}{1-s}\log\left(\int_{\mathbb{R}^n}[P(\boldsymbol{x})]^s\,d\boldsymbol{x}\right),\ \ s>0, s\neq 1. \tag{16}$$

If $s\to 1$, then (16) results in the Shannon entropy given by

$$E(P) = -\int_{\mathbb{R}^n} P(\boldsymbol{x})\log[P(\boldsymbol{x})]\,d\boldsymbol{x}. \tag{17}$$

In what follows, we obtain the Rènyi entropy UP for QCFrFT.

THEOREM 3.8. *If $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$, $\frac{1}{2} < \alpha_0 < 1$ and $\frac{1}{\alpha_0} + \frac{1}{\beta_0} = 2$, then*

$$H_{\alpha_0}(|f|^2) + H_{\beta_0}\left(\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega})\right|^2\right) \geq$$
$$\frac{2}{\alpha_0 - 1}\log(2\alpha_0) + \frac{2}{\beta_0 - 1}\log(2\beta_0) + 2\log\left(\left|\sin\left(\frac{\theta_1 + \theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2 + \theta_2'}{2}\right)\right|\right).$$

Proof. By inequality (13), we have

$$\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega})\right|^q d\boldsymbol{\omega}\right)^{\frac{1}{q}}$$
$$\leq \left|\sin\left(\frac{\theta_1 + \theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2 + \theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{\frac{1}{p}}. \tag{18}$$

Putting $p = 2\alpha_0$ and $q = 2\beta_0$, in equation (18), we have

$$\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega})\right|^{2\beta_0} d\boldsymbol{\omega}\right)^{\frac{1}{2\beta_0}}$$
$$\leq \left|\sin\left(\frac{\theta_1 + \theta_1'}{2}\right)\right|^{\frac{1}{\beta_0}-1}\left|\sin\left(\frac{\theta_2 + \theta_2'}{2}\right)\right|^{\frac{1}{\beta_0}-1}\left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^{2\alpha_0} d\boldsymbol{x}\right)^{\frac{1}{2\alpha_0}}.$$

This implies

$$\frac{\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{2-\frac{2}{\beta_0}}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{2-\frac{2}{\beta_0}}}{A_{2\alpha_0}^8}$$
$$\leq \left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^{2\alpha_0} d\boldsymbol{x}\right)^{\frac{1}{\alpha_0}}\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega})\right|^{2\beta_0} d\boldsymbol{\omega}\right)^{-\frac{1}{\beta_0}}. \tag{19}$$

Since $\frac{1}{\alpha_0} + \frac{1}{\beta_0} = 2$, we have

$$\frac{\alpha_0}{1 - \alpha_0} = \frac{\beta_0}{\beta_0 - 1}. \tag{20}$$

Raising to the power $\frac{\alpha_0}{1-\alpha_0}$ in (19) and using (20), we get

$$\frac{\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{2}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{2}}{\left(A_{2\alpha_0}^8\right)^{\frac{\alpha_0}{1-\alpha_0}}}$$
$$\leq \left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^{2\alpha_0} d\boldsymbol{x}\right)^{\frac{1}{1-\alpha_0}}\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{\omega})\right|^{2\beta_0} d\boldsymbol{\omega}\right)^{\frac{1}{1-\beta_0}}.$$

This leads to

$$2\log\left(\left|\sin\left(\frac{\theta_1 + \theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2 + \theta_2'}{2}\right)\right|\right) - \frac{8\alpha_0}{1 - \alpha_0}\log\left(A_{2\alpha_0}\right)$$
$$\leq \frac{1}{1 - \alpha_0}\log\left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^{2\alpha_0} d\boldsymbol{x}\right) + \frac{1}{1 - \beta_0}\log\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right)(\boldsymbol{\omega})\right|^{2\beta_0} d\boldsymbol{\omega}\right).$$

This implies

$$H_{\alpha_0}(|f|^2)+H_{\beta_0}\left(\left|\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right|^2\right)$$
$$\geq \frac{2}{\alpha_0-1}\log(2\alpha_0)+\frac{2}{\beta_0-1}\log(2\beta_0)+2\log\left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right). \quad (21)$$

Thus the proof is complete. □

REMARK 3.9. If $\alpha_0 \to 1$, then $\beta_0 \to 1$ and thus (21) can be written as

$$E(|f|^2)+E\left(\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{\omega})\right|^2\right)$$
$$\geq 2\log\left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)+2(2-\log 4),$$

i.e.,

$$E(|f|^2)+E\left(\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{\omega})\right|^2\right)$$
$$\geq 2\log\left(\frac{e^2}{4}\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right). \quad (22)$$

For QCFrFT, inequality (22) is the Shannon entropy UP.

4. Short time quaternion coupled fractional Fourier transform

The two sided short time quaternion coupled fractional Fourier transform (STQCFrFT) is defined and its properties are examined in this section.

DEFINITION 4.1. Let $\boldsymbol{\alpha}=(\theta_1,\theta_2), \boldsymbol{\beta}=(\theta_1',\theta_2')\in\mathbb{R}^2$, such that $\theta_1+\theta_1',\ \theta_2+\theta_2'\notin 2\pi\mathbb{Z}$. The STQCFrFT of a function $f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$ with respect to $g\in L^2_{\mathbb{H}}(\mathbb{R}^4)\cap L^\infty_{\mathbb{H}}(\mathbb{R}^4)$, called a quaternion window function (QWF), is defined by

$$\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})=\int_{\mathbb{R}^4}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)f(\boldsymbol{x})\overline{g(\boldsymbol{x}-\boldsymbol{t})}\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}, \quad (23)$$
$$(\boldsymbol{t},\boldsymbol{\omega})\in\mathbb{R}^4\times\mathbb{R}^4,$$

where $\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)$ and $\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)$ are given respectively by (7) and (8).

It is to be noted that if the kernels $\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)$ and $\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)$ both lie to the left or to the right of $f(\boldsymbol{x})\overline{g(\boldsymbol{x}-\boldsymbol{t})}$ in the integral in (23), then we have the left or right sided STQCFrFT. But our study mainly focuses on the two sided STQCFrFT.

REMARK 4.2. For $\boldsymbol{\alpha}=\boldsymbol{\beta}$, the kernels $\mathcal{K}^i_{\theta_1,\theta_1}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)$ and $\mathcal{K}^j_{\theta_2,\theta_2}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)$ are the tensor products of two one-dimensional FrFT. Thus, the STQCFrFT reduces to the STQFrFT of the function $f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$. Moreover, if $\boldsymbol{\alpha}=\boldsymbol{\beta}=(\frac{\pi}{2},\frac{\pi}{2})$, the STQCFrFT reduces to the STQFT of the function $f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

We mention below a lemma that will be used in proving some basic properties of the STQCFrFT.

Lemma 4.3. *Let* $\boldsymbol{k}{=}(\boldsymbol{k}_1,\boldsymbol{k}_2)$, $\boldsymbol{\omega}{=}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$, $\boldsymbol{x}{=}(\boldsymbol{x}_1,\boldsymbol{x}_2)\in\mathbb{R}^2{\times}\mathbb{R}^2$*. Then* $\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)$ *and* $\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)$ *satisfy the following*

$$\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1+\boldsymbol{k}_1,\boldsymbol{\omega}_1)=e^{-i\{\tilde{a}(\gamma_1)(|\boldsymbol{k}_1|+2\boldsymbol{x}_1\cdot\boldsymbol{k}_1)-\boldsymbol{k}_1\cdot M_1\boldsymbol{\omega}_1\}}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1), \tag{24}$$

and

$$\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2+\boldsymbol{k}_2,\boldsymbol{\omega}_2)=e^{-j\{\tilde{a}(\gamma_2)(|\boldsymbol{k}_2|+2\boldsymbol{x}_2\cdot\boldsymbol{k}_2)-\boldsymbol{k}_2\cdot M_2\boldsymbol{\omega}_2\}}\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2). \tag{25}$$

Proof. From the definition of $\mathcal{K}^i_{\theta_1,\theta_1'}$, we have

$$\begin{aligned}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1+\boldsymbol{k}_1,\boldsymbol{\omega}_1)&=\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)\left(|\boldsymbol{x}_1+\boldsymbol{k}_1|^2+|\boldsymbol{\omega}_1|^2\right)-(\boldsymbol{x}_1+\boldsymbol{k}_1)\cdot M_1\boldsymbol{\omega}_1\right\}}\\&=\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)\left(|\boldsymbol{x}_1|^2+|\boldsymbol{k}_1|^2+2\boldsymbol{x}_1\cdot\boldsymbol{k}_1+|\boldsymbol{\omega}_1|^2\right)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1-\boldsymbol{k}_1\cdot M_1\boldsymbol{\omega}_1\right\}}\\&=\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)\left(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2\right)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-i\left\{\tilde{a}(\gamma_1)\left(|\boldsymbol{k}_1|^2+2\boldsymbol{x}_1\cdot\boldsymbol{\omega}_1\right)-\boldsymbol{k}_1\cdot M_1\boldsymbol{\omega}_1\right\}},\end{aligned}$$

i.e., $$\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1+\boldsymbol{k}_1,\boldsymbol{\omega}_1)=e^{-i\{\tilde{a}(\gamma_1)(|\boldsymbol{k}_1|+2\boldsymbol{x}_1\cdot\boldsymbol{k}_1)-\boldsymbol{k}_1\cdot M_1\boldsymbol{\omega}_1\}}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1).$$

This proves (24). (25) follows similarly. □

The proposed STQCFrFT enjoys the following basic properties.

Theorem 4.4. *Let* $f_1,f_2,f\in L^2_{\mathbb{H}}(\mathbb{R}^4)$ *and* $g_1,g,g_2\in L^\infty_{\mathbb{H}}(\mathbb{R}^4)\cap L^2_{\mathbb{H}}(\mathbb{R}^4)$ *be QWFs. Then*

(i) Boundedness: $\left\|\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right\|_{L^\infty_{\mathbb{H}}(\mathbb{R}^4)}\le\dfrac{1}{4\pi^2\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.$

(ii) Linearity: $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}(pf_1+qf_2)=p\left[\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f_1\right]+q\left[\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f_2\right],\ p,q\in\{t+iy:t,y\in\mathbb{R}\}.$

(iii) Anti-linearity: $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},rg_1+sg_2}f=\left[\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f\right]\bar{r}+\left[\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}f\right]\bar{s},\ r,s\in\{t+jy:t,y\in\mathbb{R}\}.$

(iv) Translation: $\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}(\tau_{\boldsymbol{l}}f)\right)(\boldsymbol{t},\boldsymbol{\omega})=e^{-i\{\tilde{a}(\gamma_1)(|\boldsymbol{l}_1|+2\boldsymbol{x}_1\cdot\boldsymbol{l}_1)-\boldsymbol{l}_1\cdot M_1\boldsymbol{\omega}_1\}}\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(\boldsymbol{t}{-}\boldsymbol{l},\boldsymbol{\omega})$ $e^{-j\{\tilde{a}(\gamma_2)(|\boldsymbol{l}_2|+2\boldsymbol{x}_2\cdot\boldsymbol{l}_2)-\boldsymbol{l}_2\cdot M_2\boldsymbol{\omega}_2\}}$*, where* $(\tau_{\boldsymbol{l}}f)(\boldsymbol{x})=f(\boldsymbol{x}-\boldsymbol{l}),\ \boldsymbol{l}=(\boldsymbol{l}_1,\boldsymbol{l}_2)\in\mathbb{R}^2\times\mathbb{R}^2.$

(v) Parity: $\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},Pg}Pf\right)(\boldsymbol{t},\boldsymbol{\omega})=\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(-\boldsymbol{t},-\boldsymbol{\omega})$*, where* $(Pf)(\boldsymbol{x})=f(-\boldsymbol{x}).$

Proof. We skip the proof of (i), (ii) and (iii) as they are straightforward.

(iv) From Definition 4.1, we have

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}(\tau_{\boldsymbol{l}}f)\right)(\boldsymbol{t},\boldsymbol{\omega})=\int_{\mathbb{R}^4}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1+\boldsymbol{l}_1,\boldsymbol{\omega}_1)f(\boldsymbol{x})\overline{g(\boldsymbol{x}-(\boldsymbol{t}-\boldsymbol{l}))}\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2+\boldsymbol{l}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}.$$

By Lemma 4.3, we see that

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}(\tau_{\boldsymbol{l}}f)\right)(\boldsymbol{t},\boldsymbol{\omega})=e^{-i\{\tilde{a}(\gamma_1)(|\boldsymbol{l}_1|+2\boldsymbol{x}_1\cdot\boldsymbol{l}_1)-\boldsymbol{l}_1\cdot M_1\boldsymbol{\omega}_1\}}$$
$$\left\{\int_{\mathbb{R}^4}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)f(\boldsymbol{x})\overline{g(\boldsymbol{x}-(\boldsymbol{t}-\boldsymbol{l}))}\mathcal{K}^j_{\theta_1,\theta_1'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}\right\}e^{-j\{\tilde{a}(\gamma_2)(|\boldsymbol{l}_2|+2\boldsymbol{x}_2\cdot\boldsymbol{l}_2)-\boldsymbol{l}_2\cdot M_2\boldsymbol{\omega}_2\}}.$$

Thus, we have

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}(\tau_{\boldsymbol{l}}f)\right)(\boldsymbol{t},\boldsymbol{\omega})=$$
$$e^{-i\{\tilde{a}(\gamma_1)(|\boldsymbol{l}_1|+2\boldsymbol{x}_1\cdot\boldsymbol{l}_1)-\boldsymbol{l}_1\cdot M_1\boldsymbol{\omega}_1\}}\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(\boldsymbol{t}-\boldsymbol{l},\boldsymbol{\omega})e^{-j\{\tilde{a}(\gamma_2)(|\boldsymbol{l}_2|+2\boldsymbol{x}_2\cdot\boldsymbol{l}_2)-\boldsymbol{l}_2\cdot M_2\boldsymbol{\omega}_2\}}.$$

This proves (iv).

(v) Using the definition of STQCFrFT, we have

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},Pg}Pf\right)(\boldsymbol{t},\boldsymbol{\omega}) = \int_{\mathbb{R}^4} \mathcal{K}^i_{\theta_1,\theta_1'}(-\boldsymbol{x}_1,\boldsymbol{\omega}_1)f(\boldsymbol{x})\overline{g\,(\boldsymbol{x}+\boldsymbol{t})}\mathcal{K}^j_{\theta_2,\theta_2'}(-\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x}. \tag{26}$$

Now, it can be shown that

$$\mathcal{K}^i_{\theta_1,\theta_1'}(-\boldsymbol{x}_1,\boldsymbol{\omega}_1) = \mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,-\boldsymbol{\omega}_1) \tag{27}$$

and

$$\mathcal{K}^j_{\theta_2,\theta_2'}(-\boldsymbol{x}_2,\boldsymbol{\omega}_2) = \mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,-\boldsymbol{\omega}_2). \tag{28}$$

By virtue of (27), (28) and (26), we have

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},Pg}Pf\right)(\boldsymbol{t},\boldsymbol{\omega}) = \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(-\boldsymbol{t},-\boldsymbol{\omega}).$$

Thus the proof is complete. □

Theorem 4.5. *(Inner product relation) If $f,h\in L^2_{\mathbb{H}}(\mathbb{R}^4)$ and g_1,g_2 are two QWFs, then $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f$, $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}f\in L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)$ and*

$$\left\langle \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f, \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}h\right\rangle = \langle f(\overline{g_1},\overline{g_2}),h\rangle. \tag{29}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left|\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2 d\boldsymbol{t}\,d\boldsymbol{\omega} &= \int_{\mathbb{R}^4}\left\{\int_{\mathbb{R}^4}\left|\left(\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}}\{f(\cdot)\overline{g_1(\cdot-\boldsymbol{t})}\}\right)(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right\}d\boldsymbol{t}\\ &= \int_{\mathbb{R}^4}\left\{\int_{\mathbb{R}^4}|f(\boldsymbol{x})\overline{g_1(\boldsymbol{x}-\boldsymbol{t})}|^2\,d\boldsymbol{x}\right\}d\boldsymbol{t},\ \text{using Parseval's Identity}\\ &= \|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|g_1\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.\end{aligned}$$

Thus, $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f\in L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)$. Similarly, $\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}h\in L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)$.

Now,

$$\begin{aligned}\left\langle \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f, \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}h\right\rangle &= \mathrm{Sc}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left(\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}}\{f(\cdot)\overline{g_1(\cdot-\boldsymbol{t})}\}\right)(\boldsymbol{\omega})\overline{\left(\mathcal{F}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H}}\{h(\cdot)\overline{g_2(\cdot-\boldsymbol{t})}\}\right)(\boldsymbol{\omega})}\,d\boldsymbol{t}\,d\boldsymbol{\omega}\\ &= \mathrm{Sc}\int_{\mathbb{R}^4}\left\{\int_{\mathbb{R}^4}f(\boldsymbol{x})\overline{g_1(\boldsymbol{x}-\boldsymbol{t})}\;\overline{h(\boldsymbol{x})\overline{g_2(\boldsymbol{x}-\boldsymbol{t})}}\,d\boldsymbol{x}\right\}d\boldsymbol{t}\\ &= \mathrm{Sc}\int_{\mathbb{R}^4}f(\boldsymbol{x})\left(\int_{\mathbb{R}^4}\overline{g_1(\boldsymbol{x}-\boldsymbol{t})}g_2(\boldsymbol{x}-\boldsymbol{t})\,d\boldsymbol{t}\right)\overline{h(\boldsymbol{x})}\,d\boldsymbol{x}\\ &= \mathrm{Sc}\int_{\mathbb{R}^4}f(\boldsymbol{x})\,(\overline{g_1},\overline{g_2})\,\overline{h(\boldsymbol{x})}\,d\boldsymbol{x}.\end{aligned}$$

This implies $\left\langle \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1}f, \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2}h\right\rangle = \langle f\,(\overline{g_1},\overline{g_2})\,,h\rangle$. □

Remark 4.6. In view of (29), we can conclude that

1. If $g_1=g=g_2$ in (29), then

$$\left\langle \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f, \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}h\right\rangle = \langle f,h\rangle\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.$$

2. If $f = h$ in (29), then
$$\left\langle \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_1} f, \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g_2} f \right\rangle = \langle g_1, g_2 \rangle \|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.$$

3. If $g = g_1 = g_2$ and $f = h$ in (29), then
$$\|\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} = \|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)} \|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}. \tag{30}$$

The STQCFrFT reconstruction formula is given by the subsequent theorem.

Theorem 4.7. *If g is a QWF and $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$, then*
$$f(\boldsymbol{x}) = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)} \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)} g(\boldsymbol{x}-\boldsymbol{t})\, d\boldsymbol{t}\, d\boldsymbol{\omega}.$$

Proof. Using Theorem 4.5, we see that

$$\begin{aligned}
\langle f, h\rangle &= \mathrm{Sc} \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\left\{\int_{\mathbb{R}^4} \mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1) h(\boldsymbol{x}) \overline{g(\boldsymbol{x}-\boldsymbol{t})} \mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\, d\boldsymbol{x}\right\}}\, d\boldsymbol{t}\, d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \mathrm{Sc}\left\{\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)} g(\boldsymbol{x}-\boldsymbol{t}) \overline{h(\boldsymbol{x})} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)}\right\} d\boldsymbol{x}\, d\boldsymbol{t}\, d\boldsymbol{\omega} \\
&= \mathrm{Sc} \int_{\mathbb{R}^4} \left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)} \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)} g(\boldsymbol{x}-\boldsymbol{t})\, d\boldsymbol{t}\, d\boldsymbol{\omega}\right\} \overline{h(\boldsymbol{x})}\, d\boldsymbol{x} \\
&= \left\langle \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)} \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)} g(\cdot-\boldsymbol{t})\, d\boldsymbol{t}\, d\boldsymbol{\omega}, h(\cdot)\right\rangle.
\end{aligned}$$

Since $h \in L^2_{\mathbb{H}}(\mathbb{R}^4)$ is arbitrary, it follows that
$$f(\boldsymbol{x}) = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \overline{\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)} \left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega}) \overline{\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)} g(\boldsymbol{x}-\boldsymbol{t})\, d\boldsymbol{t}\, d\boldsymbol{\omega}.$$
This finishes the proof. □

4.1 Uncertainty principle for STQCFrFT

Similar to Heisenberg UP, which governs the localization of a function and its FT, Wilczok [32] has introduced a new form of UP that compares the localization of both a function and its windowed FT.

Here we give Lieb's UP for the STQCFrFT. In our recent work [14], we obtained an analogous result for the short-time quaternion QPFT. We first obtain Lieb's inequality for the STQCFrFT.

Lemma 4.8. *Let $2 \le q < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$ and g be a QWF. Then*
$$\begin{aligned}
&\left\|\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right\|_{L^q_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} \le \\
&\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1} \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1} \left(\frac{2}{q}\right)^{\frac{4}{q}} \|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)} \|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.
\end{aligned} \tag{31}$$

Proof. Using the definition 4.1, we have

$$\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{\omega}\right)^{\frac{1}{q}}=\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{F}_{\mathbb{H}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\{f(\cdot)\overline{g(\cdot-\boldsymbol{t})}\}\right)(\boldsymbol{\omega})\right|^q d\boldsymbol{\omega}\right)^{\frac{1}{q}}. \tag{32}$$

Using the Hausdorff-Young inequality, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{\omega}\right)^{\frac{1}{q}}\\
&\leq A_p^4\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left(\int_{\mathbb{R}^4}\left|f(\boldsymbol{x})\overline{g(\boldsymbol{x}-\boldsymbol{t})}\right|^p d\boldsymbol{x}\right)^{\frac{1}{p}}\\
&=A_p^4\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left(\int_{\mathbb{R}^4}|f(\boldsymbol{x})|^p|\tilde{g}(\boldsymbol{t}-\boldsymbol{x})|^p d\boldsymbol{x}\right)^{\frac{1}{p}},\\
&=A_p^4\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{t})\right\}^{\frac{1}{p}},
\end{aligned}$$

where $\tilde{g}(\boldsymbol{x})=g(-\boldsymbol{x})$ and $\frac{1}{p}+\frac{1}{q}=1$. This leads to

$$\begin{aligned}
&\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{t}\,d\boldsymbol{\omega}\leq\\
&A_p^{4q}\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{2-q}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{2-q}\int_{\mathbb{R}^4}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{t})\right\}^{\frac{q}{p}}\,d\boldsymbol{t}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{t}\,d\boldsymbol{\omega}\right\}^{\frac{1}{q}}\\
&\leq A_p^4\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left[\int_{\mathbb{R}^4}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{t})\right\}^{\frac{q}{p}}\,d\boldsymbol{t}\right]^{\frac{p}{q}\cdot\frac{1}{p}}\\
&=A_p^4\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\||f|^p\star|\tilde{g}|^p\|^{\frac{1}{p}}_{L_{\mathbb{H}}^{\frac{q}{p}}(\mathbb{R}^4)}.
\end{aligned}\tag{33}$$

We observe that if $l=\frac{q}{p},\ k=\frac{2}{p}$, then $k\geq 1$ and $1+\frac{1}{l}=\frac{1}{k}+\frac{1}{k}$. Since $|\tilde{g}|^p,\ |f|^p\in L_{\mathbb{H}}^k(\mathbb{R}^4)$, by Young's inequality [12], we have

$$\||f|^p\star|\tilde{g}|^p\|^{\frac{1}{p}}_{L_{\mathbb{H}}^{\frac{q}{p}}(\mathbb{R}^4)}\leq A_k^{\frac{8}{p}}A_{l'}^{\frac{4}{p}}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|\tilde{g}\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}. \tag{34}$$

From (33) and (34) it follows that

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}}f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{t}\,d\boldsymbol{\omega}\right\}^{\frac{1}{q}}\leq\\
&\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left(A_p^2A_k^{\frac{4}{p}}A_{l'}^{\frac{2}{p}}\right)^2\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)},
\end{aligned}\tag{35}$$

where $A_r = \left(\frac{r^{\frac{1}{r}}}{r'^{\frac{1}{r'}}}\right)^{\frac{1}{2}}$, $\frac{1}{r}+\frac{1}{r'}=1$. Now we have

$$\begin{aligned} A_p^2 A_k^{\frac{4}{p}} A_{l'}^{\frac{2}{p}} &= \frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}} \cdot \frac{k}{k'^{\frac{2}{k'p}}} \cdot \frac{l'^{\frac{1}{pl'}}}{\left(\frac{q}{p}\right)^{\frac{1}{q}}}, \text{ since } k=\frac{2}{q},\ l=\frac{q}{p} \\ &= \frac{p}{q^{\frac{2}{q}}} \cdot \frac{l'^{\frac{1}{pl'}}}{k'^{\frac{2}{k'p}}} = \frac{2}{q^{\frac{2}{q}}} \cdot \left(\frac{1}{2}\right)^{\frac{q-p}{pq}}, \text{ since } k'=2l' \\ &= \left(\frac{2}{q}\right)^{\frac{2}{q}}. \end{aligned} \tag{36}$$

So if we insert the equation (36) into (35), we get

$$\left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{t}\, d\boldsymbol{\omega}\right\}^{\frac{1}{q}} \leq$$
$$\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{2}{q}-1}\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{2}{q}-1}\left(\frac{2}{q}\right)^{\frac{4}{q}}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.$$

This completes the proof. □

Remark 4.9. For $\boldsymbol{\alpha}=\boldsymbol{\beta}$, the inequality (31) reduces to the Lieb's inequality for the STQFrFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$. Moreover, if $\boldsymbol{\alpha}=\boldsymbol{\beta}=(\frac{\pi}{2},\frac{\pi}{2})$, inequality (31) reduces to the Lieb's inequality for the STQFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

4.2 Lieb's uncertainty principle

Definition 4.10. Let $\Omega \subset \mathbb{R}^n$ be measurable. A function $G \in L^2_{\mathbb{H}}(\mathbb{R}^n)$ is ϵ-concentrated, $\epsilon \geq 0$, on Ω, if

$$\|\chi_{\Omega^c} G\|_{L^2_{\mathbb{H}}(\mathbb{R}^n)} \leq \epsilon \|G\|_{L^2_{\mathbb{H}}(\mathbb{R}^n)},$$

where χ_Ω takes the value 1 on Ω and 0 otherwise.

Theorem 4.11. *Let $0 \neq f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$ and $g \neq 0$ be a QWF. If $\epsilon \geq 0$ and $\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f$ is ϵ-concentrated on $\Omega \subset \mathbb{R}^4 \times \mathbb{R}^4$, then*

$$|\Omega| \geq \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^2\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^2(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\frac{8}{q-2}}, \ q>2 \tag{37}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Proof. Since $\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f$ is ϵ-concentrated on Ω, we have

$$\left\|\chi_{\Omega^c}\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} \leq \epsilon^2\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}.$$

This results in

$$\left\|\chi_{\Omega}\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} \geq (1-\epsilon^2)\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}. \tag{38}$$

Using Holder's inequality [16] with the exponents $\frac{q}{2}$ and $\frac{q}{q-2}$ we obtain

$$\left\|\chi_\Omega \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \chi_\Omega(\boldsymbol{t},\boldsymbol{\omega}) \left|\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2 d\boldsymbol{t}\, d\boldsymbol{\omega}$$

$$\leq \left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} (\chi_\Omega(\boldsymbol{t},\boldsymbol{\omega}))^{\frac{q}{q-2}}\, d\boldsymbol{t}\, d\boldsymbol{\omega}\right\}^{\frac{q-2}{q}} \left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \left(\left|\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\right)^{\frac{q}{2}} d\boldsymbol{t}\, d\boldsymbol{\omega}\right\}^{\frac{2}{q}}$$

$$= |\Omega|^{\frac{q-2}{q}} \left\|\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right\|^2_{L^q_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)}.$$

Using Lieb's inequality (31) we get

$$\left\|\chi_\Omega \mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4\times\mathbb{R}^4)} \leq$$

$$|\Omega|^{\frac{q-2}{q}} \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{4}{q}-2} \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{4}{q}-2} \left(\frac{2}{q}\right)^{\frac{8}{q}} \|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)} \|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^4)}. \tag{39}$$

From equation (38) and equation (39) we obtain

$$|\Omega|^{\frac{q-2}{q}} \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^{\frac{4}{q}-2} \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^{\frac{4}{q}-2} \left(\frac{2}{q}\right)^{\frac{8}{q}} \geq (1-\epsilon^2).$$

Since $\frac{1}{p}+\frac{1}{q}=1$, this results in

$$|\Omega| \geq \left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right| \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)^{2\left(1-\frac{2}{q}\right)\frac{q}{q-2}} (1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\frac{8}{q-2}},$$

i.e.,
$$|\Omega| \geq \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^2 \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^2 (1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\frac{8}{q-2}}.$$

This proves (37). □

Remark 4.12. If $\epsilon = 0$, from (37), we see that

$$\left|\operatorname{supp}\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},\mathrm{g}} f\right)\right| \geq \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^2 \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^2 \lim_{q\to 2+} \left(\frac{q}{2}\right)^{\frac{8}{q-2}}$$

i.e.,
$$\left|\operatorname{supp}\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},\mathrm{g}} f\right)\right| \geq \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^2 \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^2 e^4. \tag{40}$$

i.e, the measure $\operatorname{supp}\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},\mathrm{g}}\mathrm{f} \geq \left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|^2 \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|^2 \mathrm{e}^4$.

Remark 4.13. For $\boldsymbol{\alpha}=\boldsymbol{\beta}$, Theorem 4.11 gives Lieb's UP for the STQFrFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$. Moreover, if $\boldsymbol{\alpha}=\boldsymbol{\beta}=(\frac{\pi}{2},\frac{\pi}{2})$, Theorem 4.11 gives the Lieb's UP for the STQFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

4.3 Entropy uncertainty principle

Theorem 4.14. *Let $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$ and g be a QWF with $\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^4)} = 1$, then*

$$\mathcal{E}_S(f,g,\boldsymbol{\alpha},\boldsymbol{\beta}) \geq 2\left[2+\log\left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right| \left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)\right], \tag{41}$$

where $\mathcal{E}_S(f,g,\boldsymbol{\alpha},\boldsymbol{\beta}) = -\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2 \log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\right) d\boldsymbol{t}\, d\boldsymbol{\omega}.$

Proof. Define

$$I(f,g,\boldsymbol{\alpha},\boldsymbol{\beta},q) = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^q d\boldsymbol{t}\, d\boldsymbol{\omega}. \tag{42}$$

Using (42) in (30), one gets

$$I(f,g,\boldsymbol{\alpha},\boldsymbol{\beta},2) = 1. \tag{43}$$

From (31) and the hypothesis about f and g, it can also be shown that

$$I(f,g,\boldsymbol{\alpha},\boldsymbol{\beta},q) \le \left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)^{2-q}\left(\frac{2}{q}\right)^4. \tag{44}$$

For $s>0$, define

$$R(s) = \frac{I(f,g,\boldsymbol{\alpha},\boldsymbol{\beta},2) - I(f,g,\boldsymbol{\alpha},\boldsymbol{\beta},2+2s)}{s}.$$

Then

$$R(s) \ge \frac{1}{s}\left\{1 - \left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)^{-2s}\left(\frac{1}{1+s}\right)^4\right\}. \tag{45}$$

Let us assume that $\mathcal{E}_S(f,g,\boldsymbol{\alpha},\boldsymbol{\beta}) < \infty$, otherwise (41) is obvious.

From the inequality $1 + s\log a \le a^s$, $s\in\mathbb{R}$ and $a>0$ we now obtain

$$\begin{aligned} 0 &\le \frac{1}{s}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\left(1 - \left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^{2s}\right) \\ &\le -\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2 \log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\right). \end{aligned} \tag{46}$$

Since $-\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2 \log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\right)$ is integrable, we obtain using the Lebesgue dominated convergence theorem in (46):

$$\begin{aligned} \lim_{s\to 0+} R(s) &= \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \lim_{s\to 0+}\left\{\frac{1}{s}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^2\left(1 - \left|\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\right|^{2s}\right)\right\} d\boldsymbol{t}\, d\boldsymbol{\omega} \\ &= \mathcal{E}_S(f,g,\boldsymbol{\alpha},\boldsymbol{\beta}). \end{aligned} \tag{47}$$

Again from (45), we get

$$\lim_{s\to 0+} R(s) \ge 2\left[2 + \log\left(\left|\sin\left(\frac{\theta_1+\theta_1'}{2}\right)\right|\left|\sin\left(\frac{\theta_2+\theta_2'}{2}\right)\right|\right)\right]. \tag{48}$$

Based on (47) and (48), equation (41) follows. This concludes the proof. □

Remark 4.15. For $\boldsymbol{\alpha} = \boldsymbol{\beta}$, Theorem 4.14 gives the entropy UP for the STQFrFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$. Moreover, if $\boldsymbol{\alpha} = \boldsymbol{\beta} = (\frac{\pi}{2}, \frac{\pi}{2})$, Theorem 4.14 gives the entropy UP for the STQFT of the function $f \in L^2_{\mathbb{H}}(\mathbb{R}^4)$.

Example 4.16. Consider the function $f(\boldsymbol{x}) = e^{-\left(|\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2\right)}$, $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, with $\boldsymbol{x}_1=(x_1,x_2)$ and $\boldsymbol{x}_2=(x_3,x_4)$. Also consider the function

$$g(\boldsymbol{x})=\begin{cases}1, & 0\le x_1<\frac{1}{2}, 0\le x_2<\frac{1}{2}, 0\le x_3<\frac{1}{2}, 0\le x_4<\frac{1}{2}\\ -1, & \frac{1}{2}\le x_1<1, \frac{1}{2}\le x_2<1, \frac{1}{2}\le x_3<1, \frac{1}{2}\le x_4<1\\ 0, & \text{otherwise.}\end{cases}$$

Using Definition 4.1, the STQCFrFT of f with respect to the window function g is given by

$$\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(\boldsymbol{t},\boldsymbol{\omega})=\int_{\mathbb{R}^4}\mathcal{K}^i_{\theta_1,\theta_1'}(\boldsymbol{x}_1,\boldsymbol{\omega}_1)f(\boldsymbol{x})\overline{g(\boldsymbol{x}-\boldsymbol{t})}\mathcal{K}^j_{\theta_2,\theta_2'}(\boldsymbol{x}_2,\boldsymbol{\omega}_2)\,d\boldsymbol{x},\ (\boldsymbol{t},\boldsymbol{\omega})\in\mathbb{R}^4\times\mathbb{R}^4, \tag{49}$$

where $\boldsymbol{t}=(\boldsymbol{t}_1,\boldsymbol{t}_2)$, $\boldsymbol{\omega}=(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)\in\mathbb{R}^2\times\mathbb{R}^2$ with $\boldsymbol{t}_1=(t_1,t_2)$, $\boldsymbol{t}_2=(t_3,t_4)$ and $\boldsymbol{\omega}_1=(\omega_1,\omega_2)$, $\boldsymbol{\omega}_2=(\omega_3,\omega_4)$. Thus for the chosen function f and the window function g we get from (49)

$$\begin{aligned}&\left(\mathcal{S}^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathbb{H},g}f\right)(\boldsymbol{t},\boldsymbol{\omega})\\
&=\left\{\int_{t_1}^{t_1+\frac{1}{2}}\int_{t_2}^{t_2+\frac{1}{2}}\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\right\}\\
&\quad\times\left\{\int_{t_3}^{t_3+\frac{1}{2}}\int_{t_4}^{t_4+\frac{1}{2}}\tilde{d}(\gamma_2)e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}e^{-|\boldsymbol{x}_2|^2}\,d\boldsymbol{x}_2\right\}\\
&\quad-\left\{\int_{t_1+\frac{1}{2}}^{t_1+1}\int_{t_2+\frac{1}{2}}^{t_2+1}\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\right\}\\
&\quad\times\left\{\int_{t_3+\frac{1}{2}}^{t_3+1}\int_{t_4+\frac{1}{2}}^{t_4+1}\tilde{d}(\gamma_2)e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}e^{-|\boldsymbol{x}_2|^2}\,d\boldsymbol{x}_2\right\}.\end{aligned} \tag{50}$$

We first consider the integral

$$\begin{aligned}&\int_{t_1}^{t_1+\frac{1}{2}}\int_{t_2}^{t_2+\frac{1}{2}}\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\\
&=\tilde{d}(\gamma_1)\left\{\int_{t_1}^{t_1+\frac{1}{2}}e^{-i\left\{\tilde{a}(\gamma_1)\left(x_1^2+\omega_1^2\right)-x_1\left(\tilde{b}(\gamma_1,\delta_1)\omega_1+\tilde{c}(\gamma_1,\delta_1)\omega_2\right)\right\}}e^{-x_1^2}dx_1\right\}\\
&\quad\times\left\{\int_{t_2}^{t_2+\frac{1}{2}}e^{-i\left\{\tilde{a}(\gamma_1)\left(x_2^2+\omega_2^2\right)-x_2\left(-\tilde{c}(\gamma_1,\delta_1)\omega_1+\tilde{b}(\gamma_1,\delta_1)\omega_2\right)\right\}}e^{-x_2^2}dx_2\right\}.\end{aligned} \tag{51}$$

Consider the integral

$$\begin{aligned}&\int_{t_1}^{t_1+\frac{1}{2}}e^{-i\left\{\tilde{a}(\gamma_1)\left(x_1^2+\omega_1^2\right)-x_1\left(\tilde{b}(\gamma_1,\delta_1)\omega_1+\tilde{c}(\gamma_1,\delta_1)\omega_2\right)\right\}}e^{-x_1^2}dx_1\\
&=e^{-i\tilde{a}(\gamma_1)\omega_1^2}\int_{t_1}^{t_1+\frac{1}{2}}e^{-\left\{(1+i\tilde{a}(\gamma_1))x_1^2-ix_1\left(\tilde{b}(\gamma_1,\delta_1)\omega_1+\tilde{c}(\gamma_1,\delta_1)\omega_2\right)\right\}}dx_1\\
&=\frac{\sqrt{\pi}e^{-i\tilde{a}(\gamma_1)\omega_1^2-\frac{\left(\tilde{b}(\gamma_1,\delta_1)\omega_1+\tilde{c}(\gamma_1,\delta_1)\omega_2\right)^2}{4(1+i\tilde{a}(\gamma_1))}}}{2\sqrt{1+i\tilde{a}(\gamma_1)}}\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\end{aligned}$$

$$-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_1-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\Bigg], \tag{52}$$

where $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt$, $A(\theta_1,\theta_1',i) = \sqrt{1+i\tilde{a}(\gamma_1)}$ and $B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1) = \frac{\left(\tilde{b}(\gamma_1,\delta_1)\omega_1+\tilde{c}(\gamma_1,\delta_1)\omega_2\right)^2}{4(1+i\tilde{a}(\gamma_1))}$. Similarly, we have

$$\begin{aligned}
&\int_{t_2}^{t_2+\frac{1}{2}} e^{-i\left\{\tilde{a}(\gamma_1)\left(x_2^2+\omega_2^2\right)-x_2\left(-\tilde{c}(\gamma_1,\delta_1)\omega_1+\tilde{b}(\gamma_1,\delta_1)\omega_2\right)\right\}}e^{-x_2^2}dx_2\\
&=\frac{\sqrt{\pi}e^{-i\tilde{a}(\gamma_1)\omega_2^2-\frac{\left(-\tilde{c}(\gamma_1,\delta_1)\omega_1+\tilde{b}(\gamma_1,\delta_1)\omega_2\right)^2}{4(1+i\tilde{a}(\gamma_1))}}}{2\sqrt{1+i\tilde{a}(\gamma_1)}}\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\qquad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_2-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right],
\end{aligned}\tag{53}$$

where $B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1) = \frac{\left(-\tilde{c}(\gamma_1,\delta_1)\omega_1+\tilde{b}(\gamma_1,\delta_1)\omega_2\right)^2}{4(1+i\tilde{a}(\gamma_1))}$. Using equations (52) and (53) in (51), we get

$$\begin{aligned}
&\int_{t_1}^{t_1+\frac{1}{2}}\int_{t_2}^{t_2+\frac{1}{2}} \tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\\
&=J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_1-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]\\
&\times\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_2-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right],
\end{aligned}\tag{54}$$

where $J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1) = \tilde{d}(\gamma_1)\frac{\pi}{4(1+i\tilde{a}(\gamma_1))}e^{-\left\{i\tilde{a}(\gamma_1)|\boldsymbol{\omega}_1|^2+\frac{\left((\tilde{b}(\gamma_1,\delta_1))^2+(\tilde{c}(\gamma_1,\delta_1))^2\right)|\boldsymbol{\omega}_1|^2}{4(1+i\tilde{a}(\gamma_1))}\right\}}$. Similarly, we have

$$\begin{aligned}
&\int_{t_3}^{t_3+\frac{1}{2}}\int_{t_4}^{t_4+\frac{1}{2}} \tilde{d}(\gamma_2)e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}e^{-|\boldsymbol{x}_2|^2}\,d\boldsymbol{x}_1\\
&=J(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+\frac{1}{2}\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\qquad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)t_3-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]\\
&\times\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+\frac{1}{2}\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)t_4-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right].
\end{aligned}\tag{55}$$

Thus from equations (54) and (55), we have

$$\left\{\int_{t_1}^{t_1+\frac{1}{2}}\int_{t_2}^{t_2+\frac{1}{2}}\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\right\}$$
$$\times\left\{\int_{t_3}^{t_3+\frac{1}{2}}\int_{t_4}^{t_4+\frac{1}{2}}\tilde{d}(\gamma_2)e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}e^{-|\boldsymbol{x}_2|^2}\,d\boldsymbol{x}_2\right\}$$
$$= J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\left[\mathrm{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_1,\theta_1',i)t_1-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]$$
$$\times\left[\mathrm{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_1,\theta_1',i)t_2-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]J(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)$$
$$\times\left[\mathrm{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+\frac{1}{2}\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_2,\theta_2',j)t_3-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]$$
$$\times\left[\mathrm{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+\frac{1}{2}\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_2,\theta_2',j)t_4-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]. \tag{56}$$

Similarly, it can be shown that

$$\left\{\int_{t_1+\frac{1}{2}}^{t_1+1}\int_{t_2+\frac{1}{2}}^{t_2+1}\tilde{d}(\gamma_1)e^{-i\left\{\tilde{a}(\gamma_1)(|\boldsymbol{x}_1|^2+|\boldsymbol{\omega}_1|^2)-\boldsymbol{x}_1\cdot M_1\boldsymbol{\omega}_1\right\}}e^{-|\boldsymbol{x}_1|^2}\,d\boldsymbol{x}_1\right\}$$
$$\times\left\{\int_{t_3+\frac{1}{2}}^{t_3+1}\int_{t_4+\frac{1}{2}}^{t_4+1}\tilde{d}(\gamma_2)e^{-j\left\{\tilde{a}(\gamma_2)(|\boldsymbol{x}_2|^2+|\boldsymbol{\omega}_2|^2)-\boldsymbol{x}_2\cdot M_2\boldsymbol{\omega}_2\right\}}e^{-|\boldsymbol{x}_2|^2}\,d\boldsymbol{x}_2\right\}$$
$$= J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\left[\mathrm{erf}\left(A(\theta_1,\theta_1',i)(t_1+1)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]$$
$$\times\left[\mathrm{erf}\left(A(\theta_1,\theta_1',i)(t_2+1)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]J(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)$$
$$\times\left[\mathrm{erf}\left(A(\theta_2,\theta_2',j)(t_3+1)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+\frac{1}{2}\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]$$
$$\times\left[\mathrm{erf}\left(A(\theta_2,\theta_2',j)(t_4+1)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.$$
$$\left.-\,\mathrm{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+\frac{1}{2}\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]. \tag{57}$$

Thus from equations (49), (56) and (57) we obtain

$$\begin{aligned}
&\left(\mathcal{S}_{\mathbb{H},g}^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\right)(\boldsymbol{t},\boldsymbol{\omega})\\
&= J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\qquad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_1-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)t_2-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]J(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+\frac{1}{2}\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)t_3-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+\frac{1}{2}\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)t_4-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]\\
&- J(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+1\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_1+\frac{1}{2}\right)-B_1(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+1\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_1,\theta_1',i)\left(t_2+\frac{1}{2}\right)-B_2(\theta_1,\theta_1',i,\boldsymbol{\omega}_1)\right)\right]J(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+1\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_3+\frac{1}{2}\right)-B_1(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right]\\
&\quad\times\left[\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+1\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right.\\
&\quad\left.-\operatorname{erf}\left(A(\theta_2,\theta_2',j)\left(t_4+\frac{1}{2}\right)-B_2(\theta_2,\theta_2',j,\boldsymbol{\omega}_2)\right)\right].
\end{aligned}$$

5. Conclusions

In this paper, we have defined the two sided QCFrFT and obtained the Parseval's formula and the sharp Hausdorff-Young inequality, based on which we have obtained its Rènyi entropy UP. Incorporating the basic properties such as boundedness, translation, etc. of the newly proposed STQCFrFT, we have also obtained the inner product relation followed by the reconstruction formula. Moreover, using the sharp Hausdorff-Young inequality for the QCFrFT, we have also obtained the Lieb's and entropy UPs for the proposed STQCFrFT.

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References

[1] L. B. Almeida, *The fractional Fourier transform and time-frequency representations*, IEEE Trans. Signal Process., **42(11)** (1994), 3084–3091.

[2] M. Bahri, K. Abdul, A. Samsul, *Some Essential Relations for the Quaternion Quadratic-Phase Fourier Transform*, Mathematics, **11(5)** (2023), 1235.

[3] M. Bahri, R. Ashino, *Uncertainty principles related to quaternionic windowed Fourier transform*, Int. J. Wavelets Multiresolut. Inf. Process., **18(3)** (2020), 2050015.

[4] M. Bahri, R. Ashino, R. Vaillancourt, *Continuous quaternion Fourier and wavelet transforms*, Int. J. Wavelets Multiresolut. Inf. Process., **12(4)** (2014), 1460003.

[5] M. Bahri, E. Hitzer, A. Hayashi, R. Ashino, *An uncertainty principle for quaternion Fourier transform*, Comput. Math. Appl., **56(9)** (2008), 2398–2410.

[6] M. Bahri, E. Hitzer, R. Ashino, R. Vaillancourt, *Windowed Fourier transform of two-dimensional quaternionic signals*, Appl. Math. Comput., **216(8)** (2010), 2366–2379.

[7] M. Y. Bhat, A. H. Dar, I. Nurhidayat, S. Pinelas, *Uncertainty principles for the two-sided quaternion windowed quadratic-phase Fourier transform*, Symmetry, **14(12)** (2022), 2650.

[8] L. P. Chen, K. I. Kou, M. S. Liu, *Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform*, J. Appl. Math. Anal. Appl., **423(1)** (2015), 681–700.

[9] D. Cheng, K. I. Kou, *Plancherel theorem and quaternion Fourier transform for square integrable functions*, Complex Var. Elliptic Equ., **64(2)** (2019), 223–242.

[10] A. Dembo, T. M. Cover, J. A. Thomas, *Information theoretic inequalities*, IEEE Trans. Inf. Theory, **37(6)** (1991), 1501–1518.

[11] T. A. Ell, *Quaternion-Fourier transforms for analysis of two-dimensional linear time-invariant partial differential systems*, Proceedings of 32nd IEEE Conference on Decision and Control, (1993), 1830–1841.

[12] K. Gröchenig, *Foundations of time-frequency analysis*, Springer Science & Business Media, 2001.

[13] X. Guanlei, W. Xiaotong, X. Xiaogang, *Generalized entropic uncertainty principle on fractional Fourier transform*, Signal Process., **89(12)** (2009), 2692–2697.

[14] B. Gupta, A. K. Verma, *Short time quaternion quadratic phase Fourier transform and its uncertainty principles*, arXiv preprint arXiv:2204.09017, (2022).

[15] E. Hitzer, *Quaternion Fourier transform on quaternion fields and generalizations*, Adv. Appl. Clifford Algebr., **17(3)** (2007), 497–517.

[16] F. Jones, *Lebesgue integration on Euclidean space*, Jones & Bartlett Learning, 2001.

[17] R. Kamalakkannan, R. Roopkumar, *Multidimensional fractional Fourier transform and generalized fractional convolution*, Integral Transforms Spec. Funct., **31(2)** (2020), 152–165.

[18] R. Kamalakkannan, R. Roopkumar, A. Zayed, *Short time coupled fractional Fourier transform and the uncertainty principle*, Fract. Calc. Appl. Anal., **24(3)** (2021), 667–688.

[19] R. Kamalakkannan, R. Roopkumar, A. Zayed, *On the extension of the coupled fractional Fourier transform and its properties*, Integral Transforms Spec. Funct., **33(1)** (2022), 65–80.

[20] B. Kamel, E. Tefjeni, *Uncertainty principle for the two-sided quaternion windowed Fourier transform*, Integral Transforms Spec. Funct., **30(5)** (2019), 362–382.

[21] B. Kamel, E. Tefjeni, *Uncertainty principle for the two sided quaternion windowed Fourier transform*, J. Pseudo-Differ. Oper. Appl., **11(1)** (2020), 159–185.

[22] F. H. Kerr, *Namias' fractional Fourier transforms on L^2 and applications to differential equations*, J. Appl. Math. Anal. Appl., **136(2)** (1988), 404–418.

[23] F. H. Kerr, *A distributional approach to Namias' fractional Fourier transforms*, Proc. R. Soc. Edinb., Sect. A, Math., **108(1-2)** (1988), 133–143.

[24] P. Lian, *Uncertainty principle for the quaternion Fourier transform*, J. Appl. Math. Anal. Appl., **467(2)** (2018), 1258–1269.

[25] P. Lian, *Sharp Hausdorff-Young inequalities for the quaternion Fourier transforms*, Proc. Am. Math. Soc., **148** (2020), 697–703.

[26] P. Lian, *Quaternion and fractional Fourier transform in higher dimension*, Appl. Math. Comput., **389** (2021), 125585.

[27] V. Namias, *The fractional order Fourier transform and its application to quantum mechanics*, IMA J. Appl. Math., **25(3)** (1980), 241–265.

[28] F. A. Shah, W. Z. Lone, K. S. Nisar, T. Abdeljawad, *On the class of uncertainty inequalities for the coupled fractional Fourier transform*, J. Inequal. Appl., **1** (2022), 133.

[29] F. A. Shah, K. S. Nisar, W. Z. Lone, A. Y. Tantary, *Uncertainty principles for the quadratic-phase Fourier transforms*, Math. Methods Appl. Sci., **44(13)** (2021), 10416–10431.

[30] A. K. Verma, B. Gupta, *A note on continuous fractional wavelet transform in* $\mathbb{R}^n$, Int. J. Wavelets Multiresolut. Inf. Process., (2021), 2150050.

[31] D. Wei, Z. Li, *Different forms of Plancherel theorem for fractional quaternion Fourier transform*, Optik, **124(24)** (2013), 6999–7002.

[32] E. Wilczok, *New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform*, Doc. Math., **5** (2000), 201–226.

[33] A. Zayed, *Two-dimensional fractional Fourier transform and some of its properties*, Integral Transforms Spec. Funct., **29(7)** (2018), 553–570.

[34] A. Zayed, *A new perspective on the two-dimensional fractional Fourier transform and its relationship with the Wigner distribution*, J. Fourier Anal. Appl., **25** (2019), 460–487.

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