MATEMATIČKI VESNIK
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# CRITICAL POINT APPROACHES FOR IMPULSIVE STURM-LIOUVILLE DIFFERENTIAL EQUATIONS WITH NONLINEAR DERIVATIVE DEPENDENCE

**Zahra Mehraban and Shapour Heidarkhani**

**Abstract**. We guarantee the existence of multiple solutions for a class of impulsive Sturm-Liouville differential equations by considering a consequence of Bonanno's local minimum theorem on the nonlinear term and as well as, via critical point theorems due to Bonanno and another one due to Averna and Bonanno in a special case.

## 1. Introduction

In this work, we ensure the existence of multiple solutions for the following problem

$$\begin{cases} -(\phi_p(u'))' = \lambda f(t,u)h(u'), & t \neq t_j, a.e.\ t \in [0,T], \\ \Delta J(u'(t_j)) = \mu I_j(u(t_j)), & j = 1,2,\ldots,n, \\ \alpha u(0) - \beta u'(0) = 0, & \gamma u(T) + \sigma u'(T) = 0, \end{cases} \tag{1}$$

where $p > 2$, $\phi_p(s) = |s|^{p-2}s$, $\alpha$, $\gamma$, $\beta$, $\sigma > 0$, $T > 0$, $t_j$, $j = 1,2,\ldots,n$, are instants in which the impulses occur and $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = T$, $J(s) = \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta$, $\Delta J(u'(t_j)) = J(u'(t_j^+)) - J(u'(t_j^-))$, $u'(t_j^+) = \lim_{t\to t_j^+} u'(t)$, $u'(t_j^-) = \lim_{t\to t_j^-} u'(t)$, $I_j : [0,+\infty) \to [0,+\infty)$ is a Lipschitz continuous function with the Lipschitz constant $c > 0$, for $j = 1,2,\ldots,n$, $f : [0,T] \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $h : \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $\inf_{x\in\mathbb{R}} h(x) > 0$ and $\lambda$ and $\mu$ are two control parameters. The interest is that the nonlinear terms includes $u'$.

Differential equations involving impulsive effects serve as basic model to consider subject altering suddenly. There are many good monographs on the impulsive differential equations [15,20,24]. Some kinds of the processes naturally happen in dynamics, biological systems, mathematical economy, chemical technology, engineering, ecology,

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industrial robotics and so on (one can see [3,9,10,14,19,21,23].) Mathematical models of such processes are systems of impulsive differential equations.

The theory of impulsive differential equations is an important branch of the theory of differential equations. There have been a great deal of approaches to establish the existence results of differential equations with impulses, for instance: topological degree theory, comparison method, variational and so on [2, 8, 17, 18, 25, 28].

The existence of multiple solutions of impulsive Sturm-Liouville differential equations has been investigated in [7, 16, 22, 26, 27]. The authors have discussed in [26] the existence of multiple positive solutions by using a three critical points theorem for two types of impulsive Sturm-Liouville boundary value problems depending on the parameter $\lambda$. In [16], Liang and Liu based on upper and lower method and degree theory have obtained the existence of at least three solutions for a second order impulsive Sturm-Liouville boundary value problem under the assumption that the nonlinear term function satisfies a Nagumo condition with respect to the first order derivative. The authors have investigated in [27] a Sturm-Liouville boundary value problem for fourth-order impulsive differential equations applying variational methods and critical points theory. In [22], Ozkan has studied an impulsive Sturm-Liouville boundary value problem with boundary conditions containing Herglotz-Nevanlinna type rational functions of the spectral parameter and has showed that the coefficients of the problem are uniquely determined by either the Weyl function or by the Prufer angle or by the classical spectral data consisting of eigenvalues and norming constants. In particular, in [7], based on variational and critical point theory the existence of nontrivial solutions for the problem (17) has been discussed. In [13], using multiple critical points theorems, the existence of infinitely many positive solutions of a class of impulsive perturbed Sturm-Liouville differential equations with nonlinear derivative dependence has been investigated. Results on the existence of three positive solutions were also established.

Our goal in this paper is to obtain the existence of multiple solutions of the problem (2). The existence of one solution for the problem under algebraic conditions on the nonlinear term and two solutions with the classical Ambrosetti-Rabinowitz algebraic conditions on the nonlinear term are investigated by employing a consequence of Bonanno's local minimum theorem in [6]. Moreover, employing two critical point theorems, one due to Bonanno in [5] and another one due to Averna and Bonanno in [1], we consider the existence of at least two and three solutions for the problem (2) in the case $\lambda = \mu$, respectively. Here, we state a special case as a result.

THEOREM 1.1. *Suppose, there exist three positive constants $c_1$, $c_2$ and $\nu$ with the property $c_1 < \nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^{\frac{1}{q}}} < \sqrt[p]{\frac{m}{M}} c_2$. In addition,*

$$(\varrho_8) \quad mMp^2 \max\left\{ \frac{\|\chi\|_{L^1} K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2}{c_1^p}, \frac{\|\chi\|_{L^1} K(\Theta(c_2)) + \frac{cn}{2}(\Theta(c_2))^2}{c_2^p} \right\}$$
$$< \nu^p (T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p) \left( -\|\chi\|_{L^1} K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2} \sum_{j=1}^{n} t_j^2 \right).$$

*Then, for every*

$$\lambda \in \Lambda := \Bigg( \frac{mp}{\nu^p (T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)} \frac{1}{\Big(-\|\chi\|_{L^1} K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\Big)},$$

$$\min\left\{ \frac{c_1^p}{pM\left(\|\chi\|_{L^1} K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2\right)}, \frac{c_2^p}{pM\left(\|\chi\|_{L^1} K(\Theta(c_2)) + \frac{cn}{2}(\Theta(c_2))^2\right)} \right\} \Bigg),$$

*the problem* (17) *admits at least two solutions in* $X$.

## 2. Preliminaries

In order to obtain our results, the following theorems are the main tool.

THEOREM 2.1 ([6, Theorem 2.3]). *Let* $X$ *be a real Banach space and* $\Phi$, $\Psi : X \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that,* $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. *Suppose, there exist* $r > 0$ *and* $\bar{u} \in X$ *with* $0 < \Phi(\bar{u}) < r$ *such that*

*($i_1$)* $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

*($i_2$) for each* $\lambda \in \big(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\big)$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies* $(PS)^{[r]}$*-condition.*

*Then, for each* $\lambda \in \Lambda_r := (\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)})$ *there exists* $u_{0,\lambda\in\Phi^{-1}(0,r)}$ *such that* $I_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ *and* $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}(0,r)$.

THEOREM 2.2 ([6, Theorem 3.2]). *Let* $X$ *be a real Banach space and* $\Phi$, $\Psi : X \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* $\Phi$ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$. *Fix* $r > 0$ *and suppose, for each* $\lambda \in \big(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\big)$, *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies (PS)-condition and it is unbounded from below. Then, for each* $\lambda \in \big(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\big)$, *the functional* $I_\lambda$ *admits two distinct critical points.*

Now, we recall two critical point theorems as follows.

THEOREM 2.3 ([1, Theorem A]). *Let* $X$ *be a reflexive real Banach space and* $\Phi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on* $X^*$ *and* $\Psi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

*($j_1$)* $\lim_{\|u\|\longrightarrow\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda \in [0,\infty)$,

*($j_2$) there is* $r \in \mathbb{R}$ *such that,* $\inf_X \Phi < r$ *and* $\varphi_1(r) < \varphi_2(r)$, *where*

$$\varphi_1(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r)}^w}\Psi}{r - \Phi(u)},$$

$$\varphi_2(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{u\in\Phi^{-1}[r,\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

*and* $\overline{\Phi^{-1}(-\infty,r)}^{w}$ *is the closure of* $\Phi^{-1}(-\infty,r)$ *in the weak topology.*
*Then, for each* $\lambda\in(\frac{1}{\varphi_2(r)},\frac{1}{\varphi_1(r)})$*, the functional* $\Phi+\lambda\Psi$ *has at least three critical points in* $X$*.*

In Theorem 2.3, $\varphi_1(r)$ could be 0. Here and in similar cases we have $\frac{1}{0}$ as $\infty$.

THEOREM 2.4 ([5, Theorem 1.1]). *Let* $X$ *be a reflexive real Banach space and* $\Phi$, $\Psi : X \longrightarrow \mathbb{R}$ *be two sequentially weakly semiconscious and Gâteaux differentiable functions. Assume that* $\Phi$ *is (strongly) continuous and satisfies* $\lim_{\|u\|\longrightarrow\infty}\Phi(u)=\infty$. *Also suppose, there exist two constants* $r_1$ *and* $r_2$ *such that*
*($k_1$)* $\inf_X \Phi < r_1 < r_2$,

*($k_2$)* $\varphi_1(r_1) < \varphi_2^*(r_1,r_2)$,

*($k_3$)* $\varphi_1(r_2) < \varphi_2^*(r_1,r_2)$*, where* $\varphi_1$ *is defined as in Theorem 2.3 and* $\varphi_2^*(r_1,r_2)=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\sup_{v\in\Phi^{-1}[r_1,r_2)}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)}$.
*Then, for each* $\lambda\in\left(\frac{1}{\varphi_2^*(r_1,r_2)},\min\left\{\frac{1}{\varphi_1(r_1)},\frac{1}{\varphi_1(r_2)}\right\}\right)$ *the functional* $\Phi+\lambda\Psi$ *admits at least two critical points in* $\Phi^{-1}(-\infty,r_1]$ *and* $\Phi^{-1}[r_1,r_2)$*.*

Set $X = W^{1,p}([0,T])$ be the Sobolev space which endows the norm $\|u\| := \left(\int_0^T(|u'(t)|^p+|u(t)|^p)\mathrm{d}t\right)^{1/p}$. Obviously, $X$ is a reflexive real Banach space.

Let $\Delta = C([0,T])$ with the norm $\|u\|_\infty=\sup_{t\in[0,T]}|u(t)|$. There is a constant $m_2$ such that, $\|u\|_\infty < m_2\|u\|$. For any $\upsilon>0$, define $\Theta(\upsilon)=\upsilon(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}}+T^{\frac{1}{q}})$.

Corresponding to the function $f$, we have $F(t,x)=\int_0^x f(t,\xi)\mathrm{d}\xi$ for every $(\mathrm{t},\mathrm{x})\in[0,\mathrm{T}]\times[0,+\infty)$. We define $m=\inf_{x\in\mathbb{R}}h(x)$, $M=\sup_{x\in\mathbb{R}}h(x)$. So, $M\geq m>0$.

## 3. Existence of one solution

In this section, we investigate the existence of one solution for problem (2) by using Theorem 2.1.

THEOREM 3.1. *Assume, there exist three positive constants* $c_1$, $c_2$ *and* $\nu$ *with the property* $c_1<\nu\sqrt[p]{T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^{\frac{1}{q}}}<\sqrt[p]{\frac{m}{M}}c_2$*, such that*

*($\varrho_1$)* $$\frac{\int_0^T[F(t,t\nu)-f(t,0)t\nu]dt}{pM}\geq$$
$$\left(\int_0^T(\int_0^\nu J(s)ds)dt+\frac{\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}}J(s)ds\right)\frac{\int_0^T\sup_{u(t)\leq\Theta(c_2)}F(t,u^+(t))dt}{c_2^p}$$

*($\varrho_2$)* $\limsup_{|\xi(t)|\longrightarrow\infty}\frac{F(t,\xi^+(t))-f(t,0)\xi^-(t)}{|\xi(t)|^p}\leq 0$ *uniformly in* $[0,T]$*.*

*Then, for each*

$$\lambda \in \Lambda := \left( \frac{\int_0^T (\int_0^{\nu} J(s)ds)dt + \frac{\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}{\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt}, \frac{c_2^p}{pM} \frac{1}{\int_0^T \sup_{u(t)\le\Theta(c_2)} F(t,u^+(t))dt} \right)$$

*and for every Lipschitz function $I_j : [0,+\infty) \longrightarrow [0,+\infty)$ that*

$$\limsup_{|\xi(t_j)|\longrightarrow\infty} \frac{\int_0^{\xi^+(t_j)} I_j(s)ds - I_j(0)\xi^-(t_j)}{|\xi(t_j)|^p} < \infty \tag{2}$$

*for $j = 1, \ldots, n$, there exists $\delta_\lambda > 0$ given by*

$$\min\left\{ \frac{2\lambda\left(\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt\right) - 2\int_0^T (\int_0^{\nu} J(s)ds)dt - 2\frac{\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}{\nu^2 c\sum_{j=1}^n t_j}, \frac{2c_2^p - 2\lambda pM\int_0^T \sup_{u(t)\le\Theta(c_2)} F(t,u^+(t))dt}{pMcn(\Theta(c_2))^2} \right\}, \tag{3}$$

*such that for each $\mu \in [0,\delta_\lambda)$, the problem* (2) *admits at least one nontrivial solution $u_\lambda \in X$ and $\max_{t\in[0,T]} |u_\lambda(t)| \le c_2(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}})$.*

*Proof.* Our main tool is Theorem 2.1. According to [29, Lemma 3.1], we say that $u \in X$ is a solution of the problem (2) if and only if $u$ is a critical point of the Euler functional $I_\lambda = \Phi - \lambda\Psi$ such that,

$$\Phi(u) = \int_0^T \left(\int_0^{u'(t)} J(s)\mathrm{ds}\right) dt + \frac{\beta}{\alpha}\int_0^{\frac{\alpha u(0)}{\beta}} J(s)ds + \frac{\sigma}{\gamma}\int_0^{-\frac{\gamma u(T)}{\sigma}} J(s)ds \tag{4}$$

and

$$\Psi(u) = \int_0^T [F(t,u^+(t)) - f(t,0)u^-(t)]dt + \frac{\mu}{\lambda}\sum_{j=1}^n [\int_0^{u^+(t_j)} I_j(s)ds - I_j(0)u^-(t_j)] \tag{5}$$

for each $u \in X$. By using [11, 12] we find that $\Phi$ is sequentially weakly lower semi-continuous, continuous, $\lim_{||u||\to\infty} \Phi(u) = +\infty$, and its derivative at the point $u \in X$ is the functional $\Phi'(u)$ given by

$$\Phi'(u)(v) = \int_0^T J(u'(t))v'(t)\mathrm{dt} + \mathrm{J}\left(\frac{\alpha \mathrm{u}(0)}{\beta}\right)\mathrm{v}(0) - \mathrm{J}\left(\frac{-\gamma \mathrm{u}(\mathrm{T})}{\sigma}\right)\mathrm{v}(\mathrm{T})$$

for every $v \in X$ and also $\Phi' : X \to X^*$ admits a continuous inverse on $X^*$. Moreover, $\Psi$ is sequentially weakly upper semicontinuous and its derivative at the point $u \in X$ is the functional $\Psi'(u)$ given by

$$\Psi'(u)(v) = \int_0^T f(t,u^+(t))v(t)\mathrm{dt} + \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{I_i}(\mathrm{u}^+(\mathrm{t_j}))\mathrm{v}(\mathrm{t_j})$$

for every $v \in X$. Furthermore, $\Psi' : X \to X^*$ is compact. So, we should just check assumption $(i_1)$ and $(i_2)$ from Theorem 2.1. For $\lambda > 0$ the functional $I_\lambda$ is coercive. Since, $\mu < \delta_\lambda$ we can fix $\kappa$ so that, $\limsup_{|\xi(t_j)|\longrightarrow\infty} \frac{\int_0^{\xi^+(t_j)} I_j(s)ds - I_j(0)\xi^-(t_j)}{|\xi(t_j)|^p} < \kappa$ for

$j = 1, \dots, n$ and $\mu\kappa < \frac{\hbar}{nm_2}$ such that $\hbar = \min\left\{1, \frac{1}{Mp}\right\}$. Then, there is a positive constant $\iota$ that

$$\int_0^{\xi^+(t_j)} I_j(s)ds - I_j(0)\xi^-(t_j)ds \le \kappa(\xi(t_j))^p + \iota$$

for each $\xi(t_j) \in \mathbb{R}$ and $j = 1, \dots, n$. We fix a constant $0 < \varepsilon < \frac{\hbar - n\mu\kappa m_2}{\lambda m_1}$ such that $||u||_{L^p} < m_1||u||$. From the hypothesis $(\varrho_2)$ we conclude, there exists a function $\rho_\varepsilon \in L^1([0,T],[0,+\infty))$ such that $F(t, x^+(t)) - f(t,0)x^-(t) \le \varepsilon(x(t))^p + \rho_\varepsilon(t)$ for every $(t,x) \in [0,T] \times [0,+\infty)$. With a simple calculation we can see

$$\frac{1}{M\ p}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) \le \Phi(u)$$
$$\le \frac{1}{mp}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right). \tag{6}$$

Thus, for each $u \in X$,

$$\Phi(u) - \lambda\Psi(u) \ge \frac{1}{Mp}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right)$$
$$- \lambda\left(\int_0^T [\varepsilon(u(t))^p + \rho_\varepsilon(t)]dt + \frac{\mu}{\lambda}\sum_{j=1}^n [\kappa(u(t))^p + \iota]\right)$$
$$\ge \frac{1}{Mp}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) - \lambda\varepsilon\|u\|_{L^p}^p - \lambda\|\rho_\varepsilon\|_{L^1} - n\mu\kappa\|u\|_\infty^p - n\mu\iota$$
$$\ge \hbar||u||^p - \lambda\varepsilon m_1\|u\|^p - \lambda\|\rho_\varepsilon\|_{L^1} - n\mu\kappa m_2\|u\|^p - n\mu\iota - \|u\|_{L^p}^p$$
$$\ge (\hbar - \lambda\varepsilon m_1 - n\mu\kappa m_2)||u||^p - \lambda\|\rho_\varepsilon\|_{L^1} - n\mu\iota - n\mu\iota - \|u\|_{L^p}^p.$$

Hence, $\lim_{\|u\| \longrightarrow \infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$. Thus, by [4, Proposition 2.2] the functional $I_\lambda$ confirms $(PS)^r$-condition for each $r > 0$ and so condition $(i_2)$ of Theorem 2.1 is verified. Let $r_1 := \frac{c_1^p}{pM}$, $r_2 := \frac{c_2^p}{pM}$ and $w(t) = t\nu$ for all $t \in [0,T]$. Hence, by (7) we have $r_1 < \Phi(w) < r_2$. According to [11, Lemma 2.3], for every $u \in X$,

$$\Phi^{-1}(-\infty, r_2] = \{u \in X : |u(t)| \le c_2(\sqrt[p]{\frac{\beta^{p-1}}{\alpha^{p-1}}} + T^{\frac{1}{q}}) \ \text{ for } \ t \in [0,T]\}, \tag{7}$$

so that,
$$-\Theta(c_2) \le ||u||_\infty \le \Theta(c_2) \tag{8}$$

and
$$\sup_{u \in \Phi^{-1}(-\infty, r_2)} \int_0^T F(t, u^+(t))dt \le \int_0^T \sup_{u(t) \le \Theta(c_2)} F(t, u^+(t))dt. \tag{9}$$

Moreover, since $|I_j(u)| < c|u|$ for each $u \in X$, we have

$$-\frac{cn}{2}||u||_\infty^2 < \sum_{j=1}^n [\int_0^{u^+(t)} I_j(s)ds < \frac{cn}{2}||u||_\infty^2. \tag{10}$$

and it follows

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}{r_2} \le \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)}\left(\int_0^T F(t, u^+(t))dt + \frac{\mu}{\lambda}\sum_{j=1}^n \int_0^{u^+(t_j)} I_j(s)ds\right)}{r_2}$$

$$\leq \frac{pM}{c_2^p}\left(\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t,u^+(t))dt + \frac{cn\mu}{2\lambda}(\Theta(c_2))^2\right)$$

for every $u \in X$ and also by (9),

$$\begin{aligned}\Psi(w) &= \int_0^T [F(t,t\nu) - f(t,0)t\nu]dt + \frac{\mu}{\lambda}\sum_{j=1}^n [\int_0^{t_j\nu} I_j(s)ds - I_j(0)t_j\nu]\\ &\geq \int_0^T [F(t,t\nu) - f(t,0)t\nu]dt - \frac{\nu^2 c\mu}{2\lambda}\sum_{j=1}^n t_j^2.\end{aligned}$$

Therefore, we have

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt - \frac{\nu^2 c\mu}{2\lambda}\sum_{j=1}^n t_j^2}{\int_0^T (\int_0^\nu J(s)ds)dt + \frac{\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}.$$

Due to,

$$\mu < \frac{2\lambda\left(\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt\right) - 2\int_0^T (\int_0^\nu J(s)ds)dt - \frac{2\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}{\nu^2 c\sum_{j=1}^n t_j^2},$$

so,

$$\frac{\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt - \frac{\nu^2 c\mu}{2\lambda}\sum_{j=1}^n t_j^2}{\int_0^T (\int_0^\nu J(s)ds)dt + \frac{\sigma}{\gamma}\int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds} > \frac{1}{\lambda}.$$

Furthermore,

$$\frac{2c_2^p - 2\lambda pM\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t,u^+(t))dt}{pMcn(\Theta(c_2))^2} > \mu,$$

therefore,

$$\frac{pM\left(\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t,u^+(t))dt + \frac{cn\mu}{2\lambda}(\Theta(c_2))^2\right)}{c_2^p} < \frac{1}{\lambda}.$$

From $(\varrho_1)$ and above explanations, Theorem 2.1 with $\bar{u} = w$ guarantees existence of a local minimum point $u_\lambda$ for the functional $I_\lambda$ which in $\lambda \in (\frac{\Phi(w)}{\Psi(w)}, \frac{r_2}{\sup_{\Phi(u)\leq r_2}\Psi(u)})$ and $0 < \Phi(u_\lambda) < r_2$. Hence, $u_\lambda$ is a nontrivial solution of the problem (2) and also (8) shows that $\max_{t\in[0,T]} |u_\lambda(t)| \leq \Theta(c_2)$. □

Example 3.2. Put $p = 2$, $T = 1$, $\alpha = \gamma = \beta = \sigma = 1$, $n = 1$ and $t_1 = \frac{1}{2}$ . Consider the problem

$$\begin{cases} -u'' = \lambda t\sin(u)(\frac{1}{2+\cos(u')}), & t \neq t_1,\ a.e.\ t \in [0,1],\\ \Delta J(u'(\frac{1}{2})) = \mu\sin(u(\frac{1}{2})), & \\ u(0) = u'(0), & u(1) + u'(1) = 0.\end{cases} \tag{11}$$

From $h(u)$, we get $J(s) = 2s + \sin(s)$ for all $s \in \mathbb{R}$, $m = \frac{1}{3}$ and $M = 1$. By choosing $\nu = c_1 = 1$ and $c_2 = 3$, we have

$$\frac{\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt}{pM} = 0.49 \geq 0.0006$$

$$= \left( \int_0^T (\int_0^\nu J(s)ds)dt + \frac{\sigma}{\gamma} \int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds \right) \frac{\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t, u^+(t))dt}{c_2^2},$$

$$\limsup_{|u(t)|\longrightarrow\infty} \frac{F(t, u^+(t)) - f(t, 0)u^-(t)}{|u(t)|^2} = \limsup_{|u(t)|\longrightarrow\infty} \frac{u(t)(1 - \cos(u^+(t)))}{|u(t)|^2} \leq 0,$$

$$\limsup_{|u(t)|\longrightarrow\infty} \frac{\int_0^{u^+(\frac{1}{2})} I(s)ds - I(0)u^-(\frac{1}{2})}{|u(t)|^2} = \limsup_{|u(t)|\longrightarrow\infty} \frac{1 - \cos(u^+(\frac{1}{2}))}{|u(t)|^2} < \kappa,$$

$$|I(s_1) - I(s_2)| = |\sin(s_1) - \sin(s_2)| \leq |s_1 - s_2|.$$

Then, for each

$$\lambda \in \left( \frac{\int_0^T (\int_0^\nu J(s)ds)dt + \frac{\sigma}{\gamma} \int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}{\int_0^T [F(t, t\nu) - f(t, 0)t\nu]dt}, \frac{c_2^p}{pM} \frac{1}{\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t, u^+(t))dt} \right)$$

$$= \left( 4.082, 1666.7 \right)$$

and for every

$$0 < \mu < \min \left\{ \frac{2\lambda \left( \int_0^T [F(t, t\nu) - f(t, 0)t\nu]dt \right) - 2\int_0^T (\int_0^\nu J(s)ds)dt - 2\frac{\sigma}{\gamma} \int_0^{-\frac{\gamma\nu T}{\sigma}} J(s)ds}{\nu^2 c \sum_{j=1}^n t_j^2}, \right.$$

$$\left. \frac{2c_2^p - 2\lambda pM \int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t, u^+(t))dt}{pMcn(\Theta(c_2))^2} \right\} = \min \left\{ 3.92\lambda - 16.0024, \frac{18 - 0.0108\lambda}{72} \right\}$$

the problem (12) admits at least one solution in $X$.

## 4. Existence of two solutions

Now, we obtain the existence of two distinct solutions for the problem (2). To follow it up, we apply Theorem 2.2 where the assumption $(\varrho_2)$ is not required.

THEOREM 4.1. *Assume, there exist three positive constants $c_1$, $c_2$ and $\nu$ with the property $c_1 < \nu \sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^{\frac{1}{q}}} < \sqrt[p]{\frac{m}{M}} c_2$, such that*

*$(\varrho_3)$ there exist $\varpi > \max\{p, pM(\frac{1+m}{m})\}$ and $R > 0$ so that, $0 < \varpi F(t, \xi^+(t)) \leq \xi(t) f(t, \xi^+(t))$ for every $|\xi(t)| \geq R$ and $t \in [0, T]$, as well as, $0 < \varpi \int_0^{\xi^+(t_j)} I_j(s)ds \leq I_j(\xi^+(t_j))\xi(t_j)$ for every $|\xi(t_j)| \geq R$, $t_j \in [0, T]$ with $j = 1, \ldots, n$.*

*Then, for each $\lambda \in \Lambda := \left( 0, \frac{c_2^p}{pM \int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t, u^+(t))dt} \right)$ and every Lipschitz function $I_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, \ldots, n$ there exists $\delta_\lambda > 0$ given by (4) such that, for each $\mu \in [0, \delta_\lambda)$, the problem (2) admits at least two solutions $u_1$ and $u_2$ in $X$.*

*Proof.* We use Theorem 2.2 where $X$ and the functionals $\Phi$ and $\Psi$ have already been defined.

We claim that the functional $I_\lambda$ satisfies the (PS)-condition. Indeed, suppose $\{u_n\}_{n\in N} \subset X$ such that $\{I_\lambda u_n\}_{n\in N} \subset X$ is bounded and $I'_\lambda u_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, there is a positive constant $c_0$ such that for every $n \in \mathbb{N}$, $|I_\lambda(u_n)| \leq c_0, |I'_\lambda(u_n)| \leq c_0$. Therefore, according to definition of $I'_\lambda$, hypothesis $(\varrho_3)$ and (7), we have

$$\begin{aligned}
c_0+c_1||u_n|| &\geq \varpi I_\lambda(u_n) - I'_\lambda(u_n)(u_n)\\
&\geq \frac{\varpi}{Mp}\left(||u'_n||^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}|u_n(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u_n(T)|^p\right)\\
&-\left(\int_0^T J(u'_n(t))u'_n(t)dt + J\left(\frac{\alpha u_n(0)}{\beta}\right)u_n(0) - J\left(\frac{-\gamma u_n(T)}{\sigma}\right)u_n(T)\right)\\
&-\varpi\lambda\left(\int_0^T [F(t,u_n^+(t)) - f(t,0)u_n^-(t)]dt + \frac{\mu}{\lambda}\sum_{j=1}^n[\int_0^{u_n^+(t_j)} I_j(s)ds - I_j(0)u_n^-(t_j)]\right)\\
&+\lambda\left(\int_0^T f(t,u_n^+(t))u_n(t)dt + \frac{\mu}{\lambda}\sum_{j=1}^n I_j(u_n^+(t_j))u_n(t_j)\right)\\
&\geq\left(\frac{\varpi}{Mp} - \frac{1}{m}\right)||u'_n||^p_{L^p} \geq (\frac{\varpi}{Mp} - \frac{1}{m} - 1)||u_n||^p.
\end{aligned}$$

for some $c_1 > 0$. Since $\varpi > \max\{p, pM(\frac{1+m}{m})\}$, we conclude that $(u_n)$ is bounded and consequently it results that $u_n \rightharpoonup u$ in $X$. By applying $I'_\lambda(u_n) \longrightarrow 0$ we obtain $(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \longrightarrow 0$. Continuity of $f$, $I_j$ for each $j = 1,\ldots,n$ implies that

$$\int_0^T (f(t,u_n^+(t)) - f(t,u^+(t)))(u_n(t) - u(t))dt \longrightarrow 0, \quad n \longrightarrow \infty,$$

$$(I_j(u_n^+(t_j)) - I_j(u^+(t_j)))(u_n(t_j) - u(t_j)) \longrightarrow 0, \quad n \longrightarrow \infty,$$

for $j = 1,\ldots,n$. In addition,

$$\begin{aligned}
&(I'_\lambda(u_n)-I'_\lambda(u))(u_n-u)\\
&=\int_0^T J(u'_n(t))(u'_n(t)-u'(t))dt+J(\frac{\alpha u_n(0)}{\beta})(u_n(0)-u(0))-J(\frac{-\gamma u_n(T)}{\sigma})(u_n(T)-u(T))\\
&-\left(\int_0^T J(u'(t))(u'_n(t)-u'(t))dt+J(\frac{\alpha u(0)}{\beta})(u_n(0)-u(0))-J(\frac{-\gamma u(T)}{\sigma})(u_n(T)-u(T))\right)\\
&-\lambda\left(\int_0^T [f(t,u_n^+(t))-f(t,u^+(t))](u_n(t)-u(t))dt\right)\\
&-\mu\left(\sum_{j=1}^n[I_j(u_n^+(t_j))-I_j(u^+(t_j))](u_n(t_j)-u(t_j))\right) \geq \frac{1}{M}||u_n-u||.
\end{aligned}$$

Then, $u_n \longrightarrow u$ in $X$. Therefore, $I_\lambda$ satisfies the $(PS)$-condition. By $(\varrho_3)$, there exist constants $a_1, a_2 > 0$ such that, $F(t,x^+(t)) \geq a_1|x|^\varpi - a_2$, for all $t \in [0,T]$ and

$x \in [0,\infty)$. For any $u \in X\setminus\{0\}$ and each $\tau > 0$ one has

$$
\begin{aligned}
&I_\lambda(\tau u) = (\Phi - \lambda\Psi)(\tau u)\\
\leq&\frac{1}{mp}\left(||\tau u'||^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}|\tau u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|\tau u(T)|^p\right)\\
&-\lambda\left(\int_0^T [a_1|\tau u|^{\varpi} - a_2 - f(t,0)u^-(t)]dt + \frac{\mu}{\lambda}\sum_{j=1}^n[\int_0^{u_n^+(t_j)} I_j(s)ds - I_j(0)u_n^-(t_j)]\right)\\
\leq&\frac{\tau^p}{mp}\left(||u'||^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) - \tau^{\varpi}a_1\lambda\int_0^T |u(t)|^{\varpi}dt + T\lambda a_2\\
&+\lambda\int_0^T f(t,0)u^-(t)dt + \mu\left(\sum_{j=1}^n[\int_0^{u_n^+(t_j)} I_j(s)ds]\right)\\
\leq&\frac{\tau^p}{mp}\left(||u'||^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) - \tau^{\varpi}a_1\lambda\int_0^T |u(t)|^{\varpi}dt + T\lambda a_2\\
&+\lambda F(t,0)||u||_\infty + \frac{cn\mu}{2}||u||^2_\infty.
\end{aligned}
$$

Since $\varpi > \max\{p, pM(\frac{1+m}{m})\}$, this condition ensures that $I_\lambda$ is unbounded from below. So, all the assumptions of Theorem 2.2 are satisfied and hence, for each $\lambda \in \left(0, \frac{c_2^p}{pM\int_0^T \sup_{u(t)\leq\Theta(c_2)} F(t,u^+(t))dt}\right)$, the functional $I_\lambda$ has two distinct critical points that are solutions of the problem (2). □

Remark 4.2. In comparison, Theorem 3.1 ensures that the nontrivial critical point is a local minimum, information not provided by Theorem 4.1. In this sense, the conclusion of Theorem 3.1 is much more precise than that of Theorem 4.1.

## 5. Existence of multiplicity result

We investigate the existence of at least two and three solutions for the problem (2) in the case $\lambda = \mu$.

Theorem 5.1. *Suppose, there exist two positive constants $c_1$ and $\nu$ with the property $c_1 < \nu\sqrt[p]{T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^{\frac{1}{q}}}$ and let $(\varrho_2)$ in Theorem 3.1 holds. Further,*

*$(\varrho_4)$* $\int_0^T [F(t,t\nu) - f(t,0)t\nu]dt \geq 0$

*$(\varrho_5)$*
$$
\begin{aligned}
&p^2Mm\left(\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2\right)\\
&<c_1^p\nu^p(T + \frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)\left(-\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right)
\end{aligned}
$$

*Then, for each*

$$\lambda \in \Lambda := \Bigg( \frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)} \frac{1}{-\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2}, \frac{c_1^p}{pM(\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2} \Bigg),$$

*and every Lipschitz function $I_j : [0,+\infty) \longrightarrow [0,+\infty)$ for $j = 1,\dots,n$ that satisfies* (3)*, the problem* (2) *in the case $\lambda = \mu$ admits at least three solutions in $X$.*

*Proof.* Let $I_\lambda = \Phi + \lambda\Psi$ where $\Phi$ has been defined by (5) and

$$\Psi(u) = -\int_0^T [F(t,u^+(t)) - f(t,0)u^-(t)]dt - \sum_{j=1}^n [\int_0^{u^+(t_j)} I_j(s)ds - I_j(0)u^-(t_j)] \quad (12)$$

for each $u \in X$. $\Psi$ is sequentially weakly lower semicontinuous and its Gâteaux derivative at the point $u \in X$ is

$$\Psi'(u)(v) = -\int_0^T f(t,u^+(t))v(t)dt - \sum_{j=1}^n I_i(u^+(t_j))v(t_j)$$

for every $v \in X$. Now, we apply Theorem 2.3. So, it is enough to show $(j_1)$ and $(j_2)$. Furthermore, we can fix $\kappa$ which $\limsup_{|\xi(t_j)|\longrightarrow\infty} \frac{\int_0^{\xi^+(t_j)} I_j(s)ds - I_j(0)\xi^-(t_j)}{|\xi(t_j)|^p} < \kappa$ for $j = 1,\dots,n$ and $\kappa\lambda < \frac{\hbar}{m_2}$. Therefore, there is a positive constant $\iota$ such that

$$\int_0^{\xi^+(t_j)} I_j(s)ds - I_j(0)\xi^-(t_j) \leq \kappa(\xi(t_j))^p + \iota$$

for $j = 1,\dots,n$ and each $\xi(t_j) \in \mathbb{R}$. We fix a constant $0 < \varepsilon < \frac{\hbar-\kappa m_2\lambda}{\lambda m_1}$. By $(\varrho_2)$ there exists a function $\rho_\varepsilon \in L^1([0,T],[0,+\infty))$ so that, $F(t,u^+(t)) - f(t,0)u^-(t) \leq \varepsilon(u(t))^p + \rho_\varepsilon(t)$ for every $(t,x) \in [0,T]\times[0,+\infty)$. It follows that for each $u \in X$,

$$\Phi(u)+\lambda\Psi(u) \geq \frac{1}{M\ p}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right)$$

$$+\lambda\left(-\int_0^T [F(t,u^+(t)) - f(t,0)u^-(t)]dt - \sum_{j=1}^n [\int_0^{u^+(t_j)} I_j(s)ds - I_j(0)u^-(t_j)]\right) =$$

$$\frac{1}{M\ p}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right)$$

$$-\lambda\int_0^T [F(t,u^+(t)) - f(t,0)u^-(t)]dt - \lambda\sum_{j=1}^n [\int_0^{u^+(t_j)} I_j(s)ds - I_j(0)u^-(t_j)] \geq$$

$$\frac{1}{M\ p}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) - \lambda\int_0^T [\varepsilon(u(t))^p + \rho_\varepsilon(t)]dt - \lambda\sum_{j=1}^n [\kappa(u(t_j))^p + \iota] \geq$$

$$\frac{1}{M\ p}\left(||u'||_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}|u(0)|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}|u(T)|^p\right) - \lambda\varepsilon m_1\|u\|^p - \lambda\|\rho\|_{L^1} - \lambda\kappa m_2\|u\|^p - \lambda\iota \geq$$

$(\hbar-\lambda\varepsilon m_1-\lambda\kappa m_2)\|u\|^p-\lambda\|\rho\|_{L^1}-\lambda\iota-\|u\|_{L^p}^p$.

Hence, $\lim_{\|u\|\longrightarrow\infty}(\Phi(u)+\lambda\Psi(u))=\infty$. Now, it remains to show $(j_2)$. Let $r_1:=\frac{c_1^p}{pM}$, $r_2:=\frac{c_2^p}{pM}$ and $w(t)=t\nu$ for all $t\in[0.T]$. Hence, by (7) $r_1<\Phi(w)<r_2$. Due to (8) for $r=r_1$ and (11),

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r_1)}-\Psi(u)&\le\sup_{u\in\Phi^{-1}(-\infty,r_1)}\left(\int_0^T F(t,u^+(t))dt+\sum_{j=1}^n\int_0^{u^+(t_j)}I_j(s)ds\right)\\&\le\int_0^T\sup_{u(t)\le\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2.\end{aligned}$$

Moreover, by $(\varrho_4)$ and according to (8) for $r=r_1$,

$$\Psi(w)=-\int_0^T[F(t,t\nu)-f(t,0)t\nu]dt-\sum_{j=1}^n[\int_0^{t_j\nu}I_j(s)ds-I_j(0)t_j\nu].$$

Moreover, $\Phi(0)=\Psi(0)=0$, $\overline{\Phi^{-1}(-\infty,r_1)}^w=\Phi^{-1}(-\infty,r_1)$ and also from the definition of $\varphi(r_1)$, we conclude that

$$\begin{aligned}\varphi_1(r_1)&:=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\Psi(u)-\inf_{\overline{\Phi^{-1}(-\infty,r_1)}^w}\Psi}{r_1-\Phi(u)}\le\frac{-\inf_{\overline{\Phi^{-1}(-\infty,r_1)}^w}\Psi}{r_1}\\&\le\frac{pM\left(\int_0^T\sup_{u(t)\le\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2\right)}{c_1^p},\\ \varphi_2(r_1)&:=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\sup_{u\in\Phi^{-1}[r_1,\infty)}\frac{\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)}\ge\inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)}\\&\ge\frac{\inf_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)}\ge\frac{\inf_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)-\Psi(w)}{\Phi(w)}\\&\ge\frac{-\int_0^T\sup_{u(t)\le\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2}{\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}}\\&=\frac{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{mp}\left(-\int_0^T\sup_{u(t)\le\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right).\end{aligned}$$

Hence, from $(\varrho_5)$ one has $\varphi_1(r_1)<\varphi_2(r_1)$. Therefore, all the assumptions of Theorem 2.3 are fulfilled and the desired conclusion is obtained. □

EXAMPLE 5.2. Put $p=3$, $T=1$, $\alpha=\gamma=\beta=\sigma=1$, $n=2$, $t_1=\frac{1}{2}$ and $t_2=\frac{1}{4}$. Consider the problem

$$\begin{cases}-(\phi_3(u'))'=\lambda(tu)h(u'), & t\ne t_1,\ t\ne t_2,\ a.e.\ t\in[0,1],\\ \Delta J(u'(\frac{1}{2}))=\lambda u(\frac{1}{2}),\\ \Delta J(u'(\frac{1}{4}))=\lambda\sin(u(\frac{1}{4})),\\ u(0)=u'(0), & u(1)+u'(1)=0.\end{cases}\tag{13}$$

Put $$h(x)=\begin{cases}\frac{1}{2} & 0<x,\\ x+\frac{1}{2} & 0\leqslant x\leqslant 1,\\ \frac{3}{2} & x>1.\end{cases}$$

We can see $m=\frac{1}{2}$ and $M=\frac{3}{2}$. By choosing $c_1=1$, $\nu=2$ and $c_2=6$, we have

$$\int_0^T [F(t,t\nu)-f(t,0)t\nu]dt=\frac{1}{8}>0,$$

$$p^2Mm\left(\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2\right)=33.75<58$$

$$=c_1^p\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)\left(-\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right).$$

Also, $$|I_1(s_1)-I_1(s_2)|=|s_1-s_2|\leq|s_1-s_2|,$$
$$|I_2(s_1)-I_2(s_2)|=|\sin(s_1)-\sin(s_2)|\leq|s_1-s_2|$$

and $$\limsup_{|u(t)|\longrightarrow\infty}\frac{\int_0^{u^+(\frac{1}{2})}I_1(s)ds-I_1(0)u^-(\frac{1}{2})}{|u(t)|^3}=\limsup_{|u(t)|\longrightarrow\infty}\frac{(u^+(\frac{1}{2}))^2}{2|u(t)|^3}<\infty,$$

$$\limsup_{|u(t)|\longrightarrow\infty}\frac{\int_0^{u^+(\frac{1}{4})}I_2(s)ds-I_2(0)u^-(\frac{1}{4})}{|u(t)|^3}=\limsup_{|u(t)|\longrightarrow\infty}\frac{1-\cos(u^+(\frac{1}{4}))}{|u(t)|^3}<\infty.$$

Then, for each

$$\lambda\in\Bigg(\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}\ \frac{1}{-\int_0^T\sup_{u(t)\leq\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2},$$
$$\frac{c_1^p}{pM(\int_0^T\sup_{u(t)\leq\Theta(c_1)}F(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2}\Bigg)=(0.025,0.044)$$

the problem (14) admits at least three solutions.

Now, we can see an application of Theorem 2.4.

THEOREM 5.3. *Suppose, there exist three positive constants $c_1$, $c_2$ and $\nu$ with the property*

$$c_1<\nu\sqrt[p]{T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^{\frac{1}{q}}}<\sqrt[p]{\frac{m}{M}}c_2 \tag{14}$$

*such that ($\varrho_4$) of Theorem 5.1 holds and*

($\varrho_6$) $$mMp^2\max\left\{\frac{\int_0^T\sup_{u(t)\leq\Theta(c_1)}\bar{F}(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_1))^2}{c_1^p},\right.$$
$$\left.\frac{\int_0^T\sup_{u(t)\leq\Theta(c_2)}\bar{F}(t,u^+(t))dt+\frac{cn}{2}(\Theta(c_2))^2}{c_2^p}\right\}$$
$$<\nu^p(T+\tfrac{\gamma^{p-1}}{\sigma^{p-1}}T^p)\left(-\int_0^T\sup_{u(t)\leq\Theta(c_1)}\bar{F}(t,u^+(t))dt+\tfrac{cn}{2}(\Theta(c_2))^2+\tfrac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right).$$

*Then, for each*

$$\lambda \in \Lambda := \Bigg( \frac{mp}{\nu^p (T + \frac{\gamma^{p-1}}{\sigma^{p-1}} T^p)} \frac{1}{-\int_0^T \sup_{u(t)\le\Theta(c_1)} \bar{F}(t, u^+(t))dt + \frac{cn}{2}(\Theta(c_2))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2},$$
$$\min\Bigg\{ \frac{c_2^p}{pM\left(\int_0^T \sup_{u(t)\le\Theta(c_2)} \bar{F}(t, u^+(t))dt + \frac{cn}{2}(\Theta(c_2))^2\right)},$$
$$\frac{c_1^p}{pM\left(\int_0^T \sup_{u(t)\le\Theta(c_1)} \bar{F}(t, u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2\right)}\Bigg\}\Bigg),$$

*the problem* (2) *admits at least two solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *so that,* $\max_{t\in[0,T]}|u_{1,\lambda}| < c_1(\sqrt[p]{\frac{\beta^p-1}{\alpha^p-1}} + T^{\frac{1}{q}})$ *and* $\max_{t\in[0,T]}|u_{2,\lambda}| < c_2(\sqrt[p]{\frac{\beta^p-1}{\alpha^p-1}} + T^{\frac{1}{q}})$.

*Proof.* Let

$$\bar{f}(t,x) \begin{cases} f(t,\Theta(c_1)), & (t,x) \in [0,T] \times [0,\Theta(c_1)), \\ f(t,x), & (t,x) \in [0,T] \times [\Theta(c_1),\Theta(c_2)], \\ f(t,\Theta(c_2)), & (t,x) \in [0,T] \times (\Theta(c_2),\infty). \end{cases}$$

We can simply show that $\bar{f} : [0,T] \times [0,+\infty)$ is a continuous function. Now, take $\bar{F}(t,\xi) = \int_0^\xi \bar{f}(t,x)dx$ for all $(t,\xi) \in [0,T] \times [0,+\infty)$ and $X$ has been defined. Also, put $\Phi$ as (5) and

$$\Psi(u) = -\int_0^T [\bar{F}(t,u^+(t)) - \bar{f}(t,0)u^-(t)]dt - \sum_{j=1}^n [\int_0^{u^+(t_j)} I_j(s)ds - I_j(0)u^-(t_j)]$$

for all $u \in X$. We apply Theorem 2.4 for $\Phi$ and $\Psi$ that have been mentioned. $\Psi$ is a differentiable functional and its differential at the point $u \in X$ is

$$\Psi'(u)(v) = -\int_0^T \bar{f}(t,u^+(t))v(t)\mathrm{d}t - \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{I_i}(\mathrm{u}^+(\mathrm{t_j}))\mathrm{v}(\mathrm{t_j})$$

for any $v \in X$. It is sequentially weakly lower semicontinuous. Moreover, $\Psi' : X \longrightarrow X^*$ is a compact operator. So, it is enough to check $(k_1)$, $(k_2)$ and $(k_3)$. Put

$$r_1 := \frac{c_1^p}{pM}, r_2 := \frac{c_2^p}{pM}. \tag{15}$$

By (15) and (16) for $w = t\nu \in X$, we can see $r_1 < \Phi(w) < r_2$, $\inf_X \Phi < r_1 < r_2$ and

$$\varphi_1(r_1) = \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r_1)}^w} \Psi}{r_1 - \Phi(u)} \le \frac{-\inf_{\overline{\Phi^{-1}(-\infty,r_1)}^w} \Psi}{r_1}$$
$$\le \frac{pM\left(\int_0^T \sup_{u(t)\le\Theta(c_1)} \bar{F}(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2\right)}{c_1^p},$$
$$\varphi_1(r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_2)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r_2)}^w} \Psi}{r_2 - \Phi(u)} \le \frac{-\inf_{\overline{\Phi^{-1}(-\infty,r_2)}^w} \Psi}{r_2}$$

$$\leq \frac{pM\left(\int_0^T \sup_{u(t)\leq\Theta(c_2)} \bar{F}(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_2))^2\right)}{c_2^p}$$

and

$$\varphi_2^*(r_1,r_2) \geq \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)} \geq \frac{\inf_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)-\Psi(w)}{\Phi(w)}$$
$$\geq \frac{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}{mp}\left(-\int_0^T \sup_{u(t)\leq\Theta(c_1)} F(t,u^+(t))dt + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right).$$

Taking $(\varrho_4)$ and $(\varrho_6)$ into account, we obtain conditions $(k_2)$ and $(k_3)$ of Theorem 2.4. Thus, for every $\lambda \in \Lambda$, the problem (2) gets at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$. Also, according to (8), we conclude that $\max_{t\in[0,T]} |u_{1,\lambda}| < \Theta(c_1)$ and $\max_{t\in[0,T]} |u_{2,\lambda}| < \Theta(c_2)$. □

The following existence results are consequences of Theorem 5.1 and 5.3, respectively. The function $f$ has separated variables for every $(t,x) \in [0,T]\times[0,+\infty)$ in the below problem

$$\begin{cases} -(\phi_p(u'))' = \lambda\chi(t)k(u(t))h(u'), & t\neq t_j, a.e.\ t\in[0,T],\\ \Delta J(u'(t_j)) = \lambda I_j(u(t_j)), & j=1,2,\ldots,n,\\ \alpha u(0)-\beta u'(0)=0, & \gamma u(T)+\sigma u'(T)=0, \end{cases} \tag{16}$$

where $\chi : [0,T] \longrightarrow [0,+\infty)$ is a non-negative and non-zero function so that $\chi \in L^1([0,T],[0,+\infty))$. Also, $k : [0,+\infty) \longrightarrow [0,+\infty)$ is a non-negative and continuous function such that

$$K(\xi) = \int_0^\xi k(x)dx, \quad (\xi\in[0,+\infty)).$$

THEOREM 5.4. *Assume, there exist two positive constants $c_1$ and $\nu$ with the property $c_1 < \nu\sqrt[p]{T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^{\frac{1}{q}}}$. Let $(\varrho_2)$ and*

$(\varrho_9)$ $$p^2mM\left(\|\chi\|_{L^1}K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2\right)$$
$$< c_1^p\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)\left(-\|\chi\|_{L^1}K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right)$$

*hold. Then, for every*

$$\lambda\in\Lambda := \left(\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}\ \frac{1}{-\|\chi\|_{L^1}K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2},\right.$$
$$\left.\frac{c_1^p}{pM\left(\|\chi\|_{L^1}K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2\right)}\right)$$

*and every Lipschitz function $I_j : [0,+\infty) \longrightarrow [0,+\infty)$ satisfing* (3) *for $j=1,\ldots,n$, the problem* (17) *admits at least three solutions in $X$.*

We point out a special case of Theorem 1.1.

THEOREM 5.5. *Let*

$$\lim_{\xi\longrightarrow 0^+}\frac{k(\xi)}{\xi}=\lim_{|\xi|\longrightarrow\infty}\frac{k(\xi)}{|\xi|}=0 \tag{17}$$

*and there exists a positive constant* $\nu$ *such that* $F(t\nu)-f(0)t\nu>0$. *Then, for every* $\lambda>\lambda^*$, *where* $\lambda^*:=\inf_{\nu>0}\left\{\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}\times\frac{1}{\left(-\|\chi\|_{L^1}K(\Theta(c_1))+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right)}\right\}$ *and every Lipschitz function* $I_j:[0,+\infty)\longrightarrow[0,+\infty)$ *for* $j=1,\ldots,n$ *satisfing* (3) *the problem* (2) *admits at least two solutions in* $X$.

*Proof.* Fix $\lambda>\lambda^*$. Therefore, there exists $\nu>0$ such that

$$\lambda>\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)}\times\frac{1}{\left(-\|\chi\|_{L^1}K(\Theta(c_1))+\frac{cn}{2}(\Theta(c_1))^2+\frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right)}.$$

By (18) $\lim_{x\longrightarrow 0^+}\frac{\sup_{|\xi|\le x}k(\xi)}{x}=\lim_{x\longrightarrow\infty}\frac{\sup_{|\xi|\le x}k(\xi)}{x}=0$. Thus, we can choose $c_1$, $c_2>0$ so that, $c_1^p<\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^{\frac{1}{q}})<\frac{m}{M}c_2^p$, $\frac{\sup_{|\xi|\le c_1}k(\xi)}{c_1}<\frac{c_1^p}{pM\left(\|\chi\|_{L^1}K(\Theta(c_1))+\frac{cn}{2}(\Theta(c_1))^2\right)}$ and $\frac{\sup_{|\xi|\le c_2}k(\xi)}{c_2}<\frac{c_2^p}{pM\left(\|\chi\|_{L^1}K(\Theta(c_2))+\frac{cn}{2}(\Theta(c_2))^2\right)}$. Hence, we can get the result from Theorem 1.1. □

EXAMPLE 5.6. Put $p=2$, $T=1$, $\alpha=\gamma=\beta=\sigma=1$, $n=2$, $t_1=\frac{1}{2}$ and $t_2=\frac{1}{4}$. Consider the problem

$$\begin{cases}-u''=\lambda tk(u)h(u'), & t\neq t_1,\ t\neq t_2,\ a.e.\ t\in[0,1],\\ \Delta J(u'(\frac{1}{2}))=\lambda\sin(u(\frac{1}{2})),\\ \Delta J(u'(\frac{1}{4}))=\lambda\arctan(u(\frac{1}{4})),\\ u(0)=u'(0), & u(1)+u'(1)=0,\end{cases} \tag{18}$$

where $k(x)=\begin{cases}x^2, & 0\le x\le 1,\\ \frac{1}{x}, & x>1.\end{cases}$

Put $h(x)=\frac{1}{\sin(x)+2}$; thus, $J(s)=2s-\cos(s)$, $m=\frac{1}{3}$ and $M=1$. By selecting $c_1=1$ we have

$$\lim_{u(t)\longrightarrow 0^+}\frac{k(u(t))}{u(t)}=\lim_{u(t)\longrightarrow 0^+}\frac{(u(t))^2}{2u(t)}=0,\quad \lim_{|u(t)|\longrightarrow\infty}\frac{k(u(t))}{|u(t)|}=\lim_{|u(t)|\longrightarrow\infty}\frac{1}{(u(t))^2}=0,$$

$$F(t\nu)-f(0)t\nu=\frac{1}{3}+\ln(|t\nu|)>0.$$

Also,

$$|I_1(s_1)-I_1(s_2)|=|\sin(s_1)-\sin(s_2)|\le|s_1-s_2|,$$
$$|I_2(s_1)-I_2(s_2)|=|\arctan(s_1)-\arctan(s_2)|\le\pi|s_1-s_2|$$

and

$$\limsup_{|u(t)|\longrightarrow\infty}\frac{\int_0^{u^+(\frac{1}{2})}I_1(s)ds-I_1(0)u^-(\frac{1}{2})}{|u(t)|^2}=\limsup_{|u(t)|\longrightarrow\infty}\frac{1-\cos(u^+(\frac{1}{2}))}{2|u(t)|^2}<\infty,$$

$$\limsup_{|u(t)|\longrightarrow\infty} \frac{\int_0^{u^+(\frac{1}{4})} I_2(s)ds - I_2(0)u^-(\frac{1}{4})}{|u(t)|^2}$$
$$= \limsup_{|u(t)|\longrightarrow\infty} \frac{2u^+(\frac{1}{4})\arctan(u^+(\frac{1}{4})) - \ln(1+(u^+(\frac{1}{4}))^2)}{2|u(t)|^2} < \infty.$$

Then, for every

$$\lambda > \inf_{\nu>0}\left\{\frac{mp}{\nu^p(T+\frac{\gamma^{p-1}}{\sigma^{p-1}}T^p)} \times \frac{1}{\left(-\|\chi\|_{L^1}K(\Theta(c_1)) + \frac{cn}{2}(\Theta(c_1))^2 + \frac{c\nu^2}{2}\sum_{j=1}^n t_j^2\right)}\right\}$$
$$= \inf_{\nu>0}\left(\frac{32}{\nu^3(-98.54+384\pi+15\pi\nu^2)}\right),$$

the problem (19) admits at least two solutions in $X$.

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Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran

*E-mail*: mehraban.zahra@razi.ac.ir

Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran

*E-mail*: sh.heidarkhani@razi.ac.ir