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PROPERTIES OF ZERO-DIVISOR GRAPH OF THE RING $\mathbf{F}_{p^l} \times \mathbf{F}_{q^m} \times \mathbf{F}_{r^n}$

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Abstract. In this paper, we study some basic properties of the zero-divisor graph of ring $F_{p^l} \times F_{q^m} \times F_{r^n}$, where F_{p^l}, F_{q^m} and F_{r^n} are fields of order p^l, q^m and r^n, respectively, p, q and r are primes (not necessarily distinct) and $l, m, n \geq 1$ are positive numbers. Also, we discuss some topological indices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

1. Introduction

In 1988, Beck [9] began to investigate the possibility of coloring a commutative ring R by associating a *zero-divisor graph* on R, whose vertices are the elements of R, with two distinct elements x and y being adjacent if and only if $xy = 0$. While I. Beck concentrated on the connection between the clique number and the chromatic number of the graph, various works inspired by this construction have focused on the interplay of commutative rings and their zero-divisor graph. However, in 1999, Anderson and Livingston [6] modified and studied the zero-divisor graph whose vertices are the nonzero zero-divisors of the commutative ring.

Sharma *et al.* [13] studied adjacency matrices for zero-divisor graph over finite commutative rings of direct product $Z_p \times Z_p$, where p is a prime. In [2] Akgunes *et al.* examined graph parameters of zero-divisor graphs obtained from the ring $Z_p \times Z_q$, where p and q are distinct primes. In [8] Aykac and Akgunes presented some basic properties for zero-divisor graphs obtained from the ring $Z_{p^2} \times Z_{q^2}$, where p and q are distinct primes. In [3] Akgunes and Nacaroglu studied graph theoretical properties and topological index of zero-divisor graphs obtained from the ring $Z_p \times Z_q \times Z_r$, where p, q and r are primes. We will include basic definitions from graph theory as needed from [11].

Throughout the paper, we use the ring $F_{p^l} \times F_{q^m} \times F_{r^n}$, where F_{p^l}, F_{q^m} and F_{r^n} are fields of order p^l, q^m and r^n, respectively, p, q and r are primes (not necessarily distinct) and $l, m, n \geq 1$ are positive numbers.

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In this paper, we will first study the degree sequence, irregularity index, chromatic number, diameter, girth, radius, maximum and minimum degrees, domination number and clique number of the zero-divisor graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$. After that, we will discuss some topological indices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

2. Properties of $\Gamma(\mathbf{F}_{p^l} \times \mathbf{F}_{q^m} \times \mathbf{F}_{r^n})$

In this section, we will discuss some basic properties of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$. We begin our discussion with the definition of adjacent vertices of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

Definition 2.1. The adjacent vertices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ are as follows:

- $(a, 0, 0) \sim (0, b, c)$, where $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$ or $0 \neq c \in F_{r^n}$,
- $(0, b, 0) \sim (a, 0, c)$, where $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$ or $0 \neq c \in F_{r^n}$,
- $(0, 0, c) \sim (a, b, 0)$, where $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$ or $0 \neq c \in F_{r^n}$,

where p, q and r are primes (not necessarily distinct) and $l, m, n \geq 1$ are positive numbers.

Let $G(V, E)$ be a graph and $v \in V(G)$. The *degree* of a vertex v in G, denoted by $\deg(v)$, is the number of vertices adjacent to it. The *minimum* and *maximum degrees* are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

In the following theorem, we calculate the degree of every possible vertex of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

Theorem 2.2. *The degrees of vertices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ are given by*
(i) $\deg(a, 0, 0) = q^m r^n - 1$, *where* $0 \neq a \in F_{p^l}$,

(ii) $\deg(0, b, 0) = p^l r^n - 1$, *where* $0 \neq b \in F_{q^m}$,

(iii) $\deg(0, 0, c) = p^l q^m - 1$, *where* $0 \neq c \in F_{r^n}$,

(iv) $\deg(a, b, 0) = r^n - 1$, *where* $0 \neq a \in F_{p^l}$ *and* $0 \neq b \in F_{q^m}$,

(v) $\deg(0, b, c) = p^l - 1$, *where* $0 \neq b \in F_{q^m}$ *and* $0 \neq c \in F_{r^n}$,

(vi) $\deg(a, 0, c) = q^m - 1$, *where* $0 \neq a \in F_{p^l}$ *and* $0 \neq c \in F_{r^n}$,
where p, q *and* r *are primes and* $l, m, n \geq 1$ *are positive numbers.*

Proof. (i) One can see that $(a, 0, 0) \sim (0, b, c)$ because $(a, 0, 0) \cdot (0, b, c) = (0, 0, 0)$ for $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Similarly, for $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$ $(a, 0, 0) \sim (0, b, 0)$ because $(a, 0, 0) \cdot (0, b, 0) = (0, 0, 0)$. We also have $(a, 0, 0) \sim (0, 0, c)$ because $(a, 0, 0) \cdot (0, 0, c) = (0, 0, 0)$ for $0 \neq a \in F_{p^l}$ and $0 \neq c \in F_{r^n}$. As a result, the degree of the vertex $(a, 0, 0)$ is given by $\deg(a, 0, 0) = (q^m - 1)(r^n - 1) + q^m - 1 + r^n - 1 = q^m r^n - 1$.

(ii) For $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, since $(0,b,0)\cdot(a,0,c) = (0,0,0)$ we have $(0,b,0) \sim (a,0,c)$. Also, $(0,b,0) \sim (a,0,0)$ because $(0,b,0)\cdot(a,0,0) = (0,0,0)$ for $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$. In addition, as $(0,b,0)\cdot(0,0,c) = (0,0,0)$ we have $(0,b,0) \sim (0,0,c)$ for $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Therefore, the degree of vertex $(0,b,0)$ is given by $\deg(0,b,0) = (p^l-1)(r^n-1) + p^l - 1 + r^n - 1 = p^l r^n - 1$.

(iii) One can see that for $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, $(0,0,c) \sim (a,b,0)$ because $(0,0,c)\cdot(a,b,0) = (0,0,0)$. Also, since $(0,0,c)\cdot(a,0,0) = (0,0,0)$ we have $(0,0,c) \sim (a,0,0)$ for $0 \neq a \in F_{p^l}$ and $0 \neq c \in F_{r^n}$. In addition, as $(0,0,c)\cdot(0,b,0) = (0,0,0)$ we have $(0,0,c) \sim (0,b,0)$ for $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Therefore, the degree of vertex $(0,0,c)$ is given by $\deg(0,0,c) = (p^l-1)(q^m-1) + p^l - 1 + q^m - 1 = p^l q^m - 1$.

(iv) Since $(a,b,0)\cdot(0,0,c) = (0,0,0)$ for $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, $(a,b,0) \sim (0,0,c)$. As a result, the degree of the vertex $(a,0,0)$ is given by $\deg(a,b,0) = r^n - 1$.

(v) For $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, as $(0,b,c)\cdot(a,0,0) = (0,0,0)$ we have $(0,b,c) \sim (a,0,0)$. Therefore, the degree of vertex $(0,b,c)$ is given by $\deg(0,b,c) = p^l - 1$.

(vi) For $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, since $(a,0,c)\cdot(0,b,0) = (0,0,0)$ we have $(a,0,c) \sim (0,b,0)$. Therefore, the degree of vertex $(a,0,c)$ is given by $\deg(a,0,c) = q^m - 1$. □

THEOREM 2.3. *The maximum degree of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\Delta(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \max\{p^l q^m - 1, q^m r^n - 1, p^l r^n - 1\}.$$

Proof. By Theorem 2.2, we have $\deg(a,0,0) = q^m r^n - 1$, $\deg(0,b,0) = p^l r^n - 1$ and $\deg(0,0,c) = p^l q^m - 1$, where $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Hence, the maximum degree of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ is

$$\Delta(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \max\{p^l q^m - 1, q^m r^n - 1, p^l r^n - 1\}.$$ □

THEOREM 2.4. *The minimum degree of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\delta(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \min\{p^l - 1, q^m - 1, r^n - 1\}.$$

Proof. By Theorem 2.2, we have $\deg(a,b,0) = r^n - 1$, $\deg(0,b,c) = p^l - 1$, and $\deg(a,0,c) = q^m - 1$, where $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Hence, the minimum degree of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ is

$$\delta(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \min\{p^l - 1, q^m - 1, r^n - 1\}.$$ □

The *degree sequence* of a graph G, denoted by $DS(G)$, is a sequence of degrees of vertices of G. Also, *the irregularity index* of a graph G, denoted by $t(G)$, is the number of distinct terms in the degree sequence of G.

THEOREM 2.5. *The degree sequence and irregularity index of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *are given by*

$$DS(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \Big\{ \underbrace{p^l - 1}_{(q^m-1)(r^n-1) times}, \underbrace{q^m - 1}_{(p^l-1)(r^n-1) times}, \underbrace{r^n - 1}_{(p^l-1)(q^m-1) times},$$

$$\underbrace{p^l q^m - 1}_{(r^n-1) times}, \underbrace{p^l r^n - 1}_{(q^m-1) times}, \underbrace{q^m r^n - 1}_{(p^l-1) times}\Big\}$$

and $t(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 6$, *where* p, q *and* r *are distinct primes and* $l, m, n \geq 1$ *are positive numbers.*

Proof. It is clear from Theorem 2.2 that $\deg(a, 0, 0) = q^m r^n - 1$ and the number of vertices of the form $(a, 0, 0)$ is $p^l - 1$, where $0 \neq a \in F_{p^l}$. Also, $\deg(0, b, 0) = p^l r^n - 1$ and the number of these types of vertices is $q^m - 1$, for $0 \neq b \in F_{q^m}$. Moreover, the number of vertices of the form $(0, 0, c)$ is $r^n - 1$ and degree of these vertices is $p^l q^m - 1$, where $0 \neq c \in F_{r^n}$.

Again by Theorem 2.2, for $0 \neq a \in F_{p^l}$ and $0 \neq b \in F_{q^m}$, $\deg(a, b, 0) = r^n - 1$ and the number of these form of vertices is $(p^l - 1)(q^m - 1)$. Similarly, for $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$, $\deg(0, b, c) = p^l - 1$ and the number of vertices of the form $(0, b, c)$ is $(q^m - 1)(r^n - 1)$. Moreover, the number of vertices of the types $(a, 0, c)$ is $(p^l - 1)(r^n - 1)$ and $\deg(a, 0, c) = q^m - 1$, where $0 \neq a \in F_{p^l}$ and $0 \neq c \in F_{r^n}$. Also, one can see that irregularity index of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ is 6. □

Let $G(V, E)$ be a graph and $u, v \in V(G)$. The *distance* between u and v, denoted by $d(u, v)$, is the length of the shortest path connecting u and v, if such a path exists, otherwise, we set $d(u, v) = \infty$. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is defined as $\operatorname{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}$. The *eccentricity* of a vertex x is defined by $e(x) = \max\{d(x, y) : y \in V(G)\}$. The *radius* of a graph G is $\operatorname{rad}(G) = min\{e(x) : x \in V(G)\}$. Note that $\operatorname{diam}(G) = \max\{e(x) : x \in V(G)\}$.

By Definition 2.1, we can find the distance between any two vertices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ as shown in the following theorem.

Theorem 2.6. *The distance between any two vertices of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is as follows:*

- $d((a_1, 0, 0), (a_2, 0, 0)) = 2$, *where* $0 \neq a_1, a_2 \in F_{p^l}$,
- $d((a_1, 0, 0), (a_2, b, 0)) = 2$, *where* $0 \neq a_1, a_2 \in F_{p^l}$ *and* $0 \neq b \in F_{q^m}$,
- $d((a_1, 0, 0), (a_2, 0, c)) = 2$, *where* $0 \neq a_1, a_2 \in F_{p^l}$ *and* $0 \neq c \in F_{r^n}$,
- $d((0, b_1, 0), (0, b_2, 0)) = 2$, *where* $0 \neq b_1, b_2 \in F_{q^m}$,
- $d((0, b_1, 0), (a, b_2, 0)) = 2$, *where* $0 \neq a \in F_{p^l}$ *and* $0 \neq b_1, b_2 \in F_{q^m}$,
- $d((0, b_1, 0), (0, b_2, c)) = 2$, *where* $0 \neq b_1, b_2 \in F_{q^m}$ *and* $0 \neq c \in F_{r^n}$,
- $d((0, 0, c_1), (0, 0, c_2)) = 2$, *where* $0 \neq c_1, c_2 \in F_{r^n}$,
- $d((0, 0, c_1), (a, 0, c_2)) = 2$, *where* $0 \neq a \in F_{p^l}$ *and* $0 \neq c_1, c_2 \in F_{r^n}$,
- $d((0, 0, c_1), (0, b, c_2)) = 2$, *where* $0 \neq b \in F_{q^m}$ *and* $0 \neq c_1, c_2 \in F_{r^n}$,
- $d((a_1, b_1, 0), (a_2, b_2, 0)) = 2$, *where* $0 \neq a_1, a_2 \in F_{p^l}$ *and* $0 \neq b_1, b_2 \in F_{q^m}$,

- $d((0,b_1,c_1),(0,b_2,c_2)) = 2$, *where* $0 \neq b_1, b_2 \in F_{q^m}$ *and* $0 \neq c_1, c_2 \in F_{r^n}$,
- $d((a_1,0,c_1),(a_2,0,c_2)) = 2$, *where* $0 \neq a_1, a_2 \in F_{p^l}$ *and* $0 \neq c_1, c_2 \in F_{r^n}$,
- $d((a_1,b,0),(a_2,0,c)) = 3$, *where* $0 \neq a_1, a_2 \in F_{p^l}$, $0 \neq b \in F_{q^m}$ *and* $0 \neq c \in F_{r^n}$,
- $d((a,b_1,0),(0,b_2,c)) = 3$, *where* $0 \neq a \in F_{p^l}$, $0 \neq b_1, b_2 \in F_{q^m}$ *and* $0 \neq c \in F_{r^n}$,
- $d((0,b,c_1),(a,0,c_2)) = 3$, *where* $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ *and* $0 \neq c_1, c_2 \in F_{r^n}$.

The following theorem is a direct corollary of Theorem 2.6.

THEOREM 2.7. *The diameter of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\operatorname{diam}(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3.$$

THEOREM 2.8. *The radius of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\operatorname{rad}(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 2.$$

Proof. By using Theorem 2.6, we have

$$e(a,0,0) = e(0,b,0) = e(0,0,c) = 2 \quad \text{and} \quad e(a,b,0) = e(0,b,c) = e(a,0,c) = 3,$$

for $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Hence,

$$\operatorname{rad}(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \min\{e(u) : u \in V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))\} = 2. \qquad \square$$

The *girth* of a graph G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G ($\operatorname{gr}(G) = \infty$ if G contains no cycle).

THEOREM 2.9. *The girth of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$gr(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3.$$

Proof. Since $(a,0,0) \cdot (0,b,0) = (0,0,0)$, $(0,b,0) \cdot (0,0,c) = (0,0,0)$ and $(0,0,c) \cdot (a,0,0) = (0,0,0)$, for some $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Thus, $(a,0,0) \sim (0,b,0) \sim (0,0,c) \sim (a,0,0)$ is a cycle of length 3 in $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$. Hence, $\operatorname{gr}(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3$. $\square$

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V,E)$ if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. A dominating set S of minimum cardinality in G is called γ-set of G.

THEOREM 2.10. *The domination number of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\gamma(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3.$$

Proof. Clearly, $S = \{(a,0,0),(0,b,0),(0,0,c)\}$ is a dominating set of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$, where a, b and c are fixed nonzero elements of F_{p^l}, F_{q^m} and F_{r^n} respectively, since a nonzero element (x,y,z) is a vertex of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ if and only if at least one of its component is zero. If we show that no subset T of S with cardinality 2 is a dominating set of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$, then the proof is complete. Suppose on contrary that $T = \{(a,0,0),(0,b,0)\}$ is a dominating set of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$, where a and

b are fixed nonzero elements of F_{p^l} and F_{q^m}, respectively. Then, $(1,1,0)$ is a vertex of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$, which is not adjacent to any element of T, a contradiction. Therefore, $\gamma(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3$. $\square$

A *coloring* of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all points with any of the colors is independent and is called a *color class*. An *n-coloring* of a graph $G(V,E)$ uses n colors; it thereby partitions V into n color classes. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum n for which G has an n-coloring.

In the following theorem, we deternine the chromatic number of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

THEOREM 2.11. *The chromatic number of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\chi(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3.$$

Proof. Consider the vertex $(1,0,0)$ of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ and assign color t_1 to this vertex. Observe that this vertex is not adjacent to $(a,0,0)$, $(a,b,0)$, and $(a,0,c)$, for $0 \neq a \in F_{p^l}$, $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. So we can assign the same color t_1 to all these types of vertices.

Similarly, consider the vertex $(0,1,0)$ and assign a color t_2 to this vertex. Also, this vertex is not adjacent to $(0,b,0)$ and $(0,b,c)$, for $0 \neq b \in F_{q^m}$ and $0 \neq c \in F_{r^n}$. Again, we can assign the same color t_2 to all these types of vertices.

Now, choose the vertex $(0,0,1)$ and use the color t_3 for this vertex. The vertex $(0,0,1)$ is not adjacent to $(0,0,c)$, for $0 \neq c \in F_{r^n}$. Therefore, we can assign the same color t_3 to these types of vertices. Thus, all vertices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ are colored by at most 3 different colors. Hence, $\chi(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3$. $\square$

A graph is said to be *complete* if all its vertices are adjacent with each other. A complete graph with n vertices is denoted by K^n. The *clique number* of a graph G, denoted $\omega(G)$, is the maximum number of vertices in a complete subgraph of G.

THEOREM 2.12. *The clique number of the graph* $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$ *is given by*

$$\omega(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3.$$

Proof. Observe that the vertex $(1,0,0)$ is adjacent to $(0,1,0)$. Also, the vertex $(0,1,0)$ is adjacent to $(0,0,1)$. In addition, the vertex $(0,0,1)$ is adjacent to $(1,0,0)$. We can see that the graph $(1,0,0) \sim (0,1,0) \sim (0,0,1) \sim (1,0,0)$ is the maximal complete subgraph of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$. Hence, $\omega(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3$. $\square$

REMARK 2.13. From Theorems 2.11 and 2.12, we can say that

$$\omega(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \chi(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = 3,$$

which proves the perfectness property of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$.

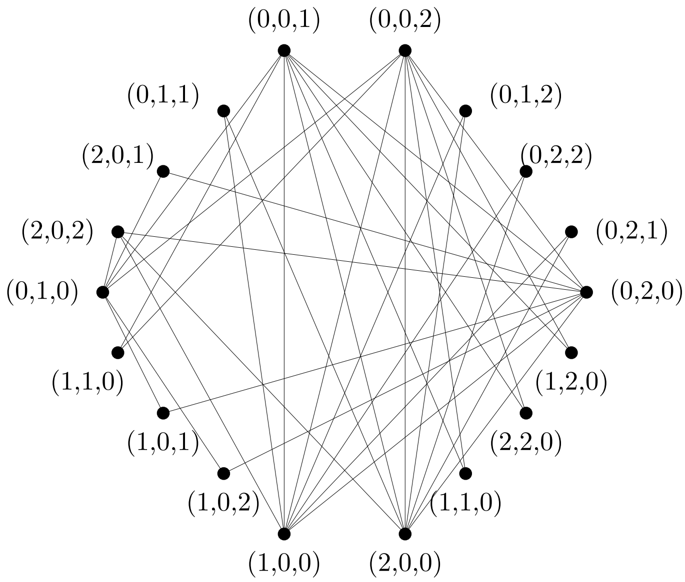

Figure 1: The zero-divisor graph of the ring $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

Example 2.14. All the properties of $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$, shown in Figure 1, are the following:

(i) $\operatorname{diam}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$,

(ii) $\operatorname{rad}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$,

(iii) $\operatorname{gr}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$,

(iv) $\Delta(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 8$,

(v) $\delta(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$,

(vi) $DS(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = \{2,2,2,2,2,2,2,2,2,2,2,2,8,8,8,8,8,8\}$,

(vii) $t(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$,

(viii) $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$,

(ix) $\chi(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$,

(x) $\omega(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) = 3$.

3. Some topological indices of $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$

In this section, we will discuss some topological indices of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$. A topological index of a graph G denoted by $\operatorname{Top}(G)$, is a number with the property that for every graph H isomorphic to G, $\operatorname{Top}(G) = \operatorname{Top}(H)$.

Some of the topological indices are define as follows:

Definition 3.1. The *Zagreb group indices* of a graph G denoted by $M_1(G)$ (*first Zagreb index*) and $M_2(G)$ (*second Zagreb index*) are defined as $M_1(G) = \sum_{u\in V(G)} d^2(u)$ and $M_2(G) = \sum_{uv\in E(G)} d(u)d(v)$, where $d(u)$ denotes the degree of the vertex u.

Definition 3.2. The *Weiner index* $W(G)$ of a graph G is defined as the sum of half of the distances between every pair of vertices of G. $W(G) = \frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d(u,v)$, where $d(u,v)$ is the distance between u and v.

Theorem 3.3. *First Zagreb index of the graph* $\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})$ *is given as*

$$M_1(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = \sum_{\substack{i,j,k\in\{p^l,q^m,r^n\}\\ i\neq j\neq k}} (ij-1)^2(k-1) + (p^l-1)(q^m-1)(r^n-1)(p^l+q^m+r^n+3),$$

where p,q *and* r *are primes and* $l,m,n\geq 1$ *are positive numbers.*

Proof. From Theorem 2.5, we have

$$M_1(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = \sum_{u\in V(G)} d^2(u)$$

$$\begin{aligned}
=&(p^l-1)^2(q^m-1)(r^n-1) + (q^m-1)^2(p^l-1)(r^n-1) + (r^n-1)^2(p^l-1)(q^m-1)\\
&+ (p^lq^m-1)^2(r^n-1) + (p^lr^n-1)^2(q^m-1) + (q^mr^n-1)^2(p^l-1)\\
=&\sum_{\substack{i,j,k\in\{p^l,q^m,r^n\}\\ i\neq j\neq k}} (ij-1)^2(k-1) + (p^l-1)(q^m-1)(r^n-1)(p^l+q^m+r^n+3). \qquad \square
\end{aligned}$$

Theorem 3.4. *Second Zagreb index of the graph* $\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})$, *where* p,q *and* r *are primes and* $l,m,n\geq 1$ *are positive numbers, is given as*

$$M_2(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = (6P+3)(P-Q) + R(P+Q-R+2) + Q^2,$$

where $P = p^lq^mr^n$, $Q = p^lq^m + q^mr^n + r^np^l$ *and* $R = p^l+q^m+r^n$.

Proof. From Definition 2.1, we have

$$\begin{aligned}
M_2(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = &\sum_{(a,0,0)\sim(0,b,0)} d(a,0,0)d(0,b,0) + \sum_{(a,0,0)\sim(0,0,c)} d(a,0,0)d(0,0,c)\\
&+ \sum_{(0,b,0)\sim(0,0,c)} d(0,b,0)d(0,0,c) + \sum_{\substack{(a,0,0)\sim(0,b,c)\\ b\neq c}} d(a,0,0)d(0,b,c)\\
&+ \sum_{\substack{(0,b,0)\sim(a,0,c)\\ a\neq c}} d(a,0,0)d(0,b,c) + \sum_{\substack{(0,0,c)\sim(a,b,0)\\ a\neq b}} d(a,0,0)d(0,b,c).
\end{aligned}$$

By applying Theorem 2.2, we get

$$\begin{aligned}
M_2(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = &(q^mr^n-1)(p^lr^n-1)(p^l-1)(q^m-1)\\
&+ (q^mr^n-1)(p^lq^m-1)(p^l-1)(r^n-1) + (q^mr^n-1)(p^l-1)(p^l-1)(q^m-1)(r^n-1)\\
&+ (p^lr^n-1)(p^lq^m-1)(q^m-1)(r^n-1) + (p^lr^n-1)(q^m-1)(q^m-1)(p^l-1)(r^n-1)\\
&+ (p^lq^m-1)(r^n-1)(r^n-1)(p^l-1)(q^m-1).
\end{aligned}$$

If we take $P = p^lq^mr^n$, $Q = p^lq^m + q^mr^n + r^np^l$ and $R = p^l+q^m+r^n$, then

$$M_2(\Gamma(F_{p^l}\times F_{q^m}\times F_{r^n})) = (6P+3)(P-Q) + R(P+Q-R+2) + Q^2. \qquad \square$$

THEOREM 3.5. *The Weiner index of the graph $\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})$, where p, q and r are primes and $l, m, n \geq 1$ are positive numbers, is given as*

$$W(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = A + 4B + 9D + E(3C - 5) - 3 \sum_{i,j,k \in \{p^l, q^m, r^n\}} i^2(j - k) + 3,$$

where $A = p^{l^2} q^{m^2} + p^{l^2} r^{n^2} + q^{m^2} r^{n^2}$, $B = p^{l^2} + q^{m^2} + r^{n^2}$, $C = p^l q^m r^n$ $D = p^l q^m + p^l r^n + q^m r^n$, *and* $E = p^l + q^m + r^n$.

Proof. From Theorem 2.6, we have

$$W(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)$$

$$= \frac{1}{2}\Bigg[\sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((a_1, 0, 0), (a, b, c)) + \sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((0, b_1, 0), (a, b, c))$$

$$+ \sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((0, 0, c_1), (a, b, c)) + \sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((a_1, b_1, 0), (a, b, c))$$

$$+ \sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((a_1, 0, c_1), (a, b, c)) + \sum_{\substack{(a,b,c) \in \\ V(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n}))}} d((0, b_1, c_1), (a, b, c))\Bigg]$$

$$= \frac{1}{2}\Bigg[(p^l - 1)\Big\{2(p^l - 1) + 1(q^m - 1) + 1(r^n - 1) + 2(p^l - 1)(q^m - 1) + 2(p^l - 1)(r^n - 1)$$

$$+ 1(q^m - 1)(r^n - 1)\Big\} + (q^m - 1)\Big\{1(p^l - 1) + 2(q^m - 1) + 1(r^n - 1) + 2(p^l - 1)(q^m - 1)$$

$$+ 1(p^l - 1)(r^n - 1) + 2(q^m - 1)(r^n - 1)\Big\} + (r^n - 1)\Big\{1(p^l - 1) + 1(q^m - 1) + 2(r^n - 1)$$

$$+ 1(p^l - 1)(q^m - 1) + 2(p^l - 1)(r^n - 1) + 2(q^m - 1)(r^n - 1)\Big\} + (p^l - 1)(q^m - 1)\Big\{2(p^l - 1)$$

$$+ 2(q^m - 1) + 1(r^n - 1) + 2(p^l - 1)(q^m - 1) + 3(p^l - 1)(r^n - 1) + 3(q^m - 1)(r^n - 1)\Big\}$$

$$+ (q^m - 1)(r^n - 1)\Big\{1(p^l - 1) + 2(q^m - 1) + 2(r^n - 1) + 3(p^l - 1)(q^m - 1)$$

$$+ 3(p^l - 1)(r^n - 1) + 2(q^m - 1)(r^n - 1)\Big\} + (p^l - 1)(q^m - 1)\Big\{2(p^l - 1) + 1(q^m - 1)$$

$$+ 2(r^n - 1) + 3(p^l - 1)(q^m - 1) + 2(p^l - 1)(r^n - 1) + 3(q^m - 1)(r^n - 1)\Big\}\Bigg].$$

On solving, we get

$$W(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = (p^{l^2} q^{m^2} + p^{l^2} r^{n^2} + q^{m^2} r^{n^2}) + 4(p^{l^2} + q^{m^2} + r^{n^2})$$
$$+ 9(p^l q^m + p^l r^n + q^m r^n) + (p^l + q^m + r^n)(3p^l q^m r^n - 5) + 3$$

$$- 3p^{l^2}(q^m + r^n) - 3q^{m^2}(p^l + r^n) - 3r^{n^2}(p^l + q^m).$$

Take $A=p^{l^2}q^{m^2}+p^{l^2}r^{n^2}+q^{m^2}r^{n^2}$, $B=p^{l^2}+q^{m^2}+r^{n^2}$, $C=p^lq^mr^n$, $D=p^lq^m+p^lr^n+q^mr^n$ and $E=p^l+q^m+r^n$, then

$$W(\Gamma(F_{p^l} \times F_{q^m} \times F_{r^n})) = A + 4B + 9D + E(3C - 5) - 3 \sum_{i,j,k \in \{p^l, q^m, r^n\}} i^2(j - k) + 3. \qquad \square$$

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References

[1] N. Akgüneş, *Analyzing special parameters over zero-divisor graphs*, AIP Conference Proceeding, **1479(1)** (2012), 390–392.

[2] N. Akgüneş, M. Togan, *Some graph theoretical properties over zero-divisor graphs of special finite commutative rings*, Adv. Studies Contemp. Math., **22(2)** (2012), 309—315.

[3] N. Akgüneş, Y. Nacaroglu, *Some properties of zero divisor graph obtained by the ring $Z_p \times Z_q \times Z_r$*, Asian-European J. Math., **12(6)** (2019), 2040001 (10 pages).

[4] D. D. Anderson, M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, **159(2)** (1993), 500–514.

[5] D. F. Anderson, A. Badawi *On the zero-divisor graph of a ring*, Commun. Algebra, **36** (2008), 3073–3092.

[6] D. F. Anderson, P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217(2)** (1999), 434–447.

[7] S. Aykaç, N. Akgüneş, *Analysis of graph parameters associated with zero-divisor graphs of commutative rings*, New Trends Math. Sci., **6(2)** (2018), 144—149.

[8] S. Aykaç, N. Akgüneş, A. S. Çevik, *Analysis of Zagreb indices over zero-divisor graphs of commutative rings*, Asian-European J. Math., **12(6)** (2019), 2040003 (19 pages).

[9] I. Beck, *Coloring of commutative rings*, J. Algebra, **116(1)** (1988), 208–226.

[10] K. C. Das, N. Akgüneş, M. Togan, A. Yurttas, I. N. Cangul, A. S. Çevik, *On the first Zagreb index and multiplicative Zagreb coindices of graphs*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat., **24(1)** (2016), 153–176.

[11] J. L. Gross, J. Yellen, *Handbook of Graph Theory* (CRC Press, 2004).

[12] S. Mukwembi, *A note on diameter and the degree sequence of a graph*, Applied Math. Letters, **25(2)** (2012), 175–178.

[13] P. Sharma, A. Sharma, R. K. Vats, *Analysis of adjacency matrix and neighborhood associated with zero divisor graph of finite commutative rings*, Int. J. Comput. Appl., **14(3)** (2011), 38—42.

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