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FIRST AND SECOND ORDER NONCONVEX SWEEPING PROCESS WITH PERTURBATION

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Abstract. We prove two existence results for functional differential inclusions governed by sweeping process. We consider the class of subsmooth moving sets. The perturbations depend on all the variables and their values are nonconvex.

1. Introduction

Let H be a separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. Let I be a closed bounded interval in $\mathbb{R}$. We denote by $\mathcal{C}(I, H)$ the Banach space of continuous functions from I to H equipped with the norm $\|x(\cdot)\|_\infty := \sup \big\{ \|x(t)\|;\ t \in I \big\}$. For a positive number a, we put $\mathcal{C}_a := \mathcal{C}([-a, 0], H)$ and for any $t \in [0, \tau]$, we define the operator $T(t)$ from $\mathcal{C}([-a, \tau], H)$ to $\mathcal{C}_a$ by $T(t)(x(\cdot))(s) := T(t)x(s) := x(t+s)$, for all $s \in [-a, 0]$. For $\varphi \in \mathcal{C}_a$ and $r > 0$, let $B_a(\varphi, r) := \big\{ \psi \in H; \|\psi - \varphi\| < r \big\}$ be the open ball centered at φ with radius r and $\overline{B}_a(\varphi, r)$ be its closure. For $\varphi \in \mathcal{C}_a$, we denote $\bar{\varphi}$ the function defined by

$$\bar{\varphi}(t) = \int_{-a}^{t} \varphi(s) ds, \quad \forall t \in [-a, 0].$$

In this paper, we present some existence results for the following functional differential inclusions governed by nonconvex sweeping process of first and second order:

$$\begin{cases} \dot{x}(t) \in -N_{C(t,x(t))}(x(t)) + G(t, T(t)x), & \text{a.e. on } [0, \tau], \\ x(t) = \varphi(t), \quad \forall t \in [-a, 0], \\ x(t) \in C(t, x(t)), \quad \forall t \in [0, \tau], \end{cases} \tag{1}$$

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and

$$\begin{cases} \ddot{x}(t) \in -N_{C(t,x(t),\dot{x}(t))}(\dot{x}(t)) + F(t, T(t)x, T(t)\dot{x}), & \text{a.e. on } [0,\tau], \\ \dot{x}(t) = \varphi(t), \quad \forall t \in [-a, 0], \\ \dot{x}(t) \in C(t, x(t), \dot{x}(t)), & \text{a.e. on } [0,\tau], \\ x(t) = \bar{\varphi}(t), \quad \forall t \in [-a, 0], \end{cases} \tag{2}$$

where $a, \tau > 0$, C, F and G are three multifunctions, φ and $\bar{\varphi}$ are two functions and $N_{C(t,x(t))}(x(t))$ (resp. $N_{C(t,x(t),\dot{x}(t))}(\dot{x}(t))$) denotes the Clarke normal cone to $C(t, x(t))$ (resp. $C(t, x(t), \dot{x}(t))$) at $x(t)$ (resp. $\dot{x}(t)$).

The evolution problems (1) and (2) are generally called the sweeping process which is related to the modelization of elasto-plastic materials (see for example [14, 15]). It has been introduced and studied by Moreau [13], in the setting where all sets $C(t)$ are assumed to be convex. Several authors have studied this problem in the case where the values of C are convex, see for example [4].

The previous results have been generalized in different papers which have treated the general case where the values of C are uniformly ρ-prox-regular, see [2, 5, 12] and the references therein. Haddad, Noel and Thibault [11] have considered the problem (1), without delay, when the values of G are convex and the sets $C(t, x)$ are ball-compact and subsmooth. Noel [17] has studied the problem 2 under the last hypotheses of the last paper. It is necessary here to notice that the class of subsmooth sets strictly contains the class of closed convex sets and the class of prox-regular sets. Recently, Aissous, Nacry and Nguyen [1] have studied the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N_{C(t,x(t))\times Q}(x(t)) + F(t, x(t)) \times \{f(t, x_1(t))\}, & \text{a.e. on } [T_0, T], \\ x(t) \in C(t, x(t)) \times Q, & \text{on } [T_0, T], \\ x(T_0) = (u_0, q_0), \end{cases}$$

where $x = (x_1, x_2) : [T_0, T] \to H^2$, the values of F are convex, f is a single-valued map, Q is a convex set of H and the sets $C(t, x_1, x_2)$ are subsmooth. The authors have used the existence of solution for the last problem, to prove the existence of solution for first and second order sweeping process.

Our main purpose in this work, is to prove the existence results for (1) and (2) when the values of G and F are nonconvex and the sets $C(t, x)$ and $C(t, x, y)$ are ball-compact and subsmooth.

The paper is organized as follows. In Section 2, we recall some important notions and introduce notations that will be used throughout the paper. The next section is devoted to Problem (1). In Section 4, we treat the existence of solutions for (2).

2. Preliminaries

Throughout the paper, B is the open unit ball of H and $B(x, r)$ (resp. $\overline{B}(x, r)$) is the open (resp. closed) ball with center $x \in H$ and radius $r > 0$. Let S be a nonempty subset of H. For an element $x \in H$, $d(x,S)$ or $d_S(x) := \inf\{\|y-x\| \mid y \in S\}$ is the distance of x from the set S. The support function of S is defined, for any $v \in H$, by $\sigma(v,S) := \sup_{s\in S}\langle v,s\rangle$ and the projection set of x into S is the set $\mathrm{Proj}_S(x) := \big\{y \in S \mid d_S(x) = \|x-y\|\big\}$. Recall that a subset S of $(H,\|\cdot\|)$ is ball-compact provided that $S \cap rB$ is compact in $(H,\|\cdot\|)$ for every real $r>0$. Note here that $\mathrm{Proj}_S(x)$ is nonempty when S is nonempty and ball-compact.

Now, we shortly review the definitions of the various notions used in this paper (see [8, 9] as general references). Let $V : H \to \mathbb{R}$ be a Lipschitz function around x. The upper generalized Clarke directional derivative $V^o(x,\cdot)$ is

$$V^o(x,v) := \limsup_{h\to 0^+ y\to x} \frac{V(y+hv)-V(y)}{h}.$$

The Clarke subdifferential of V at x is defined by

$$\partial V(x) := \big\{y \in H : \langle y,v\rangle \le V^o(x,v), \text{ for all } v \in H\big\}.$$

Note that $\partial V(x)$ is convex and closed. A vector $h \in H$ belongs to the Clarke tangent cone $T(S,x)$ when for every sequence $(x_n)_n$ in S converging to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $(x_n + t_n h_n) \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex, and its negative polar $N(S,x)$ is the Clarke normal cone to S at $x \in S$, that is, $N(S,x) = \big\{v \in H : \langle v,h\rangle \le 0, \quad \forall h \in T(S;x)\big\}$.

Next, we introduce a new class of sets via the concept of subsmoothness of the Hilbert space H (see [3]). Let S be a closed subset of H. We say that S is subsmooth at $x_0 \in S$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \xi_1 - \xi_2, x_1 - x_2\rangle \ge -\varepsilon\|x_1 - x_2\|, \tag{3}$$

whenever $x_1, x_2 \in B(x_0,\delta) \cap S$ and $\xi_i \in N(S,x_i) \cap B$, $i \in \{1,2\}$. The set S is subsmooth, if it is subsmooth at each point of S. We further say that S is uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$, such that (3) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N(S,x_i) \cap B$, $i \in \{1,2\}$. It is clear that an uniformly subsmooth set is subsmooth.

Definition 2.1. Let $\big(S(q)\big)_{q\in Q}$ be a family of closed sets of H with parameter $q \in Q$. This family is called equi-uniformly subsmooth, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $q \in Q$, the inequality (3) holds for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N(S(q),x_i) \cap B$, $i \in \{1,2\}$.

The following propositions summarize some important consequences of equi-uniformly subsmooth family needed in the sequel (see [11]).

Proposition 2.2. *Let $\{C(t,x) : (t,x) \in [0,\tau] \times H\}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let $\eta \ge 0$. Assume that there exist $L, L' \ge 0$ and a continuous function $v : [0,\tau] \to \mathbb{R}$ such that for any $x, x', y, y' \in H$ and $s,t \in [0,\tau]$: $|d(y,C(t,x)) - d(y',C(s,x'))| \le |v(t)-v(s)| + L'\|y-y'\| + L\|x-x'\|$. Then the following assertions hold:*
(a) for all $(t,x) \in [0,\tau] \times H$ and $y \in C(t,x)$, we have $\eta\partial d_{C(t,x)}(y) \subset \eta B$,

(b) for each sequence $(t_n)_n$ in $[0,\tau]$ converging to t, any sequence $(x_n)_n$ converging to x, any sequence $(y_n)_n$ converging to $y \in C(t,x)$ with $y_n \in C(t_n,x_n)$, and any $\xi \in H$, we have $\lim_{n\to+\infty} \sup \sigma(\xi, \eta\partial d_{C(t_n,x_n)}(y_n)) \le \sigma\big(\xi, \eta\partial d_{C(t,x)}(y)\big)$.

The following results will be needed. For the proof see [3, 6, 16].

PROPOSITION 2.3. *Let S be a nonempty subset of H and $x \in H$. If $y \in \mathrm{Proj}_S(x)$, then $x - y \in N_S(y)$.*

PROPOSITION 2.4. *Let S be a closed subset of H and $v_0 \in S$. If S is subsmooth at v_0, then $\partial d_S(v_0) = N_S(v_0) \cap B$.*

Next, for nonempty subsets A, B of H, we denote $e(A,B) := \sup\big\{d_B(x); x \in A\big\}$ and $H(A,B) = \max\big\{e(A,B), e(B,A)\big\}$. A multifunction is said to be measurable if its graph is measurable. For more details on measurability theory, we refer the reader to [7]. Let us recall the following lemmas that will be used in the sequel.

LEMMA 2.5 ([18]). *Let Ω be a nonempty set in H. Assume that $F : [a,b] \times \Omega \to 2^H$ is a multifunction with nonempty closed values satisfying:*
(a) for every $x \in \Omega$, $F(\cdot,x)$ is measurable on $[a,b]$;

(b) for every $t \in [a,b]$, $F(t,\cdot)$ is (Hausdorff) continuous on Ω.
Then for any measurable function $x(\cdot) : [a,b] \to \Omega$, the multifunction $F(\cdot,x(\cdot))$ is measurable on $[a,b]$.

LEMMA 2.6 ([18]). *Let $G : [a,b] \to 2^H$ be a measurable multifunction and $y(\cdot) : [a,b] \to H$ a measurable function. Then for any positive measurable function $r(\cdot) : [a,b] \to \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for almost all $t \in [a,b]$ $\|g(t) - y(t)\| \le d\big(y(t), G(t)\big) + r(t)$.*

If D is a bounded subset of H, then the Kuratowski's measure of noncompactness of D is defined by $\beta(D) = \inf\big\{d > 0 : D$ admits a finite number of sets with diameter less than $d\big\}$.

In the following lemma, we recall some useful properties for the measure of noncompactness β.

LEMMA 2.7 ([10]). *Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X. Then*
(i) $\beta(D_1) = 0 \Leftrightarrow D_1$ is relatively compact, *(ii) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$; $\lambda \in \mathbb{R}$,*

(iii) $D_1 \subseteq D_2 \Rightarrow \beta(D_1) \le \beta(D_2)$, *(iv) $\beta(D_1 + D_2) \le \beta(D_1) + \beta(D_2)$,*

(v) if $x_0 \in X$ and r is a positive real number then $\beta(B(x_0,r)) = 2r$.

3. First order perturbed nonconvex sweeping process with delay

In this section, we study the existence of solutions of (1). First, let us make the following assumptions:

(H1) $C : [0,b] \times H \to 2^H$ is a set-valued map with nonempty and closed values satisfying:

(a) the family $\{C(t,x) : (t,x) \in [0,b] \times H\}$ is equi-uniformly subsmooth,

(b) for any $I \times A \subset [0,b] \times H$, the set $C(I \times A)$ is ball-compact in H,

(c) there exist an absolutely continuous function $v : [0,b] \to \mathbb{R}$ and $K \in \mathbb{R}^+$ such that $\big|d(x, C(t,x')) - d(y, C(s,y'))\big| \leq |v(t) - v(s)| + K\|x - y\|$, for all $x, y, x', y' \in H$ and $s, t \in [0,b]$;

(H2) $G : [0,b] \times \mathcal{C}_a \to 2^H$ is a set-valued map with nonempty closed values satisfying

(i) for each $\psi \in \mathcal{C}_a$, $t \mapsto G(t,\psi)$ is measurable,

(ii) there exists a function $m(\cdot) \in L^1([0,b], \mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $\psi_1, \psi_2 \in \mathcal{C}_a$: $H\big(G(t,\psi_1), G(t,\psi_2)\big) \leq m(t)\|\psi_1 - \psi_2\|_\infty$,

(iii) for all $\varphi \in \mathcal{C}_a$, there exist $r>0$ and functions $q(\cdot)$, $p(\cdot) \in L^1([0,b], \mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $\psi \in \overline{B}_a(\varphi, r)$: $\|G(t,\psi)\| := \sup\limits_{y \in G(t,\psi)} \|y\| \leq q(t) + p(t)\|\psi\|_\infty$.

Now, we are able to state our first existence result for (1).

Theorem 3.1. *If assumptions (H1) and (H2) are satisfied, then for all $\varphi \in \mathcal{C}_a$ such that $\varphi(0) \in C(0, \varphi(0))$, there exist $\tau > 0$ and a continuous function $x(\cdot) : [-a, \tau] \to H$, that is absolutely continuous on $[0,\tau]$ such that $x(\cdot)$ is a solution of (1).*

Proof. Fix $\varphi \in \mathcal{C}_a$ such that $\varphi(0) \in C(0,\varphi(0))$. There exist $r > 0$ and $q(\cdot), p(\cdot) \in L^1([0,b], \mathbb{R}^+)$ such that $\|G(t,\psi)\| \leq q(t) + p(t)\|\psi\|_\infty$, $\forall (t,\psi) \in [0,b] \times \overline{B}_a(\varphi, r)$.

Let $\tau_1 > 0$ be such that

$$\int_0^{\tau_1} \big[|\dot{v}(t)| + (K+1)\big(q(t) + p(t)(\|\varphi\|_\infty + r)\big)\big]dt < \frac{r}{2}.$$

For $\varepsilon > 0$ set

$$\eta_1(\varepsilon) = \sup\left\{\gamma \in]0,\varepsilon] : \left|\int_{t_1}^{t_2} \big[|\dot{v}(s)| + (K+1)\big(q(s) + p(s)(\|\varphi\|_\infty + r)\big)\big]ds\right| < \varepsilon \right.$$
$$\left. \text{if } |t_1 - t_2| < \gamma \right\}$$

and $\quad \eta_2(\varepsilon) = \sup\big\{\gamma \in]0,\varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \text{ if } |t_1 - t_2| < \gamma\big\}$.

Put $\eta(\varepsilon) = \min\{\eta_1(\varepsilon), \eta_2(\varepsilon)\}$ and $\tau = \min\{\frac{1}{2}\eta(\frac{r}{2}), \tau_1, b\}$. We will use the following lemma to prove the main result of this section.

Lemma 3.2. *If assumptions (H1) and (H2) are satisfied, then for all $n \in \mathbb{N}^*$ and for all measurable function $y(\cdot) : [0,\tau] \to H$, there exist a continuous mapping $x_n(\cdot) : [-a,\tau] \to H$, step functions $\theta_n(\cdot), \overline{\theta}_n(\cdot) : [0,\tau] \to [0,\tau]$ and $g_n(\cdot) \in L^1([0,\tau], H)$ such that*

$$x_n(\cdot) = \varphi \text{ on } [-a,0],\ x_n(\overline{\theta}_n(t)) \in C(\overline{\theta}_n(t), x_n(\theta_n(t))),\ \forall t \in [0,\tau],$$
$$x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2}),\ T(\theta_n(t))x_n \in \overline{B}_a(\varphi, r),\ \forall t \in [0,\tau],$$
$$g_n(t) \in G(t, T(\theta_n(t))x_n),\ \forall t \in [0,\tau],$$

$$\|g_n(t) - y(t)\| \le d\big(y(t), G(t, T(\theta_n(t))x_n)\big) + \frac{1}{n}, \quad \textit{for almost all } t \in [0, \tau],$$
$$\dot{x}_n(t) - g_n(t) \in -N_{C(\bar{\theta}_n(t), x_n(\theta_n(t)))}\big(x_n(\bar{\theta}_n(t))\big), \quad \textit{a.e. on } [0, \tau].$$

Proof. Fix $n \in \mathbb{N}^*$ and let $y(\cdot) : [0, \tau] \to H$ be a measurable function. Consider a sequence $(P_n)_n$ of subdivisions of $[0, \tau] : P_n = \big\{t_0^n{=}0{<}t_1^n{<}\ldots{<}t_i^n{<}\ldots{<}t_{2^n}^n{=}\tau\big\}$, where $t_i^n = i\frac{\tau}{2^n}$, $0 < i < 2^n$. Let us define a sequence $(x_n)_n$ of approximate solutions as follows. Set $x_n(s) = \varphi(s)$ for all $s \in [-a, 0]$. Put $x_0^n = \varphi(0) \in C(t_0^n, x_0^n)$. The set-valued map $t \mapsto G(t, T(t_0^n)x_n)$ is measurable, then, in view of Lemma 2.6, there exists a function $g_0^n \in L^1([0, t_1^n], H)$ such that $g_0^n(t) \in G(t, T(t_0^n)x_n)$, for all $t \in [0, t_1^n]$, and $\|g_0^n(t) - y(t)\| \le d\big(y(t), G(t, T(t_0^n)x_n)\big) + \frac{1}{n}$, for almost all $t \in [0, t_1^n]$. The ball-compactness of $C(t_1^n, x_0^n)$ ensures that $\mathrm{Proj}_{C(t_1^n, x_0^n)}\big(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)ds\big) \ne \emptyset$. Then, we can choose a point $x_1^n \in \mathrm{Proj}_{C(t_1^n, x_0^n)}\big(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)ds\big)$. Hence, we have $x_1^n \in C(t_1^n, x_0^n)$ and by (H1) we deduce

$$\left\|x_1^n - \left(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)ds\right)\right\| = d_{C(t_1^n, x_0^n)}\left(x_0^n + \int_{t_0^n}^{t_1^n} g_0^n(s)ds\right)$$
$$\le \int_{t_0^n}^{t_1^n} \big[|\dot{v}(s)| + K\big(q(s) + p(s)(r + \|\varphi\|_\infty)\big)\big]ds.$$

Now, set $x_n(t){=}x_0^n{+}\frac{\alpha(t)-\alpha(t_0^n)}{\alpha(t_1^n)-\alpha(t_0^n)}\big(x_1^n{-}x_0^n{-}\int_{t_0^n}^{t_1^n} g_0^n(s)ds\big){+}\int_{t_0^n}^{t} g_0^n(s)\,ds$, for all $t{\in}[t_0^n, t_1^n]$, where $\alpha(t) = \int_0^t \big[|\dot{v}(s)|{+}K\big(q(s){+}p(s)(r{+}\|\varphi\|_\infty)\big)\big]ds$, $\forall t \in [0, \tau]$. So for all $t \in [t_0^n, t_1^n]$

$$\|x_n(t) - \varphi(0)\| \le \alpha(t) - \alpha(t_0^n) + \int_{t_0^n}^{t}(q(s) + p(s)\|\varphi\|_\infty)ds$$
$$\le \int_{t_0^n}^{t} \big[|\dot{v}(s)| + (K+1)\big(q(s) + p(s)(r + \|\varphi\|_\infty)\big)\big]ds$$

which is equivalent to $x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ for all $t \in [t_0^n, t_1^n]$. Now, we have to estimate $\|(T(t_1^n)x_n)(s) - \varphi(s)\|$ for each $s \in [-a, 0]$. If $-t_1^n \le s \le 0$, then $(t_1^n + s) \in [0, t_1^n]$. Thus, by the fact that $|s| \le t_1^n \le \tau < \eta(\frac{r}{2})$, we have

$$\|(T(t_1^n)x_n)(s){-}\varphi(s)\| = \|x_n(t_1^n{+}s){-}\varphi(s)\| \le \|x_n(t_1^n{+}s){-}\varphi(0)\|{+}\|\varphi(s){-}\varphi(0)\| \le r.$$

Therefore, $T(t_1^n)x_n \in \overline{B}_a(\varphi, r)$. Next, we reiterate this process for constructing the sequences $(g_i^n(\cdot))_i$ and $(x_i^n)_i$ and the function $x_n(\cdot)$ satisfying, for all $0 \le i \le 2^n - 1$ and for all $t \in [t_i^n, t_{i+1}^n]$, the following assertions:

$$g_i^n(t) \in G(t, T(t_i^n)x_n), \tag{4}$$
$$x_0^n \in C(t_0^n, x_0^n),\ x_{i+1}^n \in C(t_{i+1}^n, x_i^n),$$
$$\|x_n(t) - \varphi(0)\| \le \int_{t_0^n}^{t} \big[|\dot{v}(s)| + (K+1)\big(q(s) + p(s)(r + \|\varphi\|_\infty)\big)\big]ds,$$
$$x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2}),\ T(t_i^n)x_n \in \overline{B}_a(\varphi, r),$$
$$\|g_i^n(t) - y(t)\| \le d(y(t), G(t, T(t_i^n)x_n)) + \frac{1}{n},\ \text{a.e.} \tag{5}$$

$$x_{i+1}^n \in \mathrm{Proj}_{C(t_{i+1}^n, x_i^n)}\left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right), \tag{6}$$

$$\left\| x_{i+1}^n - \left(x_i^n + \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right)\right\| \leq \int_{t_i^n}^{t_{i+1}^n} \big[|\dot{v}(s)| + K\big(q(s) + p(s)(r + \|\varphi\|_\infty)\big)\big]ds,$$

$$x_n(t) = x_i^n + \frac{\alpha(t) - \alpha(t_i^n)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right) + \int_{t_i^n}^{t} g_i^n(s)ds.$$

Now, we define the functions $\theta_n(\cdot), \bar{\theta}_n(\cdot) : [0,\tau] \to [0,\tau]$ and $g_n(\cdot) \in L^1([0,\tau], H)$ by setting for all $t \in [t_i^n, t_{i+1}^n[$: $\bar{\theta}_n(t) = t_{i+1}^n$, $\theta_n(t) = t_i^n$ and $g_n(t) = g_i^n(t)$, and $\bar{\theta}(\tau) = \tau$, $\theta_n(\tau) = t_{2^n-1}^n$ and $g_n(\tau) = g_{2^n-1}^n(\tau)$. We claim that $x_n(\cdot)$ is absolutely continuous. Indeed, for all $0 \leq i \leq 2^n - 1$ and for all t and s in $[t_i^n, t_{i+1}^n]$, $s < t$, one has

$$x_n(t) - x_n(s) = \frac{\alpha(t) - \alpha(s)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(\tau)d\tau\right) + \int_s^t g_i^n(\tau)d\tau.$$

Then, by (4) and (6), we get

$$\begin{aligned}\|x_n(t) - x_n(s)\| &\leq \frac{\alpha(t) - \alpha(s)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left\|x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(\tau)d\tau\right\| + \int_s^t \|g_i^n(\tau)\|\,d\tau \\ &\leq \int_s^t \big[|\dot{v}(\tau)| + (K+1)\big(q(\tau) + p(\tau)(\|\varphi\|_\infty + r)\big)\big]d\tau. \end{aligned} \tag{7}$$

By addition this last inequality holds for all $s, t \in [0,\tau]$ with $s < t$. Hence $x_n(\cdot)$ is absolutely continuous. Remark that for all $0 \leq i \leq 2^n - 1$ and for almost every t in $[t_i^n, t_{i+1}^n]$,

$$\dot{x}_n(t) = \frac{\dot{\alpha}(t)}{\alpha(t_{i+1}^n) - \alpha(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} g_i^n(s)ds\right) + g_n(t).$$

Then, we obtain for almost every $t \in [0,\tau]$

$$\|\dot{x}_n(t) - g_n(t)\| \leq |\dot{v}(t)| + K\big(q(t) + p(t)(\|\varphi\|_\infty + r)\big).$$

Observe that by construction, we have $g_n(t) \in G\big(t, T(\theta_n(t))x_n\big)$ and

$$\|g_n(t) - y(t)\| \leq d\big(y(t), G(t, T(\theta_n(t))x_n)\big) + \frac{1}{n}.$$

Also, by construction and the relation (6), we have for almost every $t \in [0,\tau]$

$$\dot{x}_n(t) - g_n(t) \in -N_{C(\bar{\theta}_n(t), x_n(\theta_n(t)))}(x(\bar{\theta}_n(t))). \qquad \square$$

Now, we are ready to prove Theorem 3.1. The proof will be given in several steps. First, by Lemma 3.2, we can define inductively sequences $(g_n(\cdot))_{n\geq 1}, (x_n(\cdot))_{n\geq 1} \subset \mathcal{C}([-a,\tau], H)$ and $(\theta_n(\cdot))_{n\geq 1}, (\bar{\theta}_n(\cdot))_{n\geq 1} \subset S([0,\tau],[0,\tau])$, where $S([0,\tau],[0,\tau])$ denotes the space of step functions from $[0,\tau]$ into $[0,\tau]$, such that

$$\begin{aligned} &x_n(\cdot) = \varphi \text{ on } [-a,0],\ x_n(\overline{\theta}_n(t)) \in C(\overline{\theta}_n(t), x_n(\theta_n(t))),\ \forall t \in [0,\tau], \\ &x_n(t) \in \overline{B}(\varphi(0), \frac{r}{2}),\ T(\theta_n(t))x_n \in \overline{B}_a(\varphi, r),\ \forall t \in [0,\tau], \\ &g_n(t) \in G(t, T(\theta_n(t))x_n),\ \forall t \in [0,\tau], \end{aligned}$$

$$\|g_{n+1}(t) - g_n(t)\| \le d\big(g_n(t), G(t, T(\theta_{n+1}(t))x_{n+1})\big) + \frac{1}{n+1}, \quad \text{a.e. on } [0,\tau],$$
$$\dot{x}_n(t) - g_n(t) \in -N_{C(\overline{\theta}_n(t), x_n(\theta_n(t)))}\big(x_n(\bar{\theta}_n(t))\big), \quad \text{a.e. on } [0,\tau].$$

Step 1. *Convergence of $x_n(\cdot)$ to some absolutely continuous mapping $x(\cdot)$.* Fix any $t \in [0,\tau]$. We have $x_n(\overline{\theta}_n(t)) \in C\big(\overline{\theta}_n(t), x_n(\theta_n(t))\big)$, $\|x_n(\overline{\theta}_n(t))\| < \|\varphi(0)\| + 2r = \lambda$ and $\|x_n(\theta_n(t))\| < \lambda$, which implies that $x_n(\overline{\theta}_n(t)) \in C\big([0,\tau] \times \lambda B\big) \cap \lambda B$. Then $\{x_n(\overline{\theta}_n(t)) : n \ge 1\}$ is relatively compact in H, in view of (H1). Now, for all $t \in [0,\tau]$, $\beta\big\{x_n(t) : n \ge 1\big\} = \beta\big\{x_n(t) - x_n(\overline{\theta}_n(t)) + x_n(\overline{\theta}_n(t)) : n \ge 1\big\}$. From Lemma 2.7 (iv), we get $\beta\big\{x_n(t) : n \ge 1\big\} \le \beta\big\{x_n(t) - x_n(\overline{\theta}_n(t)) : n \ge 1\big\} + \beta\big\{x_n(\overline{\theta}_n(t)) : n \ge 1\big\}$. Since the set $\{x_n(\overline{\theta}_n(t)) : n \ge 1\}$ is relatively compact in H, by Lemma 2.7 (i), $\beta\big\{x_n(\overline{\theta}_n(t)) : n \ge 1\big\} = 0$. Then $\beta\big\{x_n(t) : n \ge 1\big\} \le \beta\big\{x_n(t) - x_n(\overline{\theta}_n(t)) : n \ge 1\big\} = \beta\big\{\int_{\overline{\theta}_n(t)}^{t} \dot{x}_n(s)ds : n \ge 1\big\}$. By Lemma 2.7 (v), we obtain

$$\begin{aligned}\beta\big\{x_n(t) : n \ge 1\big\} \le& \beta\left\{B\left(0, \int_t^{\overline{\theta}_n(t)} \big(|\dot{v}(s)| + (K+1)(q(s) + p(s)(\|\varphi\|_\infty + r))\big)ds\right)\right\} \\ =& 2\int_t^{\overline{\theta}_n(t)} \big(|\dot{v}(s)| + (K+1)(q(s) + p(s)(\|\varphi\|_\infty + r))\big)ds.\end{aligned}$$

Since the right term of the above inequality converges to 0 as $n \to \infty$, $\beta\big\{x_n(t) : n \ge 1\big\} = 0$. Hence $\big\{x_n(t) : n \ge 1\big\}$ is relatively compact in H. Moreover, since $\|\dot{x}_n(t)\| \le |\dot{v}(t)| + (K+1)\big(q(t) + p(t)(\|\varphi\|_\infty + r)\big)$ for almost every $t \in [0,\tau]$, by Arzelà-Ascoli's Theorem, we can select a subsequence, again denoted by $(x_n(\cdot))_n$ which converges uniformly to an absolutely continuous function $x(\cdot)$ on $[0,\tau]$. Moreover $(\dot{x}_n(\cdot))_n$ converges weakly to $\dot{x}(\cdot)$ in $L^1([0,\tau], H)$. Also, since all functions $x_n(\cdot)$ agree with $\varphi(\cdot)$ on $[-a,0]$, we can obviously say that $(x_n(\cdot))_n$ converges uniformly to $x(\cdot)$ on $[-a,0]$, if we extend $x(\cdot)$ in such a way that $x(\cdot) \equiv \varphi(\cdot)$ on $[-a,0]$. Additionally, observe that $(x_n(\overline{\theta}_n(\cdot)))_n$ converges uniformly to $x(\cdot)$ on $[0,\tau]$. Indeed, by (7), we have

$$\begin{aligned}\|x_n(\overline{\theta}_n(t)) - x(t)\| \le& \|x_n(\overline{\theta}_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\ \le& \int_t^{\overline{\theta}_n(t)} \Big(|\dot{v}(s)| + (K+1)\big(q(s) + p(s)\big)(r + \|\varphi\|_\infty)\Big)ds + \|x_n(t) - x(t)\|.\end{aligned}$$

The right term of the above inequality converge to 0. It follows that $(x_n(\overline{\theta}_n(\cdot)))_n$ converges uniformly to $x(\cdot)$ on $[0,\tau]$. Therefore, by (H1) (c), we conclude that $x(t) \in C(t, x(t))$ for all $t \in [0,\tau]$.

Step 2. *$T(\theta_n(t))x_n$ converges to $T(t)x$ in $\mathcal{C}_a$.* For all $t \in [0,\tau]$, one has

$$\begin{aligned}&\|T(\theta_n(t))x_n - T(t)x\|_\infty = \sup_{-a \le s \le 0} \|x_n(\theta_n(t) + s) - x(t+s)\| \\ \le& \sup_{-a \le s \le 0} \|x_n(\theta_n(t) + s) - x(\theta_n(t) + s)\| + \sup_{-a \le s \le 0} \|x(\theta_n(t) + s) - x(t+s)\| \\ \le& \|x_n(\cdot) - x(\cdot)\|_\infty + \sup_{-a \le s \le 0} \|x(\theta_n(t) + s) - x(t+s)\|\end{aligned}$$

Since the right term of the above relation converges to 0, $T(\theta_n(t))x_n$ converges to $T(t)x$ in $\mathcal{C}_a$.

Step 3. *$x(\cdot)$ is a solution of* (1). Let $t \in [0,\tau]$. From (H2) and (5), we have

$$\|g_{n+1}(t) - g_n(t)\| \le H\big(G(t, T(\theta_n(t))x_n), G(t, T(\theta_{n+1}(t))x_{n+1})\big) + \frac{1}{n+1}$$
$$\le m(t)\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty + \frac{1}{n+1}.$$

Since the right term of the above relation converges to 0, $(g_n(t))_{n\ge1}$ is a Cauchy sequence and $(g_n(t))_{n\ge1}$ converges to $g(t)$. Moreover, observe that

$$d\big(g(t), G(t, T(t)x)\big) \le \|g(t) - g_n(t)\| + H\big(G(t, T(\theta_n(t))x_n), G(t, T(t)x)\big)$$
$$\le \|g(t) - g_n(t)\| + m(t)\|T(\theta_n(t))x_n - T(t)x\|_\infty$$

So $g(t) \in G(t, T(t)x)$ for all $t \in [0,\tau]$. Now, the weak convergence of $\dot{x}_n(\cdot)$ to $\dot{x}(\cdot)$ in $L^1([0,\tau], H)$ and the Mazur's lemma entail $\dot{x}(t) - g(t) \in \bigcap_n \bar{co}\big\{\dot{x}_m(t) - g_m(t) : m \ge n\big\}$, for a.e. on $[0,\tau]$.

Fix $t \in [0,\tau] \setminus I$ and $y \in H$, where $[0,\tau] \setminus I$ denote the set on which the above relations hold and I is a subset of $[0,\tau]$ with null Lebesgue-measure. We have $\langle y, \dot{x}(t) - g(t)\rangle \le \inf_n \sup_{k\ge n}\langle y, \dot{x}_k(t) - g_k(t)\rangle$. On the other hand, one has $\big(\dot{x}_n(t) - g_n(t)\big) \in -N_{C(\bar\theta_n(t), x_n(\theta_n(t)))}(x_n(\bar\theta_n(t))) \bigcap \gamma(t)\overline{B}$, where $\gamma(t) = |\dot{v}(t)| + (K+1)\big(q(t) + p(t)(\|\varphi\|_\infty + r)\big)$. Hence, we get $\big(\dot{x}_n(t) - g_n(t)\big) \in -\gamma(t)\partial d_{C(\bar\theta_n(t)), x_n(\theta_n(t)))}(x_n(\bar\theta_n(t)))$. Then, we deduce

$$\langle y, \dot{x}(t) - g(t)\rangle \le \gamma(t) \limsup_{n\to\infty} \sigma\big(y, -\partial d_{C(\bar\theta_n(t)), x_n(\theta_n(t)))}(x_n(\bar\theta_n(t)))\big)$$
$$\le \gamma(t)\sigma\big(y, -\partial d_{C(t,x(t))}(x(t))\big).$$

So, the convexity and the closedness of the set $\partial d_{C(t,x(t))}(x(t))$ ensure $\big(\dot{x}(t) - g(t)\big) \in -\gamma(t)\partial d_{C(t,x(t))}(x(t)) \subset -N_{C(t,x(t))}(x(t))$. Finally, $\dot{x}(t) \in -N_{C(t,x(t))}(x(t)) + G(t, T(t)x)$ for almost all $t \in [0,\tau]$. □

4. Second order sweeping process with nonconvex perturbation

In this section, we prove the existence result of solutions for (2). Assume that the following hypothesis hold:

(H3) $C : [0,b] \times H \times H \to 2^H$ is a set-valued map with nonempty and closed values satisfying:

(a) the family $\{C(t,x,y) : (t,x,y) \in [0,b] \times H \times H\}$ is equi-uniformly subsmooth,

(b) for any $I \times A \times A' \subset [0,b] \times H \times H$, the set $C(I \times A \times A')$ is ball-compact in H,

(c) there exist an absolutely continuous function $v : [0,b] \to \mathbb{R}$ and $K, K' \in \mathbb{R}^+$ such that $\big|d(x, C(t,x',x'')) - d(y, C(s,y',y''))\big| \le |v(t) - v(s)| + K\|x' - y'\| + K'\|x - y\|$, for all $x, y, x', y', x'', y'' \in H$ and $s, t \in [0,b]$,

(H4) $F : [0,b] \times \mathcal{C}_a \times \mathcal{C}_a \to 2^H$ is a set-valued map with nonempty closed values satisfying

(i) For each $\psi, \phi \in \mathcal{C}_a$, $t \mapsto F(t,\psi,\phi)$ is measurable,

(ii) There exists a function $m(\cdot) \in L^1([0,b],\mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{C}_a$, $H\big(F(t,\psi_1,\phi_1), F(t,\psi_2,\phi_2)\big) \leq m(t)(\|\phi_1-\phi_2\|_\infty + \|\psi_1-\psi_2\|_\infty)$,
(iii) For all $\varphi \in \mathcal{C}_a$, there exist $r_1, r_2 > 0$ and three functions $g(\cdot)$, $p(\cdot)$, $q(\cdot) \in L^1([0,b],\mathbb{R}^+)$ such that for all $t \in [0,b]$ and for all $(\psi,\phi) \in B_a(\bar{\varphi}, r_1) \times B_a(\varphi, r_2)$, $\|F(t,\psi,\phi)\| \leq g(t) + p(t)\|\psi\|_\infty + q(t)\|\phi\|_\infty$.

We established the following result.

THEOREM 4.1. *If assumptions (H3) and (H4) are satisfied, then for all* $\varphi \in \mathcal{C}_a$ *such that* $\varphi(0) \in C(0,\bar{\varphi}(0),\varphi(0))$, *there exist* $\tau > 0$ *and a continuously differentiable function* $x(\cdot) : [-a,\tau] \to H$ *such that* $\dot{x}(\cdot)$ *is absolutely continuous on* $[0,\tau]$, *and* $x(\cdot)$ *is a solution of* (2).

Proof. Fix $\varphi \in \mathcal{C}_a$ such that $\varphi(0) \in C(0,\bar{\varphi}(0),\varphi(0))$. There exist $r_1, r_2 > 0$ and $g(\cdot), p(\cdot), q(\cdot) \in L^1([0,b],\mathbb{R}^+)$ such that for all $(t,\psi,\phi) \in [0,b] \times B_a(\bar{\varphi},r_1) \times B_a(\varphi,r_2)$

$$\|F(t,\psi,\phi)\| \leq g(t) + p(t)\|\psi\|_\infty + q(t)\|\phi\|_\infty.$$

Put $r = \min\{\frac{r_1}{2}, \frac{r_2}{2}\}$. For simplification, set $\delta(t) = g(t)+p(t)(\|\bar{\varphi}\|_\infty+r)+q(t)(\|\varphi\|_\infty+r)$, $\forall t \in [0,b]$. Let $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$\int_0^{\tau_1} \big(|\dot{v}(s)| + K\rho(s) + (K'+1)\delta(s)\big)ds < \frac{r}{2} \quad \text{and} \quad \int_0^{\tau_2} \rho(s)ds < \frac{r}{2},$$

where $\rho(s) = \frac{r}{2} + \|\varphi(0)\|$, for all $s \in [0,b]$. For $\varepsilon > 0$ set

$$\eta_1(\varepsilon) = \sup\Big\{\gamma \in]0,\varepsilon] : \Big|\int_{t_1}^{t_2} \big(|\dot{v}(s)| + K\rho(s) + (K'+2)\delta(s)\big)ds\Big| < \varepsilon$$
$$\text{and } \Big|\int_{t_1}^{t_2} \rho(s)ds\Big| < \varepsilon \text{ if } |t_1 - t_2| < \gamma\Big\}$$

and

$$\eta_2(\varepsilon) = \sup\big\{\gamma \in]0,\varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon$$
$$\text{and } \|\bar{\varphi}(t_1) - \bar{\varphi}(t_2)\| < \varepsilon \text{ if } |t_1 - t_2| < \gamma\big\}.$$

Put $\eta(\varepsilon) = \min\{\eta_1(\varepsilon), \eta_2(\varepsilon)\}$ and $\tau = \min\big\{\tau_1, \tau_2, \frac{1}{2}\eta(\frac{r}{2}), b\big\}$.

We will use the following lemma to prove the main result of this section.

LEMMA 4.2. *If assumptions (H3) and (H4) are satisfied, then for all* $n \in \mathbb{N}^*$ *and for all measurable function* $y(\cdot) : [0,\tau] \to H$, *there exist two continuous maps* $x_n(\cdot) : [-a,\tau] \to H$, $u_n(\cdot) : [-a,\tau] \to H$, *step functions* $\theta_n(\cdot), \bar{\theta}_n(\cdot) : [0,\tau] \to [0,\tau]$ *and* $f_n(\cdot) \in L^1([0,\tau],H)$ *such that*
(i) $f_n(t) \in F(t, T(\theta_n(t))x_n, T(\theta_n(t))u_n)$, $u_n(\bar{\theta}_n(t)) \in C(\bar{\theta}_n(t), x_n(\bar{\theta}_n(t)), u_n(\theta_n(t)))$, *for all* $t \in [0,\tau]$;
(ii) $\|f_n(t) - y(t)\| \leq d\big(y(t), F(t, T(\theta_n(t))x_n, T(\theta_n(t))u_n)\big) + \dfrac{1}{n}$ *for almost all* $t \in [0,\tau]$;
(iii) $\big(\dot{u}_n(t) - f_n(t)\big) \in -N_{C(\bar{\theta}_n(t), x_n(\bar{\theta}_n(t)), u_n(\theta_n(t)))}(u_n(\bar{\theta}_n(t)))$ *for almost all* $t \in [0,\tau]$;
(iv) $\|\dot{u}_n(t) - f_n(t)\| \leq |\dot{v}(t)| + K\rho(t) + (K'+1)\delta(t)$ *for almost every* $t \in [0,\tau]$;
(v) $x_n(t) = \bar{\varphi}(0) + \int_0^t u_n(\theta_n(s))ds$ *for every* $t \in [0,\tau]$;

(vi) $x_n(t) \in \overline{B}(\bar{\varphi}(0), \frac{r}{2})$, $u_n(t) \in \overline{B}(\varphi(0), \frac{r}{2})$, $T(\theta_n(t))x_n \in \overline{B}_a(\bar{\varphi}, r)$ *and* $T(\theta_n(t))u_n \in \overline{B}_a(\varphi, r)$ *for every* $t \in [0, \tau]$.

Proof. Fix $n \in \mathbb{N}^*$ and let $y(\cdot) : [0,\tau] \to H$ be a measurable function. Consider a sequence $(P_n)_n$ of subdivisions of $[0,\tau]$: $P = \{0 = t_0^n < t_1^n < \ldots < t_i^n < \ldots < t_{2^n}^n = \tau\}$, where $t_i^n = i\frac{\tau}{2^n}$ for $0 < i < 2^n$. Let us define the functions $x_n(\cdot)$ and $u_n(\cdot)$ as follows. Set $x_n(s) = \bar{\varphi}(s)$ and $u_n(s) = \varphi(s)$ for all $s \in [-a, 0]$. Put $x_0^n = \bar{\varphi}(0)$ and $u_0^n = \varphi(0) \in C(t_0^n, x_0^n, u_0^n)$. By Lemma 2.5, the set-valued map $t \mapsto F\big(t, T(t_0^n)x_n, T(t_0^n)u_n\big)$ is measurable. Hence, there exists $f_0^n(\cdot) \in L^1([t_0^n, t_1^n], H)$ such that $f_0^n(t) \in F(t, T(t_0^n)x_n, T(t_0^n)u_n)$ for all $t \in [t_0^n, t_1^n]$ and $\|f_0^n(t) - y(t)\| \leq d\big(y(t), F(t, T(t_0^n)x_n, T(t_0^n)u_n)\big) + \frac{1}{n}$, for almost all $t \in [t_0^n, t_1^n]$. Set $x_n(t) = x_0^n + (t - t_0^n)u_0^n$, $\forall t \in [t_0^n, t_1^n]$ and put $x_n(t_1^n) = x_1^n$. By the ball compactness of the set $C(t_1^n, x_1^n, u_0^n)$, there exists some point $u_1^n \in \mathrm{Proj}_{C(t_1^n, x_1^n, u_0^n)}\left(u_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)ds\right)$. Note that $u_1^n \in C(t_1^n, x_1^n, u_0^n)$ and

$$\left\|u_1^n - \left(u_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)ds\right)\right\| = d_{C(t_1^n, x_1^n, u_0^n)}\left(u_0^n + \int_{t_0^n}^{t_1^n} f_0^n(s)ds\right)$$

$$\leq \int_{t_0^n}^{t_1^n} \big(|\dot{v}(s)| + K'(g(s) + p(s)\|\bar{\varphi}\|_\infty + q(s)\|\varphi\|_\infty) + K\|\varphi(0)\|\big)ds$$

$$\leq \int_{t_0^n}^{t_1^n} \big(|\dot{v}(s)| + K'(g(s) + p(s)\|\bar{\varphi}\|_\infty + q(s)\|\varphi\|_\infty) + K\rho(s)\big)ds.$$

Now, set $u_n(t) = u_0^n + \dfrac{\alpha(t) - \alpha(t_0^n)}{\alpha(t_1^n) - \alpha(t_0^n)}\left(u_1^n - u_0^n - \displaystyle\int_{t_0^n}^{t_1^n} f_0^n(s)ds\right) + \displaystyle\int_{t_0^n}^{t} f_0^n(s)ds$, $\forall t \in [t_0^n, t_1^n]$,

where $\alpha(t) = \int_0^t (|\dot{v}(s)| + K\rho(s) + K'\delta(s))ds$, $\forall t \in [0, \tau]$. So, for all $t \in [t_0^n, t_1^n]$,

$$\|u_n(t) - \varphi(0)\| \leq \frac{\alpha(t) - \alpha(t_0)}{\alpha(t_1^n) - \alpha(t_0^n)}\left\|u_1^n - u_0^n - \int_{t_0^n}^{t_1^n} f_0^n(s)ds\right\| + \int_{t_0^n}^{t} \|f_0^n(s)\|ds$$

$$\leq \alpha(t) - \alpha(t_0^n) + \int_{t_0^n}^{t} \delta(s)ds \leq \int_{t_0^n}^{t} (|\dot{v}(s)| + K\rho(s) + (K'+1)\delta(s))ds \leq \frac{r}{2}$$

and $\|x_n(t) - \bar{\varphi}(0)\| \leq \int_{t_0^n}^t \rho(s)ds \leq \frac{r}{2}$, which is equivalent to $u_n(t) \in \overline{B}(\varphi(0), \frac{r}{2})$ and $x_n(t) \in \overline{B}(\bar{\varphi}(0), \frac{r}{2})$ for all $t \in [t_0^n, t_1^n]$. Now, by the same arguments us in the last section, we can prove that $T(t_1^n)x_n \in \overline{B}_a(\bar{\varphi}, r)$ and $T(t_1^n)u_n \in \overline{B}_a(\varphi, r)$.

We reiterate this process for constructing the sequences $(f_i^n(\cdot))_i$, $(x_i^n)_i$ and $(u_i^n)_i$ and the functions $x_n(\cdot)$ and $u_n(\cdot)$ satisfying, for all $0 \leq i \leq 2^n$ and for all $t \in [t_i^n, t_{i+1}^n]$, the following assertions:

$$f_i^n(t) \in F(t, T(t_i^n)x_n, T(t_i^n)u_n),$$

$$u_0^n \in C(t_0^n, x_n(t_0^n), u_n(t_0^n)),\ u_{i+1}^n \in C(t_{i+1}^n, x_n(t_{i+1}^n), u_n(t_i^n)),$$

$$\|x_n(t) - \bar{\varphi}(0)\| \leq \int_{t_0^n}^t \rho(s)ds,$$

$$\begin{aligned}
x_n(t) \in& \overline{B}(\bar{\varphi}(0), \frac{r}{2}),\ T(t^n_{i+1})x_n \in \overline{B}_a(\bar{\varphi}, r),\\
\|u_n(t) - \varphi(0)\| \leq& \int_{t^n_0}^{t} \big(|\dot{v}(s)| + K\rho(s) + (K'+1)\delta(s)\big)ds,\\
u_n(t) \in& \overline{B}(\varphi(0), \frac{r}{2}),\ T(t^n_{i+1})u_n \in \overline{B}_a(\varphi, r),\\
\|f^n_i(t) - y(t)\| \leq& d\big(y(t), F(t, T(t^n_i)x_n, T(t^n_i)u_n)\big) + \frac{1}{n},\ \text{a.e.}\\
u^n_{i+1} \in& \operatorname{Proj}_{C(t^n_{i+1}, x_n(t^n_{i+1}), u_n(t^n_i))}\left(u^n_i + \int_{t^n_i}^{t^n_{i+1}} f^n_i(s)ds\right), \qquad (8)\\
\left\|u^n_{i+1} - \left(u^n_i + \int_{t^n_i}^{t^n_{i+1}} f^n_i(s)ds\right)\right\| \leq& \int_{t^n_i}^{t^n_{i+1}} (|\dot{v}(s)| + K\rho(s) + K'\delta(s))ds,\\
u_n(t) =& u^n_i + \frac{\alpha(t) - \alpha(t^n_i)}{\alpha(t^n_{i+1}) - \alpha(t^n_i)}\left(u^n_{i+1} - u^n_i - \int_{t^n_i}^{t^n_{i+1}} f^n_i(s)ds\right) + \int_{t^n_i}^{t} f^n_i(s)ds,\\
x_n(t) =& x^n_i + (t - t^n_i)u^n_i.
\end{aligned}$$

Now, we define the functions $\theta_n(\cdot), \bar{\theta}_n(\cdot) : [0,\tau] \to [0,\tau]$ and $f_n(\cdot) \in L^1([0,\tau], H)$ by setting for all $t \in [t^n_i, t^n_{i+1}[$

$$\begin{aligned}
\bar{\theta}_n(t) =& t^n_{i+1}, \qquad \theta_n(t) = t^n_i \qquad \text{and} \qquad f_n(t) = f^n_i(t),\\
\bar{\theta}(\tau) =& \tau, \qquad \theta_n(\tau) = t^n_{2^n-1} \qquad \text{and} \qquad f_n(\tau) = f^n_{2^n-1}(\tau).
\end{aligned}$$

Next, we can easily verify that $x_n(\cdot)$ and $u_n(\cdot)$ are absolutely continuous. Now, remark that for all $0 \leq i \leq 2^n - 1$ and for almost every t in $[t^n_i, t^n_{i+1}]$,

$$\dot{u}_n(t) = \frac{\dot{\alpha}(t)}{\alpha(t^n_{i+1}) - \alpha(t^n_i)}\left(u^n_{i+1} - u^n_i - \int_{t^n_i}^{t^n_{i+1}} f^n_i(s)ds\right) + f_n(t).$$

Then, for almost every $t \in [0,\tau]$, $\|\dot{u}_n(t) - f_n(t)\| \leq |\dot{v}(t)| + K\rho(t) + (K'+1)\delta(t)$. Also, by construction and the relation (8), we have for almost every $t \in [0,\tau]$,

$$\big(\dot{u}_n(t) - f_n(t)\big) \in -N_{C(\bar{\theta}_n(t), x_n(\bar{\theta}_n(t)), u_n(\theta_n(t)))}(u_n(\bar{\theta}_n(t))).$$

Then the proof is complete. □

Now, we will prove Theorem 4.1. In view of Lemma 4.2, we can define inductively sequences $(f_n(\cdot))_{n\geq 1}$, $(x_n(\cdot))_{n\geq 1}$, $(u_n(\cdot))_{n\geq 1} \subset \mathcal{C}([-a,\tau], H)$ and $(\theta_n(\cdot))_{n\geq 1}$, $(\bar{\theta}_n(\cdot))_{n\geq 1} \subset S([0,\tau],[0,\tau])$ satisfying the assertions (i)-(vi). By the same technics of the last section, we can prove that, for every $t \in [0,\tau]$, the set $\{u_n(t) : n \geq 1\}$ is relatively compact in H. On the other hand, by (iv), we have, for almost all $t \in [0,\tau]$, $\|\dot{u}_n(t)\| \leq |\dot{v}(t)| + K\rho(t) + (K'+2)\delta(t)$. By Arzelà-Ascoli's Theorem, we can select a subsequence, again denoted by $(u_n(\cdot))_n$ which converges uniformly to an absolutely continuous function $u(\cdot)$ on $[0,\tau]$, moreover $\dot{u}_n(\cdot)$ converges weakly to $\dot{u}(\cdot)$ in $L^1([0,\tau], H)$. Also, since all functions $u_n(\cdot)$ agree with $\varphi(\cdot)$ on $[-a, 0]$, we can say, as above, that $u_n(\cdot)$ converges uniformly to $u(\cdot)$ on $[-a,\tau]$. In addition, since

$$\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(\theta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\|$$

$$\leq \int_{\theta_n(t)}^{t} \big(|\dot{v}(s)| + K\rho(s) + (K'+2)\delta(s)\big)ds + \|u_n(t) - u(t)\|$$

and the right term of the above inequality converges to 0 as $n \to \infty$, we can conclude that $(u_n(\theta_n(\cdot)))_n$ converges uniformly to $u(\cdot)$ on $[0,\tau]$. By the same argument, we can prove that $(u_n(\overline{\theta}_n(\cdot)))_n$ converges uniformly to $u(\cdot)$ on $[0,\tau]$. Next, set $x(t) = \bar{\varphi}(0) + \int_0^t u(s)ds$ for all $t \in [0,\tau]$. We have $\|x_n(t) - x(t)\| \leq \int_0^\tau \|u_n(\theta_n(s)) - u(s)\|ds$, $\forall t \in [0,\tau]$. Hence, the sequence $(x_n(\cdot))_n$ converges uniformly to a function $x(\cdot)$. Also, since all functions $x_n(\cdot)$ agree with $\bar{\varphi}(\cdot)$ on $[-a,0]$, we can obviously say that $x_n(\cdot)$ converges uniformly to $x(\cdot)$ on $[-a,\tau]$, if we extend $x(\cdot)$ in such a way that $x(\cdot) \equiv \bar{\varphi}(\cdot)$ on $[-a,0]$. Additionally, observe that for all $t \in [0,\tau]$

$$\|x_n(\bar{\theta}_n(t)) - x(t)\| \leq \|x_n(\bar{\theta}_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \leq \int_t^{\bar{\theta}_n(t)} \rho(s)ds + \|x_n(t) - x(t)\|$$

Thus $x_n(\bar{\theta}_n(\cdot))$ converges uniformly to $x(\cdot)$. By the same argument, we can prove that $x_n(\theta_n(\cdot))$ converges uniformly to $x(\cdot)$. Moreover, one has $u(t) \in C(t, x(t), u(t))$, for all $t \in [0,\tau]$. On the other hand, since $\dot{x}_n(t) = u_n(\theta_n(t))$, for almost all $t \in [0,\tau]$, the sequence $(\dot{x}_n(\cdot))_n$ converges uniformly to $u(\cdot)$. So, from

$$x_n(t) = \bar{\varphi}(0) + \int_0^t \dot{x}_n(s)ds, \quad \forall t \in [0,\tau],$$

we get

$$x(t) = \bar{\varphi}(0) + \int_0^t u(s)ds, \quad \forall t \in [0,\tau].$$

So $\dot{x}(t) = u(t)$ and $\ddot{x}(t) = \dot{u}(t)$ for almost all $t \in [0,\tau]$. Now, if we apply the same arguments of the last section, we can prove that $T(\theta_n(t))x_n$ converges to $T(t)x$ and $T(\theta_n(t))u_n$ converges to $T(t)u$ in $\mathcal{C}_a$. In the next, we shall prove that $x(\cdot)$ is a solution of (2). Let $t \in [0,\tau]$. From (i) and (ii) we deduce

$$\begin{aligned}
&\|f_{n+1}(t) - f_n(t)\| \\
&\leq H\big(F(t, T(\theta_n(t))x_n, T(\theta_n(t))u_n), F(t, T(\theta_{n+1}(t))x_{n+1}, T(\theta_{n+1}(t))u_{n+1})\big) + \frac{1}{n+1} \\
&\leq m(t)\Bigg(\|T(\theta_n(t))x_n - T(\theta_{n+1}(t))x_{n+1}\|_\infty + \|T(\theta_n(t))u_n - T(\theta_{n+1}(t))u_{n+1}\|_\infty\Bigg) + \frac{1}{n+1}
\end{aligned}$$

The right term of the above relation converges to 0. Hence $(f_n(t))_{n\geq 1}$ is a Cauchy sequence and $f_n(t)$ converges to $f(t)$. Moreover, from the following inequality

$$\begin{aligned}
&d\big(f(t), F(t, T(t)x, T(t)u)\big) \\
&\leq \|f(t) - f_n(t)\| + H\big(F(t, T(\theta_n(t))x_n, T(\theta_n(t))u_n), F(t, T(t)x, T(t)u)\big) \\
&\leq \|f(t) - f_n(t)\| + m(t)\Bigg(\|T(\theta_n(t))x_n - T(t)x\|_\infty + \|T(\theta_n(t))u_n - T(t)u\|_\infty\Bigg)
\end{aligned}$$

we get $f(t) \in F(t, T(t)x, T(t)u)$ for all $t \in [0,\tau]$. Now, as in the previous section, we can prove that $\dot{u}(t) \in -N_{C(t,x(t),u(t))}(u(t)) + F(t, T(t)x, T(t)u)$, a.e. on $[0,\tau]$. Finally, we get $\ddot{x}(t) \in -N_{C(t,x(t),\dot{x}(t))}(\dot{x}(t)) + F(t, T(t)x, T(t)\dot{x})$, and $\dot{x}(t) \in C(t, x(t), \dot{x}(t))$ for almost every $t \in [0,\tau]$. The proof is complete. □

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