MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
75, 3 (2023), 216–224
September 2023

research paper
оригинални научни рад
DOI: 10.57016/MV-rx451is8

HOPF BIFURCATIONS IN DYNAMICAL SYSTEMS VIA ALGEBRAIC TOPOLOGICAL METHOD

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Abstract. A nonlinear phenomenon in nature is often modeled by a system of differential equations with parameters. The bifurcation occurs when a parameter varies in such systems, causing a qualitative change in its solution. In this paper, we study one of the most exciting bifurcations, which is Hopf bifurcation. We use tools from algebraic topology to analyze and reveal supercritical and subcritical Hopf bifurcations.

1. Introduction

Dynamical systems describe many phenomena in different sciences, like engineering and biology. Studying a bifurcation in the dynamical system helps us to understand the phenomenon. One of the most common bifurcations is the Hopf bifurcation, which happens when an equilibrium point changes its stability, and a periodic orbit is generated to produce a local birth or death of the periodic solution. It only appears in systems of two or more dimensions. In this paper, we study the Hopf bifurcation in the planar system as the following

$$\begin{aligned} \dot{x} &= f_\mu(x,y), \\ \dot{y} &= g_\mu(x,y), \end{aligned} \tag{1}$$

where μ is a parameter. There are two basic kinds of the Hopf bifurcations.

- Supercritical Hopf bifurcation, where an unstable equilibrium point creates a stable limit cycle around it as the parameter μ passes through the bifurcation value μ_0, see picture (a) in Figure 1.
- Subcritical Hopf bifurcation, where a stable equilibrium point creates an unstable limit cycle around it as the parameter μ passes through the bifurcation value μ_0, see picture (b) in Figure 1.

2020 Mathematics Subject Classification: 55-08, 37B30, 37G15, 37M20
Keywords and phrases: Supercritical Hopf bifurcation; subcritical Hopf bifurcation; stable limit cycle; unstable limit cycle; homological Conley index; Morse sets.

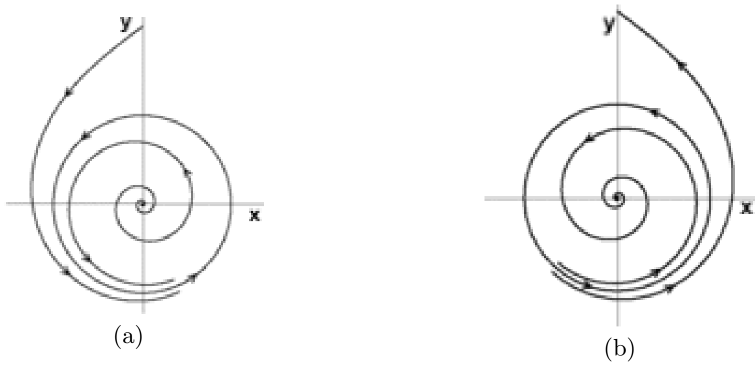

Figure 1: (a) Supercritical Hopf bifurcation and (b) subcritical Hopf bifurcation.

The Hopf bifurcation is sometimes named as a Poincaré–Andronov–Hopf bifurcation, it is prevalent in many mathematical models, such as the Lotka–Volterra model [11], the Hodgkin–Huxley model [19], the Selkov model of glycolysis [3], the Lorenz model [17], the Brusselator system [7], and the Van der Pol oscillators [21]. Recently, many approaches have been developed to study the Hopf bifurcations. In [20], numerical analysis and normal form theory have been applied to study the normal form of degenerate Hopf bifurcation. Berg et al [1] have considered the rigorous verification of Hopf bifurcations for the class of polynomial vector fields, and a blowup technique has been used to transform the Hopf bifurcation problem into a regular continuous problem. In [15], Rionero has found the conditions for the occurrence of the Hopf bifurcation in autonomous ODE systems containing n equations when $n = 2, 3, 4$. Hu has improved a general Hopf bifurcation theory for differential equations with a state-dependent delay managed by an algebraic equation, see [9]. In the homological approach, Hopf bifurcations have been detected in dynamical systems using ZigZag persistent homology called BuZZ, see [18]. In this paper, a topological method is adopted, which depends on homological Conley index theorems, Morse sets, and exact sequences. This method analyzes and evaluates the homological Conley index of both supercritical and subcritical Hopf bifurcations.

This paper is organized as follows. In the next section, we present the preliminaries of the Conley index that are necessary in this research. In Section 3, a topological investigations of both supercritical and subcritical Hopf bifurcation are explained and evaluated in the mode of homological Conley index. In Section 4, the results are summarized and some future works are suggested.

2. Preliminaries

Let Γ be a Hausdorff topological space. A dynamical system is a flow $\phi : \Gamma \times \mathbb{R} \to \Gamma$, which is a function characterized by the properties: $\phi(x, 0) = x$ and $\phi(\phi(x, t), s) = \phi(x, t + s)$ for every point $x \in \Gamma$ and all times $t, s \in \mathbb{R}$. The flow is usually obtained from a system of differential equations $\frac{dx}{dt} = f(x)$ by integration. Fixed points of a flow are points x_0 such that $\varphi(x_0, t) = x_0$ for all t, it can be obtained by solving the equation $f(x_0) = 0$.

Theorems for classifying isolated fixed points of a flow are based on the eigenvalues of the linearization of the dynamical system at the fixed point, but eigenvalues do not classify all isolated fixed points. The concept of an isolated invariant set is a generalization of the concept of an isolated fixed point of a flow.

For $\mathcal{N} \subset \Gamma$, the maximal invariant subset of $\mathcal{N}$ with respect to the flow ϕ is denoted by $\text{Inv}(\mathcal{N}) = \{x \in \mathcal{N} : \phi(x, \mathbb{R}) \subseteq \mathcal{N}\}$. The following definitions of the index pair and Conley index are given in [2, 4–6, 8, 10, 12–14, 16].

Definition 2.1. A locally compact subset $X \subset \Gamma$ is said to be a *local flow* if for every $x \in X$ there exists a neighborhood U of x in Γ and an $\varepsilon > 0$ such that $\varphi(X \cap U, [0, \varepsilon)) \subset X$.

Definition 2.2. Let $X \subset \Gamma$ be a local flow. Then a compact subset $\mathcal{N} \subset X$ is an *isolating neighborhood* in X if $\text{Inv}(\mathcal{N}) \subset \text{int}_X(\mathcal{N})$, where $\text{int}_X(\mathcal{N})$ represents the interior of $\mathcal{N}$ in the relative topology of X.

Definition 2.3. Let $X \subset \Gamma$ be a local flow. Then the subset $S \subset X$ is an *isolated invariant set* if there exists an isolating neighborhood $\mathcal{N}$ in X such that $\text{Inv}(\mathcal{N}) = S$.

The Conley index of an isolated invariant set S uses homology groups to define a generalization of the classification of an isolated fixed point. A pair of compact sets (N, L), where $L \subset N \subset X$ is called an index pair of S in $X \subset \Gamma$ if (N, L) satisfies the following conditions:

(i) $S = \text{Inv}(\overline{N \setminus L})$, and $\overline{N \setminus L}$ is an isolating neighborhood of S in X.

(ii) L is positively invariant in N, that is, given $x \in L$ and $\phi(x, [0, t]) \subset N$, then $\phi(x, [0, t]) \subset L$.

(iii) L is an exit set for N, that is, given $x \in N$ and $t_1 > 0$ such that $\phi(x, t_1) \notin N$, then there exists $t_0 \in [0, t_1]$ such that $\phi(x, [0, t_0]) \subset N$ and $\phi(x, t_0) \in L$.

The homological Conley index of S, denoted by $CH_q(S)$, is the relative homology of an index pair of S, and it is taken with $\mathbb{Z}_2$ coefficients.

Usually Γ is a metric space, and very often $\Gamma = \mathbb{R}^n$. We consider a Hausdorff space Γ with a flow to introduce basic concepts of Conley index of isolated invariant sets and attractor-repeller pairs. We say $X \subset \Gamma$ is a metric local flow if it is a local flow and a metric subspace of the Hausdorff space Γ.

Definition 2.4. Let $X \subset \Gamma$ be a metric local flow and let S be an isolated invariant set in X. Then the homotopy type $h(S) = [N/L]$ of the pointed space N/L, (N, L) being an index pair for S in X, is said to be the *homotopy Conley index* of S in X.

Definition 2.5. Let S be an isolated invariant set and (N, L) be an index pair of S in X. Then the *homology Conley index* of S is given by the homology of N relative to L: $CH_q(S) = CH_q(S, \varphi) := H_q(N, L)$.

The ω-limit set of a subset $Y \subset \Gamma$ is given by

$$\omega(Y) := \operatorname{Inv}\left(\overline{\phi(Y,[0,\infty))}\right) = \bigcap_{t>0} \overline{\phi(Y,[t,\infty))},$$

and the α-limit set (ω^*-limit set) of a subset $Y \subset \Gamma$ is given by

$$\alpha(Y) := \operatorname{Inv}\left(\overline{\phi(Y,(-\infty,0])}\right) = \bigcap_{t<0} \overline{\phi(Y,(-\infty,t])}.$$

If S_1 and S_2 are isolated invariant sets in a compact invariant set S, then we define the set of connecting orbits from S_2 to S_1 as $C(S_2, S_1) := \{x \in S \mid \omega(x) \subset S_1 \text{ and } \alpha(x) \subset S_2\}$, where $\omega(x)$ and $\alpha(x)$ denote the omega and alpha limit sets of x, respectively.

A *partial order* on a set $\mathcal{P}$ is a relation $<$ on $\mathcal{P}$ that satisfies:
(i) $\pi < \pi$ never holds for $\pi \in \mathcal{P}$; (ii) if $\pi < \pi'$ and $\pi' < \pi''$ then $\pi < \pi''$.
A subset $I \subset \mathcal{P}$ is called an *interval* if $\pi < \pi'' < \pi'$ with π, $\pi' \in I$ implies $\pi'' \in I$. The set of intervals in $<$ is denoted $I(<)$.

An interval $I \in I(<)$ is called an *attracting interval* if $\pi \in I$ and $\pi' < \pi$ imply $\pi' \in I$. The set of attracting intervals in $<$ is denoted by $A(<)$. Points π, $\pi' \in \mathcal{P}$ are called *adjacent* if $\{\pi, \pi'\} \in I(<)$.

Definition 2.6. Let S be a compact invariant subset in Γ. Then a compact invariant subset $A \subset S$ is said to be an *attractor* in S if there exists a neighborhood U of A in S such that $A = \omega(U)$.

If A is an attractor in S, the dual (complementary) *repeller* of A in S is defined by $A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}$.

Definition 2.7. Let $S \subset \Gamma$ be a compact invariant set and let A be an attractor in S. Then (A, A^*) is called an *attractor-repeller pair* in S.

Definition 2.8. Let S be a compact invariant set (not necessarily isolated). A ($<$-ordered) *Morse decomposition* of S is a collection $M(S) = \{M(p) \mid p \in \mathcal{P}\}$ of mutually disjoint compact invariant subsets of S such that if $x \in S \setminus \bigcup_{p \in \mathcal{P}} M(p)$, then there exist $q < p \quad (p, q \in \mathcal{P})$ with $\omega(x) \subset M(q)$ and $\alpha(x) \subset M(p)$, i.e., $x \in C\big(M(p), M(q)\big)$.

In general, any ordering on $\mathcal{P}$ satisfying the above property is called admissible (for the flow). The invariant sets, $M(p)$, are called Morse sets. Moreover, if S is isolated, then each $M(p)$ is also isolated.

For each $I \in I(<)$, let $M(I) = \left(\bigcup_{\pi \in I} M(\pi)\right) \cup \left(\bigcup_{\pi, \pi' \in I} C\big(M(\pi'), M(\pi)\big)\right)$. We call $M(I)$ a Morse set of the admissible ordering $<$ of M. The collection $\{M(I) \mid I \in I(<)\}$ of Morse sets of the admissible ordering $<$ is denoted by $MS(<)$. If I $\in A(<)$, then $M(I)$ is an attractor in S, and $M(\mathcal{P} \setminus I)$ is its dual repeller. Since $M(I)$ is isolated invariant set, $h\big(M(I)\big)$ is defined. Let $CH_q(I) = CH_q\big(h\big(M(I)\big); \mathbb{Z}_2\big)$ be the singular homology of the pointed space $h\big(M(I)\big)$.

If (A, A^*) is an attractor-repeller pair in S, then S decomposes into the union, $S = A \cup A^* \cup C(A^*, A; S)$. The following theorem generalizes the idea of an index pair for S to that of an index triple for (A, A^*).

THEOREM 2.9. *Assume $N_0 \subset N_1 \subset N_2$. If (N_1, N_0) is an index pair for A, and (N_2, N_0) is an index pair for S, then (N_2, N_1) is an index pair for A^*.*

We call such a triple (N_2, N_1, N_0) an index triple for the attractor-repeller pair (A, A^*) in S; it defines a sequence of maps on quotient spaces $N_1/N_0 \xrightarrow{i} N_2/N_0 \xrightarrow{p} N_2/N_1$, where i and p are the obvious inclusion and projection maps, respectively. Passing to homology, there is the following long exact homology sequence.

$$\cdots \to H_q(N_1, N_0) \xrightarrow{i} H_q(N_2, N_0) \xrightarrow{p} H_q(N_2, N_1) \xrightarrow{\partial} H_{q-1}(N_1, N_0) \to \cdots$$

This sequence is independent of the choice of index triple for (A, A*) in S and therefore defines a sequence, called the homology index sequence for (A, A^*) in S,

$$\cdots \to CH_q(A) \xrightarrow{i(A,S)} CH_q(S) \xrightarrow{p(S,A^*)} CH_q(A^*) \xrightarrow{\partial(A^*,A)} CH_{q-1}(A) \to \cdots \qquad (2)$$

The connecting homomorphism $\partial(A^*, A)$ in (2) provides information about the set of orbits connecting A^* and A in S.

THEOREM 2.10. *If $\partial(A^*, A) \neq 0$, then $C(A^*, A) \neq \emptyset$.*

For example, if $CH_*(S) = 0$ and $CH_*(A) \neq 0$, by exactness $\partial(A^*, A) \neq 0$, so $C(A^*, A) \neq \emptyset$.

THEOREM 2.11. *If (A, A^*) is an attractor-repeller pair in S and $C(A^*, A) = \emptyset$, then (A^*, A) is also an attractor-repeller pair in S. Consequently, there is the homology index sequence of the attractor-repeller pair (A^*, A)*

$$\cdots \to CH_q(A^*) \xrightarrow{i(A^*,S)} CH_q(S) \xrightarrow{p(S,A)} CH_q(A) \xrightarrow{\partial(A,A^*)} CH_{q-1}(A^*) \to \cdots$$

and $p(S, A)i(A, S) = id|H(A)$.

3. Analysing Hopf bifurcation via topological method

Consider the planar system (1) which is illustrated in Section 1 where the local birth or death of a periodic solution appear from an equilibrium point as the parameter crosses a critical value (Hopf bifurcation) $\dot{x} = f_\mu(x, y)$, $\dot{y} = g_\mu(x, y)$.

The flow of the vector field of the Hopf bifurcation makes a limit cycle and the trajectories approach the limit cycle from inside and outside in case supercritical, or move away from the limit cycle in subcritical case. In this section, we divide the region of the Hopf bifurcation into neighbourhoods of the compact invariant sets: S which is a compact invariant set of the flow, an attractor part A in S, and the repeller part A^* in S. Also, we determine the graded Morse sets in the system, then we evaluate all components of the supercritical and the subcritical Hopf bifurcation using homological Conley index and exact sequences.

3.1 Supercritical Hopf bifurcation (Stable Limit Cycle)

In this subsection, we give a description of all parts of the supercritical Hopf bifurcation in terms of homological Conley index in the following theorem

Theorem 3.1. *Let S be a supercritical Hopf bifurcation with the repeller A^*, and the attractor A. Then*

$$CH_q(S) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad CH_q(A) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$CH_q(A^*) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 2, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Consider the supercritical Hopf bifurcation as in the Figure 2, we notice that the collection $M(S) = \{M(p) \mid p \in \mathcal{P}_0 = \{1, 2\}\}$ where $M(1)$ is the attractor (stable limit cycle) and $M(2)$ is the repeller (source) with flow ordering $M(1) < M(2)$ is a Morse decomposition of S.

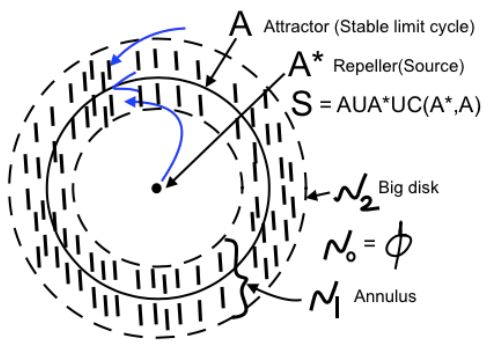

Figure 2: Supercritical Hopf bifurcation

It is clear from Figure 3 that $N_0 \subset N_1 \subset N_2$.

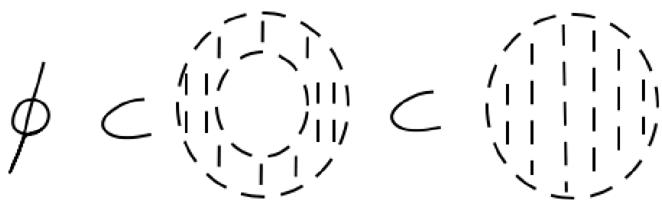

Figure 3: Neighborhoods of S, A and A^* in supercritical Hopf bifurcation

We can find the homological Conley index for each of S, A, and A^* as follows

$$CH_q(S) = H_q(N_2, N_0) = H_q(N_2, \emptyset) = H_q(\text{Attractor disk}) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$CH_q(A) = H_q(N_1, N_0) = H_q(N_1, \emptyset) = H_q(\text{annulus}) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $\quad CH_q(A^*) = H_q(N_2, N_1) = H_q(D^2, \partial D^2) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 2, \\ 0, & \text{otherwise.} \end{cases}$

We can illustrate these computations in the following exact sequence:

$$CH_2(A) \xrightarrow{i(A,S)} CH_2(S) \xrightarrow{p(S,A^*)} CH_2(A^*) \xrightarrow{\partial(A^*,A)} CH_1(A) \xrightarrow{i(A,S)} CH_1(S)$$
$$\xrightarrow{p(S,A^*)} CH_1(A^*) \xrightarrow{\partial(A^*,A)} CH_0(A) \xrightarrow{i(A,S)} CH_0(S) \xrightarrow{p(S,A^*)} CH_0(A^*)$$

which leads to the exact sequence $0 \to 0 \to \mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2 \to 0 \to 0 \to \mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2 \to 0.$ □

3.2 Subcritical Hopf bifurcation (Unstable Limit Cycle)

In this subsection, all components of the subcritical Hopf bifurcation are computed by the homological Conley index in the following theorem

THEOREM 3.2. *Let S be a subcritical Hopf bifurcation with the repeller A^*, and the attractor A. Then*

$$CH_q(S) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 2, \\ 0, & \text{otherwise,} \end{cases} \qquad CH_q(A) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $\quad CH_q(A^*) \cong \begin{cases} \mathbb{Z}_2, & \text{if } q = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$

Proof. Consider the subcritical Hopf bifurcation as in the Figure 4, we see that the (<-ordered) Morse decomposition of S is a collection $M(S) = \{M(p) \mid p \in \mathcal{P}_0 = \{1, 2\}\}$ such that $M(1)$ is an attractor (sink) and $M(2)$ is a repeller (unstable limit cycle) with flow ordering $M(1) < M(2)$. The neighborhoods N_0, N_1 and N_2 satisfy $N_0 \subset N_1 \subset N_2$, see Figure 5. We can find the homological Conley index for each of S, A, and A^* in the subcritical case.

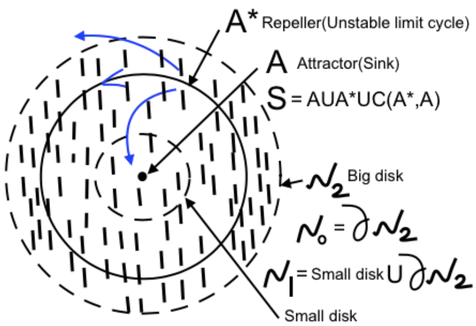

Figure 4: Subcritical Hopf bifurcation

Figure 5: Neighborhoods of S, A and A^* in subcritical Hopf bifurcation

$$CH_q(S) = H_q(N_2, N_0) = H_q(D^2, \partial D^2) = H_q(\text{Repeller disk}) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} CH_q(A) &= H_q(N_1, N_0) = H_q(N_1/N_0, N_0/N_0) \\ &= H_q(D^2 \cup \text{point}, \text{point}) = H_q(D^2) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and $$CH_q(A^*) = H_q(N_2, N_1) = H_q\Big(\overline{(N_2 - N_1)}, \partial\overline{(N_2 - N_1)}\Big) = \begin{cases} \mathbb{Z}_2, & \text{if } q = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

These results are illustrated in the following exact sequence

$$\begin{aligned} CH_2(A) &\xrightarrow{i(A,S)} CH_2(S) \xrightarrow{p(S,A^*)} CH_2(A^*) \xrightarrow{\partial(A^*,A)} CH_1(A) \xrightarrow{i(A,S)} CH_1(S) \\ &\xrightarrow{p(S,A^*)} CH_1(A^*) \xrightarrow{\partial(A^*,A)} CH_0(A) \xrightarrow{i(A,S)} CH_0(S) \xrightarrow{p(S,A^*)} CH_0(A^*), \end{aligned}$$

which is equivalent to the exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2 \to 0 \to 0 \to \mathbb{Z}_2 \xrightarrow{\approx} \mathbb{Z}_2 \to 0 \to 0.$ □

4. Conclusion

In this paper, we have shown an investigation and evaluation of both supercritical and subcritical Hopf bifurcation through topological method that depends on the homological Conley index and the exact sequences of index triple for the attractor-repeller pair (A, A^*) in S. For future work, we suggest analyzing and evaluating the Hopf bifurcation in higher dimensions.

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(received 16.11.2021; in revised form 19.04.2022; available online 25.01.2023)

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