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ON THE SUBTRACTIVE SUBSEMIMODULE-BASED GRAPH OF SEMIMODULES

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Abstract. Let M be a semimodule over a commutative semiring R and K be a subtractive subsemimodule of M with $K^* = K \setminus \{0\}$. The subtractive subsemimodule-based graph of M is defined as the simple undirected graph $\Omega = \Gamma_{K^*}(M)$ with vertex set $V(\Omega) = \{v \in M \setminus K : v + v' \in K^* \text{ for some } v \neq v' \in M \setminus K\}$, and two distinct vertices m and n are adjacent if and only if $m + n \in K^*$. In this paper, we study the interplay between semimodule properties and the properties of the graph. Among other results, we compute the diameter and the girth of $\Gamma_{K^*}(M)$.

1. Introduction

Let M be a semimodule over a commutative semiring R and K a subtractive subsemimodule of M with $K^* = K \setminus \{0\}$. The subtractive subsemimodular graph of M is defined as the simple undirected graph $\Omega = \Gamma_{K^*}(M)$ with vertex set $V(\Omega) = \{v \in M \setminus K : v + v' \in K^* \text{ for some } v \neq v' \in M \setminus K\}$, and two distinct vertices m and n are adjacent if and only if $m + n \in K^*$. In this paper, we study the interplay between semimodule properties and graph properties. Among other results, we compute the diameter and the girth of $\Gamma_{K^*}(M)$.

2. Preliminaries

First we recall some notions and algebraic notations related to graphs and the theory of semimodules. The vertex set of a simple graph Ω is denoted by $V(\Omega)$. A graph is connected if any two distinct vertices are connected by a path. The shortest path from m to n, denoted by $\mathrm{d}(m, n)$, is called the distance between m and n, if there is no such path, then $d(m, n) = \infty$. The diameter of a graph Ω is $\mathrm{diam}(\Omega) = \sup\{d(m, n) :$

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$m, n \in V(\Omega)\}$. A complete graph is a simple undirected graph in which each pair of distinct vertices is connected by exactly one edge. We define the girth of Ω, denoted by $\mathrm{gr}(\Omega)$, as the length of a shortest cycle in Ω, provided Ω contains a cycle; otherwise; $\mathrm{gr}(\Omega) = \infty$, in which case Ω is called an acyclic graph. Two (induced) subgraphs Ω_1 and Ω_2 of Ω are disjoint if they have no edges and no vertices in common. We denote the complete bipartite graph with m and n vertices by $K^{m,n}$. We say that u is a universal vertex of Ω if u is adjacent to all other vertices of Ω. A vertex v in an undirected connected graph G is a cut-point (cut vertex) of G if removing it (and the edges passing through it) disconnects the graph.

We recall here some basic notions from the theory of semimodules. For definitions of semirings, semimodules, and subsemimodules, we refer to [4–6, 11]. A subtractive subsemimodule ($= k$-subsemimodule) K is such a subsemimodule of M that if $m, m+n \in K$, then $n \in K$ (so $\{0_M\}$ is a subtractive subsemimodule of M). An element m of M is called a zero-sum in M if $m+n = 0$ for every $n \in M$. We use $S(M)$ to denote the set of all zero-sum elements of M. If L is a subset of M, then $S(L) = \{a \in L : a + b = 0 \text{ for any } b \in L\}$.

We say that a subsemimodule P of M is a partitioning subsemimodule ($= Q_M$-subsemimodule) if $M = \bigcup\{q+P : q \in Q_M\}$ for a subset Q_M of M, and if $q, q' \in Q_M$, then $(q + P) \cap (q' + P) \neq \emptyset$ if and only if $q = q'$. Let P be a Q_M-subsemimodule of M and let $M/P = \{q + P : q \in Q_M\}$. Then M/P forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q+P) \oplus (q'+P) = q'' + P$, where $q'' \in Q_M$ is the only such element that $q+q'+P \subseteq q''+P$ and $r \odot (q+P) = q^* + P$, where $r \in R$ and $q^* \in Q_M$ is the unique element such that $rq + P \subseteq q^* + P$. This R-semimodule M/P is called the quotient semimodule of M by P (see [4]). The single element $q_0 \in Q_M$ exists such that $q_0 + P = P$ by [4, Lemma 2.3]. It is shown that every partitioning subsemimodule is a subtractive subsemimodule by [4, Theorem 3.2]. The subset $\{r \in R : rM \subseteq P\}$ is denoted by $(P :_R M)$ or $(P : M)$. It is clear that if P is a subsemimodule of M, $(P : M)$ is an ideal of R.

In this work, R is a commutative semiring, M is an R-semimodule, and $K^* = K \setminus \{0\}$ for each subsemimodule K of M.

3. Some properties of $\Gamma_{K^*}(M)$

This section is devoted to some properties of the $\Gamma_{K^*}(M)$-graph. Let us start with the following lemma, which will be used throughout this paper.

LEMMA 3.1. *Let K be a subtractive subsemimodule of R-semimodule M. Then:*
(i) If w and u are adjacent for some $w, u \in \Gamma_{K^*}(M)$, then $2w + u \notin K$.

(ii) If $y \in V(\Gamma_{K^*}(M))$ and $y + y' = 0$ for some $y' \in M$, then $y' \notin K$.

(iii) Let $v, w \in V(\Gamma_{K^*}(M))$ with $v \neq w$ and $NB_\Omega(v) \cap NB_\Omega(w) \neq \emptyset$. If $w + w' = 0$ for some $w' \in M$, then $v + w' \in K^*$.

(iv) Let $v, w \in V(\Gamma_{K^*}(M))$ with $v + v' = 0$ and $w + w' = 0$ for some $v', w' \in M$, then $v + w \in K^*$ if and only if $v' + w' \in K^*$.

Proof. (i) Let $2w + u \in K$. Then $2w + u = w + (w + u) \in K$. This implies that $w \in K$, since $w + u \in K$ and K is a subtractive subsemimodule of M, which contradicts our assumption.

(ii) The result is obvious since K is a subtractive subsemimodule.

(iii) Assume that $t \in NB_\Omega(v) \cap NB_\Omega(w)$. So $v + t, w + t \in K^*$. Let $v + w' = a$ for some $a \in M$. This implies that $v + t = v + w' + w + t = a + w + t \in K^*$. So $a \in K$ since K is subtractive. If $v + w' = 0$, then $v = v + w' + w = w$, which is a contradiction. Hence $v + w' \in K^*$.

(iv) It is clear that $v + w = 0$ if and only if $v' + w' = 0$. Now, let $v + w \in K^*$. Then $v + w = a$ for some $0 \neq a \in K$. Therefore $a + v' + w' = v + w + v' + w' = 0 \in K$, thus $v' + w' \in K$, since K is a subtractive subsemimodule. Similarly the other side holds. □

Remark 3.2. Let K be a subtractive subsemimodule of the R-semimodule M and $K' = \{x \in M : x + z \in K \text{ for some } z \in M \setminus K \text{ where } z \neq x\}$. If K is a subtractive subsemimodule of the R-semimodule M, then $K' \subseteq M \setminus K$.

Now, we consider the conditions under which $V(\Gamma_{K^*}(M)) \neq \emptyset$.

Proposition 3.3. *Let K be a subtractive subsemimodule of R-semimodule M with $|K^*| \geq 2$. If $S(M) \cap K' \neq \emptyset$, then $V(\Gamma_{K^*}(M)) \neq \emptyset$.*

Proof. Let $k \in S(M) \cap K'$. Then $k + l = 0$ for some $l \in M \setminus K$. If $k = l + e$ for every $0 \neq e \in K$, then $k = l + e = l + f$ for some distinct nonzero elements $e, f \in K$. So we get $e = k + l + e = k + l + f = f$, which is a contradiction. Thus we have $k \neq l + n$ for some $0 \neq n \in K$. Hence $k + l + n = n \in K^*$ and so $k, l + n \in V(\Gamma_{K^*}(M))$. □

The next result is used to identify the adjacency between the vertices of the $\Gamma_{K^*}(M)$.

Theorem 3.4. *Let K be a subtractive subsemimodule of R-semimodule M. If $x_1, x_2 \in V(\Gamma_{K^*}(M))$ are distinct vertices connected by a path of length 3 and $x_1 + x_2 \neq 0$, then x_1 and x_2 are adjacent.*

Proof. Let $x_1 - m_1 - m_2 - x_2$ be a path of length 3 between x_1 and x_2 for distinct vertices $x_1, m_1, m_2, x_2 \in V(\Gamma_{K*}(M))$. So $x_1 + m_1, m_1 + m_2, m_2 + x_2 \in K^*$. Therefore we have $(x_1 + x_2) + (2(m_1 + m_2)) = (x_1 + m_1) + (m_1 + m_2) + (m_2 + x_2) \in K$. Then $x_1 + x_2 \in K^*$ since $m_1 + m_2 \in K$ and K is a subtractive subsemimodule. Thus x_1 and x_2 are adjacent. □

Now, we extend a result of Abbasi and Jahromi [1, Theorem 2.7] for generalized graph of modules to the subtractive subsemimodule-based graph of semimodules.

Theorem 3.5. *Let K be a subtractive subsemimodule of R-semimodule M. If $x, y \in V(\Gamma_{K^*}(M))$ are distinct vertices connected by a path of length 4, then these vertices are connected by a path of length 2. In particular, $NB_\Omega(x) \cap NB_\Omega(y) \neq \emptyset$.*

Proof. Assume that $x - m_1 - m_2 - m_3 - y$ is a path from x to y with the distinct vertices $x, m_1, m_2, m_3, y \in V(\Gamma_{K*}(M))$. If either $x+m_3 \neq 0$ or $m_1+y \neq 0$, then x and m_3 or m_1 and y are adjacent by Theorem 3.4. So we get $x+m_3 = 0$ and $m_1+y = 0$. If $x = m_1+m_2+m_3$, then $x+y = m_1+m_2+m_3+y = m_2+m_3 \in K^*$. This implies that x and y are adjacent, a contradiction. Similarly, if $y = m_1+m_2+m_3$, then $x+y = x+m_1+m_2+m_3 = m_1+m_2 \in K^*$ which is also a contradiction. Therefore $x-(m_1+m_2+m_3)-y$ is a path of length 2 between x and y. □

Corollary 3.6. *Let K be a subtractive subsemimodule of R-semimodule M. If P is a path of length 4 in $\Gamma_{K^*}(M)$, then $\Gamma_{K^*}(M)$ has a cycle.*

Proof. Let $x - m_1 - m_2 - m_3 - y$ be a path of length 4 between x and y for distinct vertices $x, m_1, m_2, m_3, y \in V(\Gamma_{K*}(M))$. If either $x + m_3 \neq 0$ or $m_1 + y \neq 0$, then x and m_3 or m_1 and y are adjacent by Theorem 3.4, as we desired. So we may assume that $x + m_3 = 0$ and $y + m_1 = 0$. Then $x - (m_1 + m_2 + m_3) - y$ is a path of length 2 between x and y by Theorem 3.5. If $m_1 + m_2 + m_3 = m_1$, then $y+m_1+m_2+m_3 = y+m_1$ and $m_2+m_3 = 0$, a contradiction. Thus $m_1+m_2+m_3 \neq m_1$. Similarly, $m_1 + m_2 + m_3 \neq m_3$. If $m_1 + m_2 + m_3 = m_2$, then $x - m_1 - m_2 - x$ and $m_2 - m_3 - y - m_2$ are two cycles of length 3 in $\Gamma_{K^*}(M)$. If $m_1 + m_2 + m_3 \neq m_2$, then $x - m_1 - m_2 - m_3 - y - (m_1 + m_2 + m_3) - x$ is a cycle of length 6 in $\Gamma_{K^*}(M)$, as we desired. □

4. The maximum distance and the shortest cycle in $\Gamma_{K^*}(M)$

In this section we compute $diam(\Gamma_{K^*}(M))$ and $gr(\Gamma_{K^*}(M))$, the length of the maximum distance and the shortest cycle in $\Gamma_{K^*}(M)$, respectively. In the next example, we introduce an R-semimodule such that $\Gamma_{K^*}(M)$ is a disconnected graph and is a union of two disjoint complete subgraphs.

Example 4.1. Let $R = Z^+ \cup \{0\}$. It is clear that R is an R-semimodule. Assume that $K = 4Z^+ \cup \{0\}$, so K is a subtractive R-subsemimodule. Assume that $x \in V(\Gamma_{K^*}(M))$, then $x \neq 4t$ and so either $x = 4t+1$, $x = 4t+2$ or $4t+3$ for some nonnegative integer t. Let $V_1 = \{x \in V(\Gamma_{K^*}(M)) : x = 4t+2$ for some nonnegative integer $t\}$ and $V_2 = \{x \in V(\Gamma_{K^*}(M)) : x = 4t+1$ or $x = 4t+3$ for some nonnegative integer $t\}$. It is easy to see that the subgraph $\Gamma^{V_1}_{K^*}(M)$ of $\Gamma_{K^*}(M)$ is a complete graph and the subgraph $\Gamma^{V_2}_{K^*}(M)$ of $\Gamma_{K^*}(M)$ is a bipartite complete graph and these two subgraphs are disjoint. So $gr(\Gamma_{K^*}(M)) = 3$ and $diam(\Gamma_{K^*}(M)) = \infty$.

Theorem 4.2. *Let K be a subtractive subsemimodule of R-semimodule M. Then $diam(\Gamma_{K^*}(M)) \in \{1, 2, 3, \infty\}$.*

Proof. Applying Theorem 3.5, every path of length greater 3 can reduce to a path of length at most 3. □

The subtractive condition of subsemimodule K in Theorem 4.2 is necessary. To see this, consider the following example.

Example 4.3. Consider the idempotent semiring $R = \{0, 1, c\}$ in which $1 + c = c + 1 = c$. Then the subsemimodule $N = \{0, c\}$ of R-semimodule R is not subtractive [11, Example 6.39]. It is clear that $V(\Gamma_{N^*}(M)) = \{1\}$ and $\Gamma_{N^*}(M) = K_1$. So $diam(\Gamma_{N^*}(M)) = 0$.

Theorem 4.4. *Let K be a subtractive subsemimodule of R-semimodule M and $\Gamma_{K^*}(M)$ be a connected graph. If $2t = 0$ for every $t \in V(\Gamma_{K*}(M))$, then $\Gamma_{K^*}(M)$ is a complete graph.*

Proof. By Theorem 4.2, we have $diam(\Gamma_{K^*}(M)) \leq 3$. First suppose that $z, w \in V(\Gamma_{K^*}(M))$ with $z \neq w$ and $d(z, w) = 2$, so there exists a path $z - t - w$ in $\Gamma_{K^*}(M)$. Hence $z + w = z + w + 2t = (z + t) + (w + t) \in K$. If $z + w = 0$, it follows that $w = 2z + w = z$, a contradiction. So $z + w \neq 0$ and z and w are adjacent. If $d(z, w) = 3$, then z and w are adjacent by Theorem 3.4. □

In the following example we introduce an R-semimodule M such that $\Gamma_{K^*}(M)$ is a complete graph but $2t \neq 0$ for every $t \in V(\Gamma_{K*}(M))$ showing that the converse of [1, Theorem 2.11] can fail for semimodules.

Example 4.5. Let $R = \{[0, a], +, .\}$ where $a \in Z^+ \cup \{0\}$ be an interval semiring and $M = R$. Then subsemimodule $K = \{[0, a] : a \in 2Z^+ \cup \{0\}\}$ is subtractive. Assume that $x \in V(\Gamma_{K^*}(M))$, so $x \neq [0, 2k]$ and $x = [0, 2k+1]$ for some nonnegative integer k. Let $x, y \in V(\Gamma_{K^*}(M))$. Then $x = [0, 2a+1]$ and $y = [0, 2a'+1]$ for some nonnegative integers a and a'. So $x + y = [0, 2a + 1] + [0, 2a' + 1] = [0, 2(a + a' + 1)] \in K^*$, thus $\Gamma_{K^*}(M)$ is a complete graph with $diam(\Gamma_{K^*}(M)) = 1$ and $gr(\Gamma_{K^*}(M)) = 3$.

Our next goal is to determine an upper bound for the girth of $\Gamma_{K^*}(M)$.

Theorem 4.6. *Let K be a subtractive subsemimodule of R-semimodule M and $\Gamma_{K^*}(M)$ has a cycle. Then $gr(\Gamma_{K^*}(M)) \leq 6$.*

Proof. Let $x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_1$ be a cycle of length 7 in $\Gamma_{K^*}(M)$. Then there exists a path $x_1 - t - x_5$ of length 2 between x_1 and x_5 by Theorem 3.5. If $t \notin \{x_2, x_3, x_4\}$, then $x_1 - t - x_5 - x_6 - x_7 - x_1$ is a cycle of length 5 in $\Gamma_{K^*}(M)$. Now suppose that $t \in \{x_2, x_3, x_4\}$. Then we have a cycle with length less than 6. □

Theorem 4.7. *Let K be a subtractive subsemimodule of R-semimodule M. If $|S(M) \setminus K| \geq 4$ and $2t \in K^*$ for every $0 \neq t \in R$, then $gr(\Gamma_{K^*}(M)) \leq 4$.*

Proof. Let $0 \neq x \in S(M) \setminus K$. Then there exists $y \in S(M)$ such that $x + y = 0$. It is clear that $y \notin K^*$ since K is a subtractive subsemimodule. If $x = y$, then $2x = 0$ which contradicts our assumption. Now let $z \in S(M) \setminus K$ and $z \neq x, y$. Similarly $z + w = 0$ for some $w \in S(M) \setminus K$ and $w \notin \{x, y, z\}$.
Case 1. If $x + z \in K$, then $y + z = x + z + 2y \in K$ since $2y \in I$. If $y + z = 0$, then $y = y + z + w = w$, a contradiction. This implies that $y + z \in K^*$. By a similar argument we can show that $y + w, w + x \in K^*$. So in this case we have a cycle $x - w - y - z - x$ of length 4 in $\Gamma_{K^*}(M)$.

Case 2. Assume that $x+z \notin K$. If $x+w \in K$, then $x+z = x+w+2z \in K$ by assumption which is a contradiction. So we have $x+w \notin K$. Similarly we can show that $y+w, y+z \notin K$. Now we show that $(x+z)-(x+w)-(y+w)-(y+z)-(x+z)$ is a cycle of length 4 in $\Gamma_{K^*}(M)$. It suffices to show that $x+z \neq y+w$ and $x+w \neq y+z$. Suppose that $x+z = y+w$. Then $2x = 2x+z+w = x+z+x+w = y+w+x+w = 2w$. This implies that $2(x+z) = 2x+2z = 2w+2z = 2(w+z) = 0$ which contradicts our assumption. So $x+z \neq y+w$. If $x+w = y+z$, then $2x = 2x+z+w = x+w+x+z = y+z+x+z = 2z$ and we have $2(x+w) = 2x+2w = 2z+2w = 0$, a contradiction. So we have a cycle $(x+z)-(x+w)-(y+w)-(y+z)-(x+z)$ of length 4 in $\Gamma_{K^*}(M)$. □

THEOREM 4.8. *Let K be a subtractive subsemimodule of R-semimodule M and $|K^* \setminus S(M)| \geq 2$. Then $\Gamma_{K^*}(M)$ is an acyclic graph if and only if it is a distinct union of some star components.*

Proof. Let $\Gamma_{K^*}(M)$ be an acyclic graph. If $\Gamma_{K^*}(M)$ has no star components, then there exists a path $x-w_1-w_2-y$ of length 3 in $\Gamma_{K^*}(M)$. If $x+y \neq 0$, then x and y are adjacent by Theorem 3.4. So we have a cycle in $\Gamma_{I^*}(R)$ which is a contradiction. Then we may assume that $x+y=0$.

Let $\alpha, \beta \in K^* \setminus S(M)$ and $\alpha \neq \beta$. If $x+\alpha = w_1$, then $y+w_1 = y+x+\alpha = \alpha \in K^*$. So we have a cycle $w_1-w_2-y-w_1$ in $\Gamma_{I^*}(R)$, a contradiction. Now assume that $x+\alpha = y$, then $2x+\alpha = x+y = 0$. This implies that $\alpha \in S(R)$ which contradicts the assumption. Therefore $x+\alpha \notin \{x, w_1, y\}$. By a similar argument we have $x+\beta \notin \{x, w_1, y\}$ and $y+\alpha \notin \{y, x, w_2\}$. If either $x+\alpha \neq w_2$ or $y+\alpha \neq w_1$, then $x-w_1-w_2-y-(x+\alpha)$ or $(y+\alpha)-x-w_1-w_2-y$ is a path of length 4 in $\Gamma_{I^*}(R)$. Thus we have a cycle in $\Gamma_{I^*}(R)$ by Corollary 3.6. So we may assume that $x+\alpha = w_2$ and $y+\alpha = w_1$. If $x+\beta = w_2 = x+\alpha$, then we have $\alpha = \beta$, a contradiction. Then $x+\beta \neq w_2$ and so $x-w_1-w_2-y-(x+\beta)$ is a path of length 4 in $\Gamma_{I^*}(R)$. So we have a cycle in $\Gamma_{I^*}(R)$ by Corollary 3.6 which is a contradiction. □

LEMMA 4.9. *Let K be a subtractive subsemimodule of R-semimodule M with $2v \in K^*$ for all $v \in V(\Gamma_{K^*}(M))$. Then:*

(i) If x and u are adjacent with $x \neq u$ and $x+x' = 0$ for some $x' \in M$, then $u+x' \in K^*$.

(ii) If $u-x-s$ is the shortest path between u and s and $2x+u = 2u+x = x$, then $u+s=0$ and $2x+s = 2s+x = x$.

(iii) If $x-z-y$ is the shortest path from x to y, then $x+y=0$.

Proof. (i) It is clear that $u+x' \neq 0$, since $x \neq u$ by assumption. Suppose that $u+x' = n$ for some $n \in M$, then $u+x = u+x'+2x = n+2x$. So $u+x' \in K$, since K is a subtractive subsemimodule of M and $u+x, 2x \in K$.

(ii) Assume that $2x+u = x$, so $2x+u+s = x+s \in K$. This implies that $u+s \in K$, since K is subtractive and $2x \in K$. Therefore $u+s=0$ since $d(u,s)=2$. Also $x+u = 2u+s+x = s+x$ so we can conclude that $2s+x = s+2x = x$.

(iii) Let $x - z - y$ be the shortest path from x to y. Then $2z + x + y = x + z + z + y \in K$ and so $x + y \in K$ since $2z \in K$ and K is a subtractive subsemimodule of M. Therefore $x + y = 0$ since x and y are not adjacent. □

THEOREM 4.10. *Let K be a subtractive subsemimodule of R-semimodule M. If $\Gamma_{K^*}(M)$ is a connected graph with $|M \setminus K^*| \geq 4$ and $2v \in K^*$ for all $v \in M$, then $\Gamma_{K^*}(M)$ has no cut-points.*

Proof. Suppose that c is a cut-point of $\Gamma_{K^*}(M)$. So there exist vertices $u, w \in V(\Gamma_{K^*}(M))$ such that $c \neq u, w$ and c lies on every path between u to w. The shortest path between u to w is of length 2 or 3 by Theorem 4.2.
Case 1. Suppose that $u - c - y - w$ is a path of the shortest length between u and w. So $u + w = 0$ by Theorem 3.4. Also, $2c + u, 2y + w \notin K$ by Lemma 3.1. First suppose that $c \notin S(M)$ and $y \notin S(M)$. If $2c + u = c$, then $c + w = 2c + u + w = 2c \in K^*$ follows that c and w are adjacent, a contradiction. Similarly, $2y + w \neq c$. If $2c + u = 2c + w$, then $u - (2c + u) - w$ is a path of length 2, which is a contradiction. Hence $u - (2c + u) - (2y + w) - w$ is a path between u and w, which contradicts our assumption. Now, suppose that either $c \in S(M)$ or $y \in S(M)$. If $c \in S(M)$, then $u - c' - w$ is a path from u to w by Lemma 3.1 and Lemma 4.9, where $c + c' = 0$, which is a contradiction. The case $y \in S(M)$ is similar.

Case 2. Assume that $u - c - w$ is a path of the shortest length from u to w. First suppose that $c \in S(M)$. So $c + c' = 0$ for some $c' \in M \setminus K$ by Lemma 3.1. Therefore $u - c' - w$ is a path from u to w by Lemma 4.9, which contradicts our assumption. Now assume that $c \notin S(M)$. If either $2c + u \neq c$ or $c + 2u \neq c$, then $u - (2c + u) - w$ or $u - (2u + c) - w$ is a path between u and w, a contradiction. So we may assume that $2c + u = 2u + c = c$. Since $|M \setminus K^*| \geq 4$ and $\Gamma_{K^*}(M)$ is a connected graph, there exists $m \in R \setminus \{u, c, w\}$ such that $m \notin K^*$ and m is adjacent to one vertex of path $u - c - w$. First suppose that m and c are not adjacent and $m - u - c - w$ is a path from m to w. If $m - u - c - w$ is the shortest path from m to w, then by Case 1., we have a path P from m to w which $c \notin V(P)$, so $P' = P \cup \{u, m\}$ is a path from u to w which is a contradiction. Now, assume that $m - t - w$ is the shortest path from m to w where $t \neq c$. Thus $u - m - t - w$ is a path from u to w, a contradiction. If $u - c - w - m$ is a path, the proof is similar. So we may assume that m and c are adjacent and $m + c \in K^*$. This implies that $2c + u + m = c + m \in K$, thus $u + m \in K$. If $u + m = 0$, then $m = u + m + w = w$, a contradiction. Thus $u + m \in K^*$. Now we show that $w + m \in K^*$. We have $w + 2c + m = w + c + c + m \in K$. So $w + m \in K$ since $2c \in K$ and K is a k-ideal. If $w + m = 0$, then $u = u + w + m = m$, a contradiction. Therefore $u - m - w$ is a path from u to w which is also a contradiction. □

5. Partitioning subsemimodule-based graphs of semimodules

In this section we assume that P is a partitioning subsemimodule (Q_M-subsemimodule) of M and we shall describe the $\Gamma_{P^*}(M)$ graph with its structure, girth and diameter. First, we have the following lemma.

LEMMA 5.1. *Let P be a partitioning subsemimodule of R-semimodule M with $Q'_M = Q_M \setminus \{q_0\}$. Then the following hold:*
(i) $Q'_M \subseteq M \setminus P$.

(ii) If $q \in Q'_M$ and $q + q^* = 0$ for some $q^* \in M$, then $q^* \in M \setminus P$.

(iii) Let $q \in Q'_M \cap S(M)$ and $a + q$ and $b + q$ are adjacent in $\Gamma_{P^*}(M)$ for some $a, b \in P$. Then $2q = 0$.

Proof. (i) Let $q \in Q'_M$. If $q \in P$, then $q \in P \cap Q_M$. So $q + q_0 \in (q + P) \cap (q_0 + P)$ and $(q + P) \cap (q_0 + P) \neq \emptyset$. We have $q = q_0$, which is a contradiction.

(ii) It is clear by part (i) and since P is a subtractive subsemimodule by [4, Theorem 3.2].

(iii) Let $a + q$ and $b + q$ be adjacent. Then $2q + a + b = k$ for some $k \in P^*$. Since $q \in S(M)$, so $q + p = 0$ for some $p \in M \setminus P$. We have $p \in q' + P$ for some $q' \in Q'_R$. Then $p = q' + n$ for some $n \in P$. Hence $q + q' + n = 0$ and $q+a+b = q+q'+n+q+a+b = q'+2q+a+b+n = q'+n+n \in (q+P)\cap(q'+P)$. Then $(q + P) \cap (q' + P) \neq \emptyset$ and so $q = q'$. Thus, we have $2q = 0$. □

In the following lemma, we see that if $2 \in (K : M)$, then the adjacent vertices of $\Gamma_{P^*}(M)$ are in a same coset of partitioning subsemimodule P of M.

LEMMA 5.2. *Let P be a partitioning subsemimodule of R-semimodule M with $2 \in (P : M)$. Then:*
(i) If v and y are adjacent in $\Gamma_{P^*}(M)$, then there exists $q \in Q_M \setminus \{q_0\}$ such that $v, y \in q + P$.

(ii) If $x \in V(\Gamma_{P^*}(M))$, then $NB_\Omega(x) \subseteq q + P$ for some $q \in Q_M \setminus \{q_0\}$.

(iii) If $q \in Q'_M \cap S(M)$ and $|P^*| \geq 1$, then $2q + m = 0$ for $m \in P$.

Proof. (i) Let $v \in q_1 + P$ and $y \in q_2 + P$ for some $q_1, q_2 \in Q_M \setminus \{q_0\}$. We show that $q_1 = q_2$ and so v and y are in a same coset. Assume that $v = q_1 + a$ and $y = q_2 + b$ for some $a, b \in P$. Therefore we have $q_1 + q_2 + a + b = v + y \in P^*$, since v and y are adjacent. Hence $q_1 + q_2 \in P$ since $a + b \in P$ and P is a subtractive subsemimodule by [4, Theorem 3.2]. So $q_2 + 2q_1 = q_1 + (q_1 + q_2) \in q_1 + P$. Likewise, $q_2 + 2q_1 \in q_2 + P$ since $2 \in (P : M)$. So $q_2 + 2q_1 \in (q_1 + P) \cap (q_2 + P)$; hence $q_1 = q_2$.

(ii) It is clear from part (i).

(iii) Let $q + p = 0$ for some $p \in M \setminus P$. So $p \in q' + P$ for some $q' \in Q'_M$. Then $p = q' + m$ for some $m \in P$. Hence $q + q' + m = 0$ and $q + q' + m + n = n \in P^*$ for every $n \in P^*$. So q and $q'+m+n$ are adjacent vertices and there exists $q'' \in Q_R \setminus \{q_0\}$ such that $q, q' + m + n \in q'' + P$ by (i). This implies that $q \in (q + P) \cap (q'' + P)$ and $(q+P)\cap(q''+P) \neq \emptyset$. Thus $q = q''$. On the other hand, $q'+m+n \in (q'+P)\cap(q''+P)$ and then $(q+P)\cap(q'+P) \neq \emptyset$. So we have $q' = q''$ and $2q+m = q+q'+m = 0$. □

We can now prove the following theorem that provides a characterization of $\Gamma_{P^*}(M)$ when $2 \in (P : M)$.

THEOREM 5.3. *Let P be a partitioning subsemimodule of R-semimodule M with $2 \in (P : M)$ and $Q'_M = Q_M \setminus \{q_0\}$. If $|P^*| \geq 1$, $\alpha = |Q'_M \setminus S(M)|$ and $\beta = |Q'_M \cap S(M)|$,* *then $\Gamma_{P^*}(M)$ is a union of disjoint α complete subgraphs and β connected subgraphs with a universal vertex.*

Proof. Let $q \in Q'_M$. First suppose that $q \notin S(M)$, then $q+n+q+n' = 2q+n+n' \in P^*$ for every $n, n' \in P$. So the induced subgraph $\Gamma^{q+P}_{P^*}(M)$ of $\Gamma_{P^*}(M)$ is a complete subgraph. So we have α disjoint complete subgraphs by Lemma 5.2. Now, assume that $q \in S(M)$. So $2q + m = 0$ for some $m \in P$ by Lemma 5.2. Then the induced subgraph $\Gamma^{q+P}_{P^*}(M)$ of $\Gamma_{P^*}(M)$ is a connected graph and $q + m$ is a universal vertex of this subgraph. Also, these subgraphs are disjoint by Lemma 5.2. $\square$

PROPOSITION 5.4. *Let P be a partitioning subsemimodule of R-semimodule M with $Q'_M = Q_M \setminus \{q_0\}$ and $|P^*| \geq 1$. If $q + q' \in P$ for some $q, q' \in Q'_M$, then the induced subgraphs of $\Gamma_{P^*}(M)$ with vertices set $V = (q+P) \cup (q'+P)$ are connected subgraphs.*

Proof. First suppose that $q + q' = 0$. Then $q + (q' + x^*) = (q + x^*) + q' = x^* \in P^*$ for every $x^* \in P^*$. This implies that every element of $q + P$ are adjacent to q' in $q' + P$ and also, every element of $q' + P$ is adjacent to q in $q + P$. Therefore the induced subgraph with vertices set $(q + P) \cup (q' + P)$ is a connected subgraph. Now suppose that $q + q' \neq 0$. We divide the proof into two cases:
Case 1. If $S(P) = P$, then $q + q' + y^* = 0$ for some $y^* \in P$. Now let $0 \neq n \in P$. Then $q + n + q' + y^* = q + y^* + q' + n = n \in P^*$. It means that every element of $q + P$ are adjacent to $q' + y^*$ in $q' + P$ and also, every element of $q' + P$ is adjacent to $q + y^*$ in $q + P$. Therefore the induced subgraph with vertices set $(q + P) \cup (q' + P)$ is a connected subgraph.

Case 2. Assume that $S(P) \neq P$, then there exists $m \in P$ such that $m + p' \neq 0$ for every $p' \in P$. This implies that $q + m + q' + p' = q + p' + q' + m \in P^*$. Hence every element of $q + P$ are adjacent to $q' + m$ in $q' + P$ and also, every element of $q' + P$ is adjacent to $q + m$ in $q + P$, as required. $\square$

THEOREM 5.5. *Let P be a partitioning subsemimodule of R-semimodule M. Then $diam(\Gamma_{P^*}(M)) = \{1, 2, 3, \infty\}$ and $gr(\Gamma_{P^*}(M)) = \{3, 4, \infty\}$.*

Proof. Let $2 \notin (P : M)$ and $q \in Q'_M = Q_M \setminus \{q_0\}$. If either $q \in S(M)$ or $NB_\Omega(q) \neq \emptyset$, then $q + q' \in P$ for some $q' \in Q_M$. So the induced subgraphs of $\Gamma_{P^*}(M)$ with vertices set $V = (q + P) \cup (q' + P)$ is a connected subgraph by Proposition 5.4. Now assume that $q \notin S(M)$ and $NB_\Omega(q) = \emptyset$. If $2q \in P$, then $q + n_1 + q + n_2 = 2q + n_1 + n_2 \in P^*$ for every $n_1, n_2 \in P$. So the induced subgraph $\Gamma^{q+P}_{P^*}(M)$ is a complete graph. Now, we may assume that $2q \notin P$, then $q + n_1 + q + n_2 = 2q + n_1 + n_2 \notin P$ for every $n_1, n_2 \in P$, since P is a subtractive subsemimodule. Therefore the induced subgraph with vertices set $q + P$ is a totally disconnected subgraph and the proof is complete by Theorem 5.3. $\square$

We end the paper with the following example.

EXAMPLE 5.6. (i) Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = \mathbb{Z}_6$. Then $(R, +, .)$ is a commutative semiring and $(M, +_6)$ is an R-semimodule. Set $P = \{0, 2, 4\}$ and $Q_M = \{0, 1\}$.

Then P is a partitioning subsemimodule of M. It is easy to see that $2 \in (P : M)$. Since $|M/P| = 2$, so $\Gamma_{P^*}(M)$ is a complete graph by Theorem 5.3 with $diam(\Gamma_{P^*}(M)) = 1$ and $gr(\Gamma_{P^*}(M)) = 3$.

(ii) Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$ and $M = R$. Then for each $m \in R \setminus \{0\}$, Rm is a Q_M-subsemimodule of M where $Q_M = \{0, 1, 2, \ldots, m-1\}$. If $P = 3R = \{0, 3, 6, 9, \ldots\}$, then P is a partitioning subsemimodule of M and $2 \notin (P : M)$. Then $\Gamma_{P^*}(M)$ is a graph with vertices set $(1 + P) \cup (2 + P)$. It is easy to see that this graph is bipartite with $diam(\Gamma_{P^*}(M)) = 2$ and $gr(\Gamma_{P^*}(M)) = 4$.

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