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COUPON COLLECTOR PROBLEM WITH PENALTY COUPON

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Abstract. In this paper we consider a generalization of the coupon collector problem where we assume that the set of available coupons consists of standard coupons and an additional penalty coupon, which does not belong to the collection and interferes with collecting standard coupons. Applying Markov chain approach the following problem is solved: how many coupons (on average) one has to purchase in order to complete a collection without interference or to collect n more penalty coupons than standard coupons. Also, we obtain additional results related to the distribution of the waiting time until the collection is sampled without interference or until n more penalty coupons than standard coupons is sampled.

1. Introduction

The classical coupon collector problem (CCCP) belongs to the family of urn problems, together with the birthday problem and the occupancy problem. Formally, the coupon collector problem (CCP) can be defined as follows: Consider a person (collector) who collects different types of coupons from the finite set $\mathbb{N}_n = \{1, 2, \dots n\}$, where each coupon can be drawn with a certain probability. The CCP consists of determining the distribution or expected value of the number of coupons that must be drawn with replacement until the collector obtains a complete collection of coupons.

The CCP has been treated in several papers since the 1960s. The waiting time to complete a collection with unequal probabilities was first determined in [10] and the waiting time to complete two collections of coupons in the same case was calculated in [8]. In the 1990s, several authors made further contributions to this classical combinatorial problem (see, e.g. [2, 5]).

There are several generalizations of CCP, and some of them are based on the idea of introducing one or more additional coupons (apart from the original collection) with a specific purpose. One of these generalizations is the CCP with a null coupon, i.e., a coupon that never belongs to the collection. A group of authors first determined the

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distribution and the expectation of the waiting time until a given portion of coupons is collected (see [1,2]). In [7], we consider a variant of the CCP that introduces a bonus, where once a collector receives one of the bonus coupons, he immediately collects another coupon. In [11], coupons are assumed to have more than one purpose, and it is shown how the expected number of coupons to be drawn can be determined by enumerating transversals of hypergraphs where coupons can be drawn either with or without replacement.

In this work, we consider the following generalization of CCP: suppose that, in addition to elements from the set $\mathbb{N}_n$, the set contains a penalty coupon that is not part of the collection and interferes with the collection of standard coupons. Thus, we have an expanded set of available coupons $\mathbb{N}_n^\diamond = \{1, 2, \ldots, n, \diamond\}$, where $\diamond$ denotes the penalty coupon. Suppose that sampling with replacement occurs, that a particular coupon is drawn with probability $p_\diamond < 1$, and that each coupon $k \in \mathbb{N}_n$ is drawn with probability $p = \frac{1-p_\diamond}{n}$. We call this version of the problem the coupon-collector problem with penalty coupon and refer to it as CCPPC in the remainder of this paper.

Let W_n be the waiting time until n elements of $\mathbb{N}_n$ are sampled without interference, or until $m \leq n$ elements of $\mathbb{N}_n$ (called standard coupons) are sampled, and $m+n$ penalty coupons are sampled. Thus, the random variable W_n can be defined as

$$W_n = \min\{t \geq 0 : |Y_t - Z_t| = n\}, \tag{1}$$

where Y_t is the number of different types of standard coupons obtained after drawing t coupons, and Z_t is the number of penalty coupons obtained after drawing t coupons.

The structure of this paper is as follows. In Section 2 we explain how CCPPC fits into the Markov chain approach. Exact formulas for the expectation and variance of the waiting time W_n are obtained in Section 3. In Section 4 we use the Markov chain approach to obtain additional results related to the waiting time W_n. The relationship between this variant of CCP and the random walk (gambler's ruin problem) is explained in Section 5. A numerical example is given in Section 6, and the conclusion is found in Section 7.

2. Markov chain approach

The Markov chain approach has already been successfully applied to the CCP (see [4]) and to several variants of this problem.

Consider the two-dimensional discrete-time Markov chain $\{X_t,\ t \in \mathbb{N}\}$ defined as $X_t = (Y_t, Z_t)$, where Y_t is the number of different types of standard coupons obtained after drawing t coupons, and Z_t is the number of penalty coupons obtained after drawing t coupons. In this case, the state space is

$$S = \{T_0, T_1, \ldots, T_{2n-1}, A\},$$

where

$$T_0 = \{(0,0), \ldots, (n-1,0)\}, \quad |T_0| = n$$

$$T_i = \{(0,i), \ldots, (n,i)\}, \quad |T_i| = n+1 \quad i \in \{1, 2, \ldots n-1\}$$

$$T_i = \{(i-n+1,i), \ldots, (n,i)\}, \quad |T_i| = 2n-i \quad i \in \{n, n+1, \ldots 2n-1\}$$

are sets of transient states and

$$A = \{(n,0),\ (0,n),\ (1,n+1)\dots(n,2n)\}$$

is the set of absorbing states. Therefore, $|S| = \frac{3n^2+3n-2}{2} + n + 2$. Now we introduce some notations. In the rest of the paper, we denote by $\mathbf{I}$ the identity matrix and by $\mathbf{0}$ the matrix with all entries equal to 0.

The transition probability matrix for one step of the considered Markov chain is

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q}^*_{\frac{3n^2+3n-2}{2}\times\frac{3n^2+3n-2}{2}} & \mathbf{R}_{\frac{3n^2+3n-2}{2}\times(n+2)} \\ \mathbf{0}_{(n+2)\times\frac{3n^2+3n-2}{2}} & \mathbf{I}_{n+2} \end{pmatrix},$$

where the matrix $\mathbf{Q}^*$ describes transitions between transient states and has the following form:

$$\mathbf{Q}^* = \begin{array}{c} \\ T_0 \\ T_1 \\ T_2 \\ \vdots \\ T_{n-1} \\ T_n \\ T_{n+1} \\ \vdots \\ T_{2n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} T_0 & T_1 & T_2 & \dots & T_{n-1} & T_n & T_{n+1} & \dots & T_{2n-1} \end{array} \\ \begin{pmatrix} \mathbf{A}^{(0)}_{n-1} & p_\diamond\mathbf{I}_n & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{(0)}_n & p_\diamond\mathbf{I}_{n+1} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{(0)}_n & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}^{(0)}_n & p_\diamond\mathbf{I}_n & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}^{(1)}_n & \mathbf{B}_n & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{A}^{(2)}_n & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}^{(n)}_n \end{pmatrix} \end{array}.$$

Matrix $\mathbf{A}^{(l)}_k,\ k \in \{0,1,\dots,n\},\ l \in \{0,1,\dots k\}$ describes transitions between sets of states $T_i,\ i \in \{1,2,\dots,2n-1\}$ and has the form:

$$\mathbf{A}^{(l)}_k = \begin{pmatrix} lp & 1-p_\diamond-pl & 0 & \dots & 0 & 0 \\ 0 & (l+1)p & 1-p_\diamond-(l+1)p & \dots & 0 & 0 \\ 0 & 0 & (l+2)p & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (k-1)p & 1-p_\diamond-(k-1)p \\ 0 & 0 & 0 & \dots & 0 & kp \end{pmatrix}_{(k+1-l)\times(k+1-l)}.$$

Matrix $\mathbf{B}_k,\ k \in \{2,\dots,n\}$ describes transitions from sets of states T_i to sets of states $T_{i+1},\quad i \in \{n,n+1,\dots 2n-1\}$ and has the form:

$$\mathbf{B}_k = \begin{pmatrix} \mathbf{0}_{1\times(k-1)} \\ p_\diamond\mathbf{I}_{(k-1)} \end{pmatrix}_{k\times(k-1)}.$$

Matrix $\mathbf{R}$ is related to transitions from transient to absorbing states, and has the form:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_0 \\ \mathbf{0}_{(n^2-n-2)\times(n+2)} \\ \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_{n+1} \end{pmatrix}_{\frac{3n^2+3n-2}{2}\times(n+2)}, \tag{2}$$

where $\mathbf{R}_0 = \begin{pmatrix} \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{(n-1)\times(n+1)} \\ p & \mathbf{0}_{1\times(n+1)} \end{pmatrix}$,

$$\mathbf{R}_k = \begin{pmatrix} \mathbf{0}_{1\times k} & p_\diamond & \mathbf{0}_{1\times(n+1-k)} \\ \mathbf{0}_{(n+1-k)\times k} & \mathbf{0}_{(n+1-k)\times 1} & \mathbf{0}_{(n+1-k)\times(n+1-k)} \end{pmatrix},\ 1 \le k \le n+1.$$

3. Properties of the waiting time W_n

In this section we obtain expressions for the distribution function and the first and second moments of the waiting time W_n and analyze their properties.

THEOREM 3.1. *For $0 < p_\diamond < 1$, the following relations hold for the waiting time W_n:*

$$\begin{aligned} P\{W_n > t\} = & \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\left(\frac{k(1-p_\diamond)}{n}\right)^t + \sum_{i=0}^{n-1}\binom{t}{n+i}p_\diamond^{n+i}(1-p_\diamond)^{t-n-i} \\ & - \sum_{i=0}^{n-1}\sum_{k=0}^{i}(-1)^{i-k}\binom{n-k-1}{n-i-1}\binom{n}{k}\left(\frac{k}{n}\right)^{t-i-n}\binom{t}{n+i}p_\diamond^{n+i}(1-p_\diamond)^{t-n-i}, \end{aligned} \tag{3}$$

$$E(W_n) = \frac{1}{p_\diamond}\left(n+1-\frac{\Gamma(n+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)} - \sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^{k-1}}{(1+k\alpha)^{2n}}\right), \tag{4}$$

$$\begin{aligned} E(W_n^2) = & \frac{1}{p_\diamond}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)} - \frac{2}{p_\diamond^2}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)}\sum_{k=1}^{n}\frac{1}{\alpha k+1} \\ & + \frac{2}{p_\diamond^2}\left(\frac{3}{2}n^2+\frac{1}{2}n-\frac{1}{2}np_\diamond+1-\frac{1}{2}p_\diamond\right) \\ & + \left(2(2n+1)-\frac{2}{p_\diamond^2\alpha}-\frac{1}{p_\diamond}\right)\sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^{k-1}}{(1+k\alpha)^{2n+1}} \\ & + \frac{2}{p_\diamond^2}\sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^{k-2}}{(1+k\alpha)^{2n+1}} - \left(\frac{1}{p_\diamond}+\frac{2}{\alpha}\right)\sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^k}{(1+k\alpha)^{2n+1}}, \end{aligned} \tag{5}$$

where $\alpha = \frac{1-p_\diamond}{np_\diamond}$ and $\Gamma(\cdot)$ denotes the gamma function.

Proof. The statement (3) follows from the representation (1) and the following relations:

$$P\{W_n > t\} = P\{|Y_t - Z_t| < n\}$$

$$=P\{|Y_t-Z_t|<n|Z_t=0\}P\{Z_t=0\}+\sum_{i=0}^{n-1}P\{|Y_t-Z_t|<n|Z_t=n+i\}P\{Z_t=n+i\}$$
$$=P\{Y_t<n|Z_t=0\}P\{Z_t=0\}+\sum_{i=0}^{n-1}P\{Y_t\geq +1|Z_t=n+i\}P\{Z_t=n+i\}$$
$$=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\left(\frac{k(1-p_\diamond)}{n}\right)^t$$
$$+\sum_{i=0}^{n-1}\left(1-\sum_{k=0}^{i}(-1)^{i-k}\binom{n-k-1}{n-i-1}\binom{n}{k}\left(\frac{k}{n}\right)^{t-i-n}\right)\binom{t}{n+i}p_\diamond^{n+i}(1-p_\diamond)^{t-n-i}.$$

The last line follows from [2, Theorem 1]. Using notation introduced in [2], we have that $P\{Y_t<u|Z_t=v\}=P\{T_{u,n}(\mathbf{p})>t-v\}$, where $\mathbf{p}=\left(\frac{1}{n},\dots,\frac{1}{n}\right)$.

The first moment of W_n is:

$$E(W_n)=\sum_{t\geq 0}P\{W_n>t\}$$
$$=\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\sum_{t\geq 0}\left(\frac{k(1-p_\diamond)}{n}\right)^t+\sum_{i=0}^{n-1}\sum_{t\geq 0}\binom{t}{n+i}p_\diamond^{n+i}(1-p_\diamond)^{t-n-i}$$
$$-\sum_{i=0}^{n-1}\sum_{k=0}^{i}(-1)^{i-k}\binom{n-k-1}{n-i-1}\binom{n}{k}p_\diamond^{n+i}\sum_{t\geq 0}\binom{t}{n+i}\left(\frac{k}{n}(1-p_\diamond)\right)^{t-i-n}.$$

Using the fact
$$\sum_{t=i}^{+\infty}\binom{t}{i}a^{t-i}=\frac{1}{(1-a)^{i+1}},\quad |a|<1,$$

and by simple transformations of the sums and using the relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\alpha k+1}=\frac{\Gamma(n+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)},\tag{6}$$

which is easily proved using mathematical induction, we obtain the required statement. The second moment of W_n is:

$$E(W_n^2)=\sum_{t\geq 0}P\{W_n>t\}+2\sum_{t\geq 0}tP\{W_n>t\}.$$

Using (4), (6) and the relations:

$$\sum_{t=i}^{+\infty}t\binom{t}{i}a^{t-i}=\frac{i}{(1-a)^{i+1}}+\frac{a(i+1)}{(1-a)^{i+2}},\quad |a|<1,$$
$$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}k(f(x))^k=nf(x)(1-f(x))^{n-1},$$

we obtain the required statement. □

Remark 3.2. An expression for the variance of the waiting time W_n follows directly from (4) and (5).

Corollary 3.3. *For* $0 < p_\diamond < 1$, *we have*

$$E(W_n) \leq \frac{1}{p_\diamond}\left(n+1-\frac{\Gamma(n+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(n+\frac{1}{\alpha}+1\right)}-\frac{n}{(1+\alpha)^{2n}}\right).$$

Proof. The sum in (4) satisfy the following inequalities:

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^{k-1}}{(1+k\alpha)^{2n}} &\geq \sum_{k=1}^{n}\binom{n}{k}\frac{(k\alpha)^{k-1}}{(1+\alpha)^{2nk}} \geq \sum_{k=1}^{n}\binom{n}{k}\frac{\alpha^{k-1}}{(1+\alpha)^{2nk}}\\ &= \frac{1}{\alpha}\left(1+\frac{\alpha}{(1+\alpha)^{2n}}\right)^{n}-\frac{1}{\alpha} \geq \frac{1}{\alpha}\left(1+n\frac{\alpha}{(1+\alpha)^{2n}}\right)-\frac{1}{\alpha},\end{aligned}$$

which yields the required statement. □

4. Fundamental matrix and its applications

In this section, the fundamental matrix $\mathbf{F} = (\mathbf{I}-\mathbf{Q})^{-1}$ is determined using the Markov chain approach and matrix calculus (see [6]). The fundamental matrix provides another way to obtain the expectation and variance of the waiting time W_n, as well as some more precise results regarding the type of coupons sampled. First, we give a few useful lemmas.

Lemma 4.1. *For an upper-triangular block matrix* $\mathbf{M} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{C}\end{pmatrix}$, *the following equality holds:*

$$\mathbf{M}^{-1} = \begin{pmatrix}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{-1}\\ \mathbf{0} & \mathbf{C}^{-1}\end{pmatrix}. \tag{7}$$

Lemma 4.2. *Element in the m-th row and j-th column of the matrix* $(\mathbf{I}_{k-l+1}-\mathbf{A}_k^{(l)})^{-s}$, $k \in \{0,1,\dots,n\}$, $l \in \{0,1,\dots,k\}$, $s \in \mathbb{N}$, *is*

$$\left((\mathbf{I}_{k-l+1}-\mathbf{A}_k^{(l)})^{-s}\right)_{m,j} = \sum_{i=0}^{k-l}(-1)^{j-1+i}\binom{n-l-m+1}{i-m+1}\binom{n-l-i}{j-1-i}\frac{1}{(1-(l+i)p)^s}. \tag{8}$$

Proof. First, notice that the matrix $(\mathbf{I}_{k-l+1}-\mathbf{A}_k^{(l)})^{-1}$ is upper-triangular, therefore, eigenvalues are equal to diagonal elements $\lambda_i = \frac{1}{1-(l+i)p}$, $i \in \{0,1,\dots,k-l\}$. Straightforward calculation of the corresponding eigenvectors leads to representation:

$$(\mathbf{I}_{k-l+1}-\mathbf{A}_k^{(l)})^{-s} = \mathbf{M}_k^{(l)}\mathbf{J}_{k-l}^{(s)}(\mathbf{M}_k^{(l)})^{-1}, \tag{9}$$

where

$$\mathbf{J}_{k-l}^{(s)} = \begin{pmatrix}\lambda_0^s & 0 & 0 & \dots & 0\\ 0 & \lambda_1^s & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & \dots & \lambda_{k-l}^s\end{pmatrix},$$

and elements in the m-th row and j-th column of matrices $\mathbf{M}_k^{(l)}$ and $(\mathbf{M}_k^{(l)})^{-1}$ are given by:

$$\left(\mathbf{M}_k^{(l)}\right)_{m,j} = \begin{cases} \binom{n-l-(m-1)}{j-1,} & m \leq j; \\ 0, & m > j, \end{cases}$$

and
$$\left((\mathbf{M}_k^{(l)})^{-1}\right)_{m,j} = \begin{cases} (-1)^{m+j}\binom{n-l-(m-1)}{j-1}, & m \leq j; \\ 0, & m > j, \end{cases}$$

respectively. Multiplying matrices in (9) leads to the expression (8). □

Before we give an expression for fundamental matrix, notice that matrix $\mathbf{Q}_n^*$ is upper-triangular block matrix and has the form:

$$\mathbf{Q}_n^* = \begin{pmatrix} \mathbf{A}_{n-1}^{(0)} & \mathbf{B}^* \\ \mathbf{0} & \mathbf{C}_n \end{pmatrix}_{\frac{3n^2+3n-2}{2} \times \frac{3n^2+3n-2}{2}}, \tag{10}$$

where

$$\mathbf{B}^* = \begin{pmatrix} p_\diamond \mathbf{I}_n & \mathbf{0} \end{pmatrix}_{n \times \frac{3n^2+n-2}{2}}, \quad \mathbf{C}_n = \begin{pmatrix} \alpha_n^{(n-1)} & \beta^* \\ \mathbf{0} & \mathbf{D}_n^{(n)} \end{pmatrix}_{\frac{3n^2+n-2}{2} \times \frac{3n^2+n-2}{2}},$$

$$\alpha_n^{(k)} = \begin{pmatrix} \mathbf{A}_n^{(0)} & p_\diamond \mathbf{I}_{n+1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_n^{(0)} & p_\diamond \mathbf{I}_{n+1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_n^{(0)} & p_\diamond \mathbf{I}_{n+1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_n^{(0)} \end{pmatrix}_{k(n+1) \times k(n+1)}, \tag{11}$$

$$\beta^* = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ p_\diamond \mathbf{I}_n & \mathbf{0} \end{pmatrix}_{(n^2-1) \times \frac{n(n+1)}{2}},$$

$$\mathbf{D}_n^{(k)} = \begin{pmatrix} \mathbf{A}_n^{(1)} & \mathbf{B}_n & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_n^{(2)} & \mathbf{B}_{n-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_n^{(k-1)} & \mathbf{B}_{n-k+2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_n^{(k)} \end{pmatrix}_{\frac{k(2n-k+1)}{2} \times \frac{k(2n-k+1)}{2}}. \tag{12}$$

LEMMA 4.3. *For upper-triangular block matrix $\alpha_n^{(k)}$ defined in* (11) *the following equality holds for $k \geq 1$:*

$$(\mathbf{I} - \alpha_n^{(k)})^{-1} = \begin{pmatrix} \mathbf{V}_n^{(1)} & \mathbf{V}_n^{(2)} & \dots & \mathbf{V}_n^{(k)} \\ \mathbf{0} & \mathbf{V}_n^{(1)} & \dots & \mathbf{V}_n^{(k-1)} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_n^{(1)} \end{pmatrix}_{k(n+1) \times k(n+1)}, \tag{13}$$

where $\mathbf{V}_n^{(l)} = p_\diamond^{l-1}\left((\mathbf{I}_{n+1} - \mathbf{A}_n^{(0)})^{-1}\right)^l$.

Proof. We use mathematical induction on k to prove the lemma.

For $k=1$, $(\mathbf{I}-\alpha_n^{(1)})^{-1} = (\mathbf{I}-\mathbf{A}_n^{(0)})^{-1} = \mathbf{V}_n^{(1)}$.

Assuming that (4.2) holds for all dimensions less than k, we prove that it holds for k using Lemma 4.1. Note that

$$(\mathbf{I}-\alpha_n^{(k)})^{-1} = \begin{pmatrix} \mathbf{V}_n^{(1)} & \mathbf{X} \\ \mathbf{0} & (\mathbf{I}-\alpha_n^{(k-1)})^{-1} \end{pmatrix},$$

where $\mathbf{X} = \mathbf{V}_n^{(1)}\mathbf{Y}_n(\mathbf{I}-\alpha_n^{(k-1)})^{-1}$, $\quad \mathbf{Y}_n = (p_\diamond \mathbf{I}_{n+1} \quad \mathbf{0})_{(n+1)\times(k-1)(n+1)}$. Let

$$\mathbf{U} = \begin{pmatrix} p_\diamond \mathbf{V}_n^{(1)} & \mathbf{0} \end{pmatrix}_{(n+1)\times(k-1)(n+1)}.$$

Then, we have $\mathbf{X} = \mathbf{U}(\mathbf{I}-\alpha_n^{(k-1)})^{-1} = \begin{pmatrix} \mathbf{V}_n^{(2)} & \mathbf{V}_n^{(3)} & \dots & \mathbf{V}_n^{(k)} \end{pmatrix}$, which completes the proof. □

LEMMA 4.4. *For upper-triangular block matrix $\mathbf{D}_n^{(k)}$ defined in* (12) *the following equality holds for $k \geq 1$:*

$$(\mathbf{I}-\mathbf{D}_n^{(k)})^{-1} = \begin{pmatrix} \mathbf{U}_n^{(1,0)} & \mathbf{U}_n^{(1,1)} & \dots & \mathbf{U}_n^{(1,k-1)} \\ \mathbf{0} & \mathbf{U}_n^{(2,0)} & \dots & \mathbf{U}_n^{(2,k-2)} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{U}_n^{(k,0)} \end{pmatrix}_{\frac{k(2n-k+1)}{2}\times\frac{k(2n-k+1)}{2}}, \tag{14}$$

where

$$\mathbf{U}_n^{(i,m)} = p_\diamond^m (\mathbf{I}-\mathbf{A}_n^{(i)})^{-1} \begin{pmatrix} \mathbf{0}_{1\times(n-i)} \\ \mathbf{I}_{n-i} \end{pmatrix} (\mathbf{I}-\mathbf{A}_n^{(i+1)})^{-1} \begin{pmatrix} \mathbf{0}_{1\times(n-i-1)} \\ \mathbf{I}_{n-i-1} \end{pmatrix} \dots (\mathbf{I}-\mathbf{A}_n^{(m+i)})^{-1},$$

for $i=1,2,\dots k$ and $m=0,1,\dots,k-i$.

Proof. Again, we use mathematical induction on k to prove the lemma.

For $k=1$ equality (14) holds: $(\mathbf{I}-\mathbf{D}_n^{(1)})^{-1} = \mathbf{U}_n^{(1,0)}$. Assuming that (14) holds for all dimensions less than k, we prove that it holds for k using Lemma 4.1. We have that:

$$(\mathbf{I}-\mathbf{D}_n^{(k)})^{-1} = \begin{pmatrix} (\mathbf{I}-\mathbf{D}_n^{(k-1)})^{-1} & \mathbf{Y} \\ \mathbf{0} & \mathbf{U}_n^{(k,0)} \end{pmatrix},$$

where

$$\mathbf{Y} = (\mathbf{I}-\mathbf{D}_n^{(k-1)})^{-1}\mathbf{Z}_n\mathbf{U}_n^{(k,0)}, \quad \mathbf{Z}_n = \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{n-k+2} \end{pmatrix}_{\frac{(2n-k+1)k}{2}\times(n-k+1)}.$$

Let

$$\mathbf{T} = \begin{pmatrix} \mathbf{0} \\ p_\diamond \mathbf{U}_n^{(k,0)} \end{pmatrix}_{\frac{(2n-k+1)k}{2}\times(n-k+1)}.$$

Then, we have:

$$\mathbf{Y} = (\mathbf{I}-\mathbf{D}_n^{(k-1)})^{-1}\mathbf{T} = \begin{pmatrix} \mathbf{U}_n^{(1,k-1)} \\ \mathbf{U}_n^{(2,k-2)} \\ \vdots \\ \mathbf{U}_n^{(k-1,1)} \end{pmatrix},$$

which completes the proof. □

THEOREM 4.5. *Fundamental matrix* $\mathbf{F} = \mathbf{F}_n^* = (\mathbf{I} - \mathbf{Q}_n^*)^{-1}$ *can be partitioned as follows:*

$$\mathbf{F}_n^* = \begin{pmatrix} \mathbf{V}_{n-1}^{(1)} & \mathbf{K}_n^{(1)} & \mathbf{K}_n^{(2)} & \dots & \mathbf{K}_n^{(n-1)} & \mathbf{L}_n^{(1)} & \mathbf{L}_n^{(2)} & \dots & \mathbf{L}_n^{(n)} \\ \mathbf{0} & \mathbf{V}_n^{(1)} & \mathbf{V}_n^{(2)} & \dots & \mathbf{V}_n^{(n-1)} & \mathbf{M}_n^{(1,1)} & \mathbf{M}_n^{(1,2)} & \dots & \mathbf{M}_n^{(1,n)} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_n^{(1)} & \dots & \mathbf{V}_n^{(n-2)} & \mathbf{M}_n^{(2,1)} & \mathbf{M}_n^{(2,2)} & \dots & \mathbf{M}_n^{(2,n)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_n^{(1)} & \mathbf{M}_n^{(n-1,1)} & \mathbf{M}_n^{(n-1,2)} & \dots & \mathbf{M}_n^{(n-1,n)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{U}_n^{(1,0)} & \mathbf{U}_n^{(1,1)} & \dots & \mathbf{U}_n^{(1,n-1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{U}_n^{(2,0)} & \dots & \mathbf{U}_n^{(2,n-2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{U}_n^{(n,0)} \end{pmatrix}, \quad (15)$$

where $\mathbf{K}_n^{(l)} = \begin{bmatrix} p_\diamond \mathbf{U}_n^{(0,0)} & \mathbf{0}_{n\times 1} \end{bmatrix} \mathbf{V}_n^{(l)}$, $\mathbf{L}_n^{(l)} = \begin{bmatrix} p_\diamond^2 \mathbf{U}_n^{(0,0)} & \mathbf{0}_{n\times 1} \end{bmatrix} \mathbf{V}_n^{(n-2)} \mathbf{U}_n^{(0,l)}$ *and* $\mathbf{M}_n^{(i,j)} = (p_\diamond \mathbf{V}_n^{(n-2)})^{n-1-i} \mathbf{U}_n^{(0,j)}$.

Proof. The transition probability matrix for one step $\mathbf{Q}_n^*$ is upper-triangular block matrix and has representation given in (10), so using (7) we obtain that fundamental matrix has representation:

$$(\mathbf{I}-\mathbf{Q}_\mathbf{n}^*)^{-1} = \begin{pmatrix} (\mathbf{I}-\mathbf{A}_{n-1}^{(0)})^{-1} & (\mathbf{I}-\mathbf{A}_{n-1}^{(0)})^{-1}\mathbf{B}^*(\mathbf{I}-\mathbf{C}_n)^{-1} \\ \mathbf{0} & (\mathbf{I}-\mathbf{C}_n)^{-1} \end{pmatrix}_{\frac{3n^2+3n-2}{2}\times\frac{3n^2+3n-2}{2}}, \quad (16)$$

where

$$(\mathbf{I}-\mathbf{C}_n)^{-1} = \begin{pmatrix} (\mathbf{I}-\alpha_n^{(n-1)})^{-1} & (\mathbf{I}-\mathbf{A}_n^*)^{-1}\beta^*(\mathbf{I}-\mathbf{D}_n^{(n)})^{-1} \\ \mathbf{0} & (\mathbf{I}-\mathbf{D}_n^{(n)})^{-1} \end{pmatrix}_{\frac{3n^2+n-2}{2}\times\frac{3n^2+n-2}{2}}. \quad (17)$$

Note that the expressions for matrices $(\mathbf{I} - \alpha_n^{(n-1)})^{-1}$ and $(\mathbf{I} - \mathbf{D}_n^{(n)})^{-1}$ are given in (13) and (14), respectively. Expressions for matrices $(\mathbf{I} - \mathbf{A}_{n-1}^{(0)})^{-1}\mathbf{B}^*(\mathbf{I} - \mathbf{C}_n)^{-1}$ and $(\mathbf{I} - \mathbf{A}_n^*)^{-1}\beta^*(\mathbf{I} - \mathbf{D}_n^{(n)})^{-1}$ are trivially confirmed by multiplying the matrices. So, fundamental matrix has the form given by (16) which completes the proof of the theorem. □

Now, we can provide another way to compute the expectation and the variance of the waiting time W_n.

We introduce some more notation. Let $\mathbf{A}^{(sq)}$ denote the matrix whose elements are squared elements of matrix $\mathbf{A}$, $S_1(\mathbf{A})$ denote the sum of entries of the first row of the matrix $\mathbf{A}$, and $\mathbf{1}$ denote matrix with all entries equal to 1.

From [6, Theorem 3.3.5], it follows that the expectation of the waiting time W_n is equal to

$$E(W_n) = S_1(\mathbf{E}), \quad (18)$$

and the variance of the waiting time W_n is equal to

$$Var(W_n) = S_1(\mathbf{V}), \tag{19}$$

where
$$\mathbf{E} = \mathbf{F}_n^* \mathbf{1}_{\frac{3n^2+3n-2}{2}\times 1}, \quad \mathbf{V} = (2\mathbf{F}_n^* - \mathbf{I})\mathbf{E} - \mathbf{E}^{(sq)},$$

and matrix $\mathbf{F}_n^*$ is defined in (15).

Combining Theorem 4.5 with the representation (2), one can obtain several results related to the type of coupons collected. One result of this type is formulated in the next theorem.

Theorem 4.6. *Probability that all coupons from the set $\mathbb{N}_n$ are sampled before a penalty coupon is equal to*

$$p_{(n,0)} = (1 - p_\diamond) \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n}{j} \frac{n-j}{n - j(1 - p_\diamond)}.$$

Proof. The first row of the matrix $\mathbf{FR}$ consists of probabilities that the chain starting in the transient state $(0, 0)$ ends up in an absorbing state (see in [6, Theorem 3.3.7]). Also, notice that the probability that exactly n coupons and 0 penalty coupon are sampled, if the full collection is sampled, is the element of the first row and first column of the matrix $\mathbf{FR}$. Using the representation of the fundamental matrix $\mathbf{F}$ obtained in Theorem 4.5, and the form of the matrix $\mathbf{R}$ in (2), we obtain the result. □

Example 4.7. Let $p_\diamond = p_j = 1/(n + 1),\ j \in \mathbb{N}_n$. Probability that all coupons from the set $\mathbb{N}_n$ are sampled before a penalty coupon is equal to

$$\begin{aligned} p_{(n,0)} &= \frac{1}{n+1} \sum_{j=0}^{n-1} (-1)^{n-1-j} (n - j) \binom{n}{j} \frac{1}{1 - \frac{j}{n+1}} \\ &= \frac{1}{n+1} \sum_{j=0}^{n-1} (-1)^{n-1-j} (n - j) \binom{n+1}{j} = \frac{1}{n+1}. \end{aligned}$$

The last equality follows from the identity (see [2]):

$$\sum_{i=0}^{c-1} (-1)^{c-1-i} \binom{n-i-1}{n-c} \binom{n}{i} = 1.$$

5. Connection to random walk

Introducing a penalty coupon into the traditional CCP setting, leads to an interesting connection with random walk (or, alternatively, with gambler's ruin problem). The CCPPC can be considered as a special case of 1-dimensional random walk with absorbing barriers at $-n$ and n. More precisely, a particle starts at the origin of the straight line, and, in consecutive time intervals makes a unit step in positive or negative direction, or remains in the same position, with specific probabilities. The walk is confined to the set $S = \{-(n-1), -(n-2), \dots, 0, \dots, n-2, n-1\}$ and we assume that when the particle exits this set, it gets absorbed.

In the case of the CCP, the corresponding probabilities are $p^{(i)}_{k,k-1} = p_\diamond$, $p^{(i)}_{k,k} = (1-p_\diamond)\frac{i}{n}$ and $p^{(i)}_{k,k+1} = (1-p_\diamond)\left(1-\frac{i}{n}\right)$, where i steps in positive direction are already made.

We can notice that this walk is not symmetric, the probability of the step in negative direction is constant, and the probabilities of moving forward or remaining in the same point linearly depend on the number of steps in positive direction already made. It is also clear that at most n positive steps can be made.

The CCCP is obtained for $p_\diamond = 0$ and also fits into this setting. The corresponding probabilities are $q^{(i)}_{k,k} = \frac{i}{n}$ and $q^{(i)}_{k,k+1} = \left(1-\frac{i}{n}\right)$, where i steps in positive direction are already made. Notice that in this case the only absorbing barrier is at n.

If $p_\diamond = \frac{1}{2}$, and we slightly modify the problem considered, defining the waiting time W^*_n as follows: $W^*_n = \min\{t \geq 0 : |Y^*_n - Z_t| = n\}$, where Y^*_n is the total number of standard coupons (of any type) obtained after having drawn t coupons and Z_t is the number of penalty coupons drawn, we obtain the symmetric random walk with absorbing barriers. This problem can be equivalently described as a variant of the gambler's ruin problem.

6. Numerical example

In this section, we present numerical results for the CCPPC considered in this paper. We assume that the set of available coupons is $\mathbb{N}_3^{\star,\circ} = \{1, 2, 3, \diamond\}$.

If a penalty coupon is drawn with probability $p_\diamond = 1/4$, then any coupon $k \in \mathbb{N}_3$ is drawn with probability $p = p_k = 1/4$. In that case, the transition probability matrix for one step is

$$\mathbf{P} = \begin{pmatrix}
0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Next, we consider different combinations of sampling probabilities $p_\diamond$ and, by direct matrix manipulation, provide the expectation and variance of the waiting time W_3 using formulas (18) and (19). The results are presented in Table 1. Statistical Software R was used for all calculations.

	$p_\diamond=\frac{2}{3}$	$p_\diamond=\frac{1}{2}$	$p_\diamond=\frac{1}{4}$	$p_\diamond=\frac{1}{8}$	$p_\diamond=\frac{1}{10}$	$p_\diamond=\frac{1}{20}$	$p_\diamond=\frac{1}{50}$	$p_\diamond=\frac{1}{100}$
$E(W_3)$	4.59	7.26	14.94	27.96	34.21	64.79	155.19	305.35
$Var(W_3)$	9.93	22.59	116.22	510.61	763.57	2222.07	7025.53	15219.95

Table 1: Expectation and variance of the waiting time W_3

Next, we show how the expectation and the variance of the waiting time W_n depend on the probability $p_\diamond$ for different values of n.

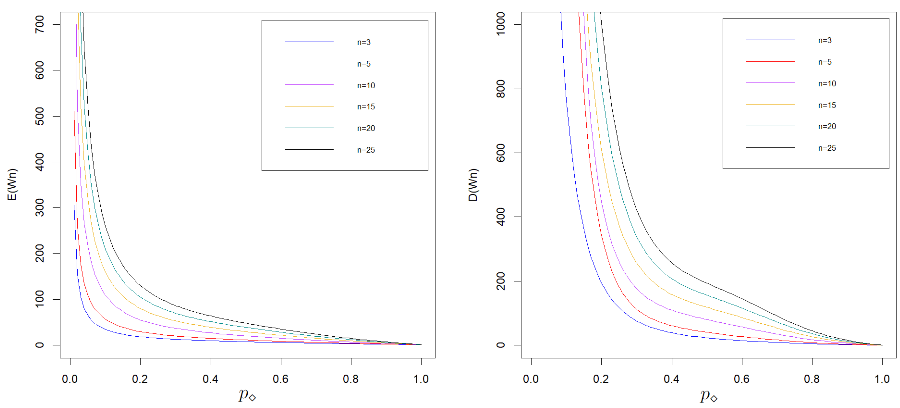

Figure 1: Expectation and variance of the waiting time W_n in terms of $p_\diamond$ for different values of n

Note that the behavior of the expectation and variance is consistent with the theoretical results in Section 3. The expectation and variance of the waiting time W_n decrease as the probability $p_\diamond$ increases. Also, the expectation and variance of the waiting time W_n increase when n increases, as we expected.

For comparison, we show the corresponding results for the CCCP and the symmetric random walk with absorbing barriers at $-n$ and n.

Using the formula for the expectation and variance of the waiting time W_n for the CCCP (see, e.g., [4, 12]), we obtain that $E(W_3) = 5.5$ and $Var(W_3) = 6.75$.

Using the formula for the expectation and variance for the waiting time W_n^* (see, e.g., [3, 9]), we get that $E(W_n^*) = 15$ and $Var(W_n^*) = 48$.

7. Conclusion

The introduction of additional coupons with specific purposes leads to new generalizations and modifications of the CCP, and one of them is presented in this paper. We assume that the additional coupon interferes with the collection of the standard coupons, and that the experiment (coupon collecting) is terminated when either the complete collection (without interference) is sampled or the interference reaches a certain value. The problem can also be formulated as a special case of 1-dimensional random walk with absorbing barriers, with the additional condition that the probability of steps in the positive direction depends linearly on the number of steps already taken.

We are interested in the numerical characteristics of the waiting time until the collection is over. Using the Markov chain approach, we provide two ways to compute the expectation and variance of this waiting time: by reference to the CCCP and by determining the exact form of the fundamental matrix. We provide a numerical comparison of the expectation and variance of the corresponding waiting times for several variants of the CCP.

The problem considered in this paper leads to several possibilities for future research. One obvious task would be to obtain accurate estimates for the expectation and variance of the waiting time W_n, taking into account the relationship between $p_\diamond$ and n, since $n \to \infty$ (i.e., if $np_\diamond \to 0$, $np_\diamond = O(1)$, or $np_\diamond \to \infty$).

Future work can also be devoted to further generalizing the problem in two ways: either by replacing the penalty coupon with another collection that would interfere with the collection of the main one, or by considering a new variant of the random walk such that the probability of steps in each direction depends on the number of steps in a more general way.

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References

[1] E. Anceaume, Y. Busnel, E. Schulte-Geers, B. Sericola, *Optimization results for a generalized coupon collector problem*, J. Appl. Probab., **53(2)** (2016), 622-–629.

[2] E. Anceaume, Y. Busnel, B. Sericola, *New Results on a Generalized Coupon Collector Problem Using Markov Chains*, J. Appl. Probab., **52(2)** (2015), 405-–418.

[3] J. Anděl, Š. Hudecová, *Variance of the game duration in the gambler's ruin problem*, Stat. Probab. Lett., **82(9)** (2012), 1750–1754.

[4] M. Ferrante, M. Saltalamacchia, *The Coupon Collector's problem*, Materials matemàtics, (2014), 1–35.

[5] P. Flajolet, D. Gardy, L. Thimonier, *Birthday Paradox, Coupon Collectors, Caching Algorithms and Self-Organizing Search*, Discret. Appl. Math., **39** (1992), 207–229.

[6] J. G. Kemeny, L. J. Snell, *Finite Markov chains*, Springer - Verlag, New York - Heidelberg, 1976.

[7] T. Nakata, I. Kubo, *A coupon collector's problem with bonuses*, Discrete Math. Theor. Comput. Sci., (2006), 215–224.

[8] D. J. Newman, *The double dixie cup problem*, Am. Math. Mon., **67** (1960), 58.

[9] G. Orosi, *Duration of play in the gambler's ruin problem: a novel derivation*, Appl. Math. E-Notes, **18** (2018), 268–274.

[10] H. Von Schelling, *Coupon collecting for unequal probabilities*, Am. Math. Mon., **61(5)** (1954), 306–311.

[11] M. Wild, S. Janson, S. Wagner, D. Laurie, *Coupon collecting and transversals of hypergraphs*, Discrete Math. Theor. Comput. Sci., **15(2)** (2013), 259–270.

[12] Y. Yu, *Minimum variance in the coupon collector's problem*, J. Appl. Probab., **54(2)** (2017), 655–656.

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