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ON SOLVABILITY OF QUADRATIC HAMMERSTEIN INTEGRAL EQUATIONS IN HÖLDER SPACES

Mohamed Abdalla Darwish, Mohamed M. A. Metwali and Donal O'Regan

Abstract. Using Schauder's fixed point theorem we consider the solvability of a quadratic Hammerstein integral equation in the space of functions satisfying a Hölder condition. An example is included to illustrate our results.

1. Introduction

In this paper, we investigate the existence of solutions of the following quadratic integral equation of Hammerstein type

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau)\, f(\tau, (\Lambda x)(\tau))\, d\tau,\ t \in [0,1], \tag{1}$$

where Λ is a general operator.

If $f(t,y) = y$ we get an equation in [13], if $f(t,y) = y$ and $\Lambda x = \max\{|x(\tau)| : 0 \le \tau \le r(t)\}$, where $r : [0,1] \to [0,1]$ is a continuous and nondecreasing function we obtain an equation studied in [6] and if $f(t,y) = y$ and $(\Lambda x)(t) = x(r(t))$, where $r : [0,1] \to [0,1]$ is a measurable function, we obtain an equation studied in [5]. When $\Lambda y = y$ and $f(t,y) = -y$, (1) becomes

$$x(t) + x(t) \int_0^1 k(t,\tau)\, x(\tau)\, d\tau = p(t),\ t \in [0,1].$$

This equation is a generalization of a famous equation in transport theory, the so-called Chandrasekhar H-equation in which $p(t) = 1$, x must be identified with the H-function and for a nonnegative characteristic function ϕ, $k(t,\tau) = \frac{t\,\phi(t)}{t+\tau}$; see for example [7, 10, 12] and the references therein. Quadratic integral equations arise in the theory of radiative transfer, in the theory of neutron transport and in the theory of traffic; see [1, 4, 8, 9, 11, 14] and the references therein.

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In the space of functions satisfying a Hölder condition, Schauder's fixed point theorem and the relative compactness in these spaces are the main tools used to prove our main result.

2. Preliminaries

We denote by $C[a,b]$ the space of all continuous functions $x:[a,b]\to\mathbb{R}$ equipped with the norm $\|x\|_\infty=\sup\{|x(t)|:t\in[a,b]\}$ for $x\in C[a,b]$. Let $H_\alpha[a,b]$, $\alpha\in(0,1]$ be the collection of all real functions x defined on $[a,b]$ which satisfies a Hölder condition

$$|x(t)-x(\tau)|\leq H_x^\alpha|t-\tau|^\alpha,\ \ \forall(t,\tau)\in[a,b]^2, \tag{2}$$

where H_x^α is the least possible constant for which inequality (2) is satisfied, i.e.,

$$H_x^\alpha=\sup\left\{\frac{|x(t)-x(\tau)|}{|t-\tau|^\alpha}:t,\tau\in[a,b],\ t\neq\tau\right\}.$$

The spaces $H_\alpha[a,b]$, $0<\alpha\leq1$, equipped with the norm $\|x\|_\alpha=|x(a)|+H_x^\alpha$ are Banach spaces.

LEMMA 2.1. *The norm $\|\cdot\|_\infty$ is dominated by the norm $\|\cdot\|_\alpha$, i.e., for an arbitrarily fixed $x\in H_\alpha[a,b]$ and for an arbitrary $t\in[a,b]$, the following inequality holds $\|x\|_\infty\leq\max\{1,(b-a)^\alpha\}\|x\|_\alpha$.*

LEMMA 2.2. *For $0<\alpha<\beta\leq1$, we have $H_\beta[a,b]\subset H_\alpha[a,b]\subset C[a,b]$. Moreover, for $x\in H_\beta[a,b]$ the following inequality is satisfied $\|x\|_\alpha\leq\max\left\{1,(b-a)^{\beta-\alpha}\right\}\|x\|_\beta$.*

The authors in [2] established a sufficient condition for relative compactness in the spaces $H_\alpha[a,b]$, $\alpha\in(0,1]$.

THEOREM 2.3. *Let $0<\alpha<\beta\leq1$ and let B be a bounded subset in $H_\beta[a,b]$ (this means that $\|x\|_\beta\leq M$ for certain constant $M>0$, for any $x\in B$). Then B is a relatively compact subset of $H_\alpha[a,b]$.*

3. Main results

In this section we discuss the solvability of (1) in Hölder spaces.

We assume the following are satisfied.

(a1) $p\in H_\beta[0,1]$, $0<\beta\leq1$.

(a2) The function $k:[0,1]^2\to\mathbb{R}$ is a continuous function and there exists a constant $\kappa_\beta>0$ such that $|k(t,\tau)-k(s,\tau)|\leq\kappa_\beta\,|t-s|^\beta$ for any $t,\tau,s\in[0,1]$.

(a3) The function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a nondecreasing function $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $|f(t,x)|\leq\Psi(|x|)$, $\forall(t,x)\in([0,1],\mathbb{R})$.

(a4) The operator $\Lambda:H_\beta[0,1]\to C[0,1]$ is continuous and there exists a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that for any $x\in H_\beta[0,1]$, $\|\Lambda x\|_\infty\leq\psi(\|x\|_\beta)$.

(a5) Let r be a positive solution of the following equation $\|p\|_\beta + (K+\kappa_\beta)\Psi(\psi(r))r \le r$, where $K = \sup\left\{\int_0^1 |k(t,\tau)|\, d\tau : t \in [0,1]\right\}$.

THEOREM 3.1. *Under assumptions (a1)–(a5),* (1) *has at least one solution* $x \in H_\alpha[0,1]$ *(here α is an arbitrarily fixed number satisfying* $0 < \alpha < \beta$*).*

Proof. Consider the operator $\mathfrak{T}$ defined on $H_\beta[0,1]$ by

$$(\mathfrak{T}x)(t) = p(t) + x(t)\int_0^1 k(t,\tau)\ f(\tau,(\Lambda x)(\tau))\ d\tau,\ \ t\in[0,1].$$

We claim that $\mathfrak{T}$ maps the space $H_\beta[0,1]$ into itself. Take $x \in H_\beta[0,1]$ and $t, s \in [0,1]$ with $t \neq s$. Then, by assumptions (a1) and (a2), we have

$$\begin{aligned}
&\frac{|(\mathfrak{T}x)(t)-(\mathfrak{T}x)(s)|}{|t-s|^\beta}\\
&=\frac{\left|p(t)+x(t)\int_0^1 k(t,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau - p(s)-x(s)\int_0^1 k(s,\tau)f(\tau,\Lambda x(\tau))\ d\tau\right|}{|t-s|^\beta}\\
&\le\frac{|p(t)-p(s)|}{|t-s|^\beta}+\frac{\left|x(t)\int_0^1 k(t,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau - x(s)\int_0^1 k(t,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau\right|}{|t-s|^\beta}\\
&\quad+\frac{\left|x(s)\int_0^1 k(t,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau - x(s)\int_0^1 k(s,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau\right|}{|t-s|^\beta}\\
&\le\frac{|p(t)-p(s)|}{|t-s|^\beta}+\frac{|x(t)-x(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)|\ |f(\tau,\Lambda x(\tau))|\ d\tau\\
&\quad+\frac{|x(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)-k(s,\tau)|\ |f(\tau,\Lambda x(\tau))|\ d\tau\\
&\le\frac{|p(t)-p(s)|}{|t-s|^\beta}+\frac{|x(t)-x(s)|}{|t-s|^\beta}\Psi(\|\Lambda x\|_\infty)\int_0^1 |k(t,\tau)|\ d\tau\\
&\quad+\frac{\|x\|_\infty\Psi(\|\Lambda x\|_\infty)}{|t-s|^\beta}\int_0^1 |k(t,\tau)-k(s,\tau)|\ d\tau\\
&\le\frac{|p(t)-p(s)|}{|t-s|^\beta}+K\Psi(\psi(\|x\|_\beta))\frac{|x(t)-x(s)|}{|t-s|^\beta}+\kappa_\beta\Psi(\psi(\|x\|_\beta))\|x\|_\infty\frac{\int_0^1 |t-s|^\beta\ d\tau}{|t-s|^\beta}.
\end{aligned}$$

Thus $H^\beta_{\mathfrak{T}x} \le H^\beta_p + \left(KH^\beta_x + \kappa_\beta\|x\|_\beta\right)\Psi(\psi(\|x\|_\beta))$, so

$$\begin{aligned}
&\|\mathfrak{T}x\|_\beta = |(\mathfrak{T}x)(0)| + H^\beta_{\mathfrak{T}x}\\
&\le |p(0)|+|x(0)|\int_0^1 |k(0,\tau)|\ |f(\tau,\Lambda x(\tau))|\ d\tau + H^\beta_p + \left(KH^\beta_x+\kappa_\beta\|x\|_\beta\right)\Psi(\psi(\|x\|_\beta))\\
&\le \|p\|_\beta + K|x(0)|\Psi(\psi(\|x\|_\beta)) + \left(KH^\beta_x+\kappa_\beta\|x\|_\beta\right)\Psi(\psi(\|x\|_\beta))\\
&\le \|p\|_\beta + (K+\kappa_\beta)\,\|x\|_\beta\Psi(\psi(\|x\|_\beta)). \qquad (3)
\end{aligned}$$

This proves that the operator $\mathfrak{T}$ maps $H_\beta[0,1]$ into itself.

Using assumption (a5) and inequality (3), we deduce that $\mathfrak{T}$ maps the closed ball $B^{\beta}_{r_0} = \{x \in H_{\beta}[0,1] : \|x\|_{\beta} \leq r_0\}$ into itself, for any r_0 satisfying $\|p\|_{\beta} + (K+\kappa_{\beta})\, r_0 \Psi(\psi(r_0)) \leq r_0$. Theorem 2.3 guarantees that the set $B^{\beta}_{r_0}$ is relatively compact in $H_{\alpha}[0,1]$ for any $0<\alpha<\beta\leq 1$. Moreover, it is easy to see that $B^{\beta}_{r_0}$ is a compact subset in $H_{\alpha}[0,1]$ for any $0<\alpha<\beta\leq 1$; see the Appendix in [5].

We now prove that the operator $\mathfrak{T}$ is continuous on $B^{\beta}_{r_0}$ with respect to the norm $\|\cdot\|_{\alpha}$, where $0<\alpha<\beta\leq 1$. Fix $\varepsilon>0$ and take $x,y \in B^{\beta}_{r_0}$ with $\|x-y\|_{\alpha} \leq \varepsilon$. Then, for $t \in [0,1]$, we have

$$
\begin{aligned}
&\frac{|[(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)]-[(\mathfrak{T}x)(s)-(\mathfrak{T}y)(s)]|}{|t-s|^{\alpha}}\\
&=\frac{1}{|t-s|^{\alpha}}\left|\left(x(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda y(\tau))\, d\tau\right)\right.\\
&\quad\left.-\left(x(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda y(\tau))\, d\tau\right)\right|\\
&=\frac{1}{|t-s|^{\alpha}}\left|\left(x(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau\right)\right.\\
&\quad+\left(y(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(t)\int_0^1 k(t,\tau)\, f(\tau,\Lambda y(\tau))\, d\tau\right)\\
&\quad-\left(x(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau\right)\\
&\quad\left.-\left(y(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau - y(s)\int_0^1 k(s,\tau)\, f(\tau,\Lambda y(\tau))\, d\tau\right)\right|\\
&=\frac{1}{|t-s|^{\alpha}}\left|(x(t)-y(t))\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau\right.\\
&\quad+y(t)\int_0^1 k(t,\tau)(f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau)))\, d\tau\\
&\quad-(x(s)-y(s))\int_0^1 k(s,\tau)\, (f(\tau,\Lambda x(\tau))\, d\tau\\
&\quad\left.-y(s)\int_0^1 k(s,\tau)\, (f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau)))\, d\tau\right|\\
&=\frac{1}{|t-s|^{\alpha}}\left|((x(t)-y(t))-(x(s)-y(s)))\int_0^1 k(t,\tau)\, f(\tau,\Lambda x(\tau))\, d\tau\right.\\
&\quad+(x(s)-y(s))\int_0^1 (k(t,\tau)-k(s,\tau))\, (f(\tau,\Lambda x(\tau))\, d\tau\\
&\quad+(y(t)-y(s))\int_0^1 k(t,\tau)(f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau)))\, d\tau\\
&\quad\left.+y(s)\int_0^1 (k(t,\tau)-k(s,\tau))\, (f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau)))\, d\tau\right|
\end{aligned}
$$

$$\leq \frac{|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)|\ |f(\tau,\Lambda x(\tau))|\ d\tau$$
$$+\frac{|x(s)-y(s)|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)-k(s,\tau)|\ |f(\tau,\Lambda x(\tau))|\ d\tau$$
$$\frac{|y(t)-y(s)|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)|\ |f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau))|\ d\tau$$
$$+\frac{|y(s)|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)-k(s,\tau)|\ |f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau))|\ d\tau$$
$$\leq \frac{K|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^{\alpha}} \Psi(\psi(\|x\|_{\beta}))+K_{\beta}\|x-y\|_{\infty}\Psi(\psi(\|x\|_{\beta})) \int_0^1 |t-s|^{\beta-\alpha} d\tau$$
$$+\frac{K|y(t)-y(s)|}{|t-s|^{\alpha}}\gamma_f(\varepsilon)+K_{\beta}\|y\|_{\infty}\gamma_f(\varepsilon) \int_0^1 |t-s|^{\beta-\alpha} d\tau$$
$$\leq \left(\frac{K|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^{\alpha}}+K_{\beta}\|x-y\|_{\alpha}\right)\Psi(\psi(\|x\|_{\beta}))$$
$$+\left(\frac{K|y(t)-y(s)|}{|t-s|^{\alpha}}+K_{\beta}\|y\|_{\alpha}\right)\gamma_f(\varepsilon),$$

where, $\gamma_f(\varepsilon)=\sup\{|f(t,y_1)-f(t,y_2)|: t\in[0,1], y_1,y_2\in[0,\psi(r_0)], \|y_1-y_2\|\leq\varepsilon\}$. Hence,

$$H^{\alpha}_{\mathfrak{T}x-\mathfrak{T}y} \leq \left(KH^{\alpha}_{x-y}+K_{\beta}\|x-y\|_{\alpha}\right)\Psi(\psi(\|x\|_{\beta}))+\left(KH^{\alpha}_{y}+K_{\beta}\|y\|_{\alpha}\right)\gamma_f(\varepsilon). \qquad (4)$$

Also, we have

$$|(\mathfrak{T}x)(0)-(\mathfrak{T}y)(0)|$$
$$=\left|x(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau-y(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda y(\tau))\ d\tau\right|$$
$$\leq\left|x(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau-y(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau\right|$$
$$+\left|y(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda x(\tau))\ d\tau-y(0)\int_0^1 k(0,\tau)\ f(\tau,\Lambda y(\tau))\ d\tau\right|$$
$$\leq|x(0)-y(0)|\int_0^1 |k(0,\tau)||f(\tau,\Lambda x(\tau))|d\tau+|y(0)|\int_0^1 |k(0,\tau)||f(\tau,\Lambda x(\tau))-f(\tau,\Lambda y(\tau))|d\tau$$
$$\leq K|x(0)-y(0)|\Psi(\psi(\|x\|_{\beta}))+K|y(0)|\gamma_f(\varepsilon). \qquad (5)$$

Add (4) and (5), and we obtain

$$\begin{aligned}\|\mathfrak{T}x-\mathfrak{T}y\|_{\alpha} &\leq \left(KH^{\alpha}_{x-y}+K_{\beta}\|x-y\|_{\alpha}\right)\Psi(\psi(\|x\|_{\beta}))+\left(KH^{\alpha}_{y}+K_{\beta}\|y\|_{\alpha}\right)\gamma_f(\varepsilon)\\ &\quad +K|x(0)-y(0)|\Psi(\psi(\|x\|_{\beta}))+K|y(0)|\gamma_f(\varepsilon)\\ &=(K+K_{\beta})\,\Psi(\psi(\|x\|_{\beta}))\|x-y\|_{\alpha}+(K+K_{\beta})\,\|y\|_{\alpha}\gamma_f(\varepsilon)\\ &\leq (K+K_{\beta})\,\Psi(\psi(r_0))\varepsilon+(K+K_{\beta})\,r_0\gamma_f(\varepsilon)\to 0 \text{ as } \varepsilon\to 0,\end{aligned}$$

where, we used the fact that $\gamma_f(\varepsilon)\to 0$ since the function f is uniformly continuous on the set $[0,1]\times[0,\psi(r_0)]$. Therefore, $\mathfrak{T}$ is continuous on $B^{\beta}_{r_0}$.

Apply the Schauder fixed point theorem (recall $B^{\beta}_{r_0}$ is compact in $H_{\alpha}[0,1]$) to obtain the desired result. □

4. Example

Here we illustrate our theory with an example.

Example 4.1. Consider the quadratic integral equation

$$x(t)=\sqrt[8]{m\cos^2 t+n}+x(t)\int_0^1 \sqrt[6]{l\sin^2 t+\tau}\ \arctan\left(\frac{\tau^2x(\tau)}{1+\tau^2}\right)^{\frac{1}{3}}\,d\tau,\ t\in[0,1],\quad (6)$$

where, l, m and n are nonnegative constants.

Note that (6) is a special case of (1), where $p(t)=\sqrt[8]{m\cos^2 t+n}$, $k(t,\tau)=\sqrt[6]{l\sin t^2+\tau}$, $f(\tau,y)=\arctan(\tau y)^{\frac{1}{3}}$ and $\Lambda x=\frac{\tau x}{1+\tau^2}$.

One can easily check that:

$$\begin{aligned}|p(t)-p(s)|&=\left|\sqrt[8]{\left(\sqrt{m}\cos t\right)^2+n}-\sqrt[8]{\left(\sqrt{m}\cos s\right)^2+n}\right|\le\sqrt[8]{|\sqrt{m}\cos t-\sqrt{m}\cos s|^2}\\&=\sqrt[8]{m|\cos t-\cos s|^2}=\sqrt[8]{m}\sqrt[4]{|\cos t-\cos s|}=\sqrt[8]{m}\ |\cos t-\cos s|^{\frac{1}{4}}\\&\le\sqrt[8]{m}\ |t-s|^{\frac{1}{8}}\ |t-s|^{\frac{1}{8}}\le\sqrt[8]{m}\ |t-s|^{\frac{1}{8}},\end{aligned}$$

for $t,s\in[0,1]$, where we use [3, Theorem 2.1]. Thus $p\in H_{\frac{1}{8}}[0,1]$ and $H_p^{\frac{1}{8}}=\sqrt[8]{m}$. Therefore, the assumption (a1) of Theorem 3.1 is satisfied with $0<\alpha<\beta=\frac{1}{8}$ and $\|p\|_{\frac{1}{8}}=|p(0)|+H_p^{\frac{1}{8}}=\sqrt[8]{m+n}+\sqrt[8]{m}$. Moreover, we have

$$\begin{aligned}|k(t,\tau)-k(s,\tau)|&=\left|\sqrt[6]{l\sin t^2+\tau}-\sqrt[6]{l\sin s^2+\tau}\right|\le\sqrt[6]{|l\sin t^2-l\sin s^2|}\le\sqrt[6]{l}\ \sqrt[6]{|t^2-s^2|}\\&=\sqrt[6]{l}\ \sqrt[6]{t+s}\ \sqrt[6]{|t-s|}\le\sqrt[6]{l}\ \sqrt[6]{2}\ |t-s|^{\frac{1}{6}}=\sqrt[6]{2l}\ |t-s|^{\frac{1}{8}}\ |t-s|^{\frac{1}{24}}\le\sqrt[6]{2l}\ |t-s|^{\frac{1}{8}},\end{aligned}$$

where the inequality $\left|\sqrt[6]{l\sin t^2+\tau}-\sqrt[6]{l\sin s^2+\tau}\right|\le\sqrt[6]{|l\sin t^2-l\sin s^2|}$ follows from [3, Theorem 2.1]. Therefore, the assumption (a2) of Theorem 3.1 is satisfied with $\kappa_\beta=\kappa_{\frac{1}{8}}=\sqrt[6]{2l}$.

Now, since $|f(\tau,x)|=\left|\arctan(\tau x)^{\frac{1}{3}}\right|\le|\tau x|^{\frac{1}{3}}\le|x|^{\frac{1}{3}}$, then $f(\tau,x)=\arctan(\tau x)^{\frac{1}{3}}$, satisfies the assumption (a3) of Theorem 3.1 with a nondecreasing function $\Psi(r)=r^{\frac{1}{3}}$.

Also, we have $\|\Lambda x\|_\infty\le\sup_{\tau\in[0,1]}\frac{\tau|x(\tau)|}{1+\tau^2}\le\frac{1}{2}\|x\|_\infty\le\frac{1}{2}\|x\|_\beta$, so the assumption (a4) is satisfied with $\psi(t)=\frac{1}{2}t$.

Next, we will show that the operator $\Lambda:H_\beta[0,1]\to C[0,1]$ is continuous with respect to the norm $\|\cdot\|_\alpha$. Take $x,y\in H_\beta[0,1]$ and $\tau\in[0,1]$, and we have

$$\left|\frac{\tau x(\tau)}{1+\tau^2}-\frac{\tau y(\tau)}{1+\tau^2}\right|=\frac{\tau}{1+\tau^2}|x(\tau)-y(\tau)|\le\frac{1}{2}|x(\tau)-y(\tau)|\le\frac{1}{2}\|x-y\|_\infty\le\frac{1}{2}\|x-y\|_\alpha.$$

Then $\|\Lambda x-\Lambda y\|_\infty\le\frac{1}{2}\|x-y\|_\alpha$.

Note that the constant K satisfies

$$K=\sup\left\{\int_0^1|k(t,\tau)|\,d\tau:t\in[0,1]\right\}=\sup\left\{\int_0^1\sqrt[6]{l\sin t^2+\tau}\,d\tau:t\in[0,1]\right\}$$

$$= \sup \left\{ \frac{6}{7} \left(\sqrt[6]{(l \sin t^2 + 1)^7} - \sqrt[6]{l^7 \sin^7 t^2} \right) : t \in [0,1] \right\} \leq \frac{6}{7} \left(\sqrt[6]{(l+1)^7} - \sqrt[6]{l^7} \right),$$

so for the inequality appearing in the assumption (a5), we could consider the inequality

$$\sqrt[8]{m+n} + \sqrt[8]{m} + \left(\frac{6}{7} \left(\sqrt[6]{(l+1)^7} - \sqrt[6]{l^7} \right) + \sqrt[6]{2l} \right) \sqrt[3]{\frac{r}{2}}\, r \leq r. \tag{7}$$

Choosing suitable values for the constants m, n and l, one can find a positive solution of inequality (7) and then all the assumptions of Theorem 3.1 will be satisfied and (6) will have at least one solution $x \in H_\alpha[0,1]$, where $0 < \alpha < \frac{1}{8}$.

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Department of Mathematics, Faculty of Sciences, Damanhour University, Egypt

E-mail: dr.madarwish@gmail.com

Department of Mathematics, Faculty of Sciences, Damanhour University, Egypt

E-mail: m.metwali@yahoo.com

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

E-mail: donal.oregan@nuigalway.ie