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APPROXIMATION OF GENERALIZED PĂLTĂNEA AND HEILMANN-TYPE OPERATORS

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Abstract. In this paper, we study the approximation on differences of two different positive linear operators (generalized Păltănea type operators and M. Heilmann type operators) with same basis functions. We estimates a quantitative difference of these operators in terms of modulus of continuity and Peetre's K-functional. We represent the rate of convergence, using modulus of continuity and Peetre's K-functional. Also, we represent Heilmann-type operators in terms of hypergeometric series.

1. Introduction

In order to discuss the approximation properties of linear positive operators the rate of convergence is one of the important characteristics. Several methods and techniques have been applied to investigate and estimate the rate of convergence of positive linear operators. Motivated by an article of Acu and Rasa [1], we establish some quantitative results on the difference of generalized Păltănea type operators (1) and Heilmann type operators (4). In our estimation positive linear functionals in quantitative form play an important role. So, we are ready to obtain some information about the rate of convergence of the quantitative difference of these positive linear operators. Recently, the difference of positive linear operators have been discussed by many researchers [2,3,5,7,8,11].

For $f \in C^{\gamma}[0,\infty) = \{f \in C^{\gamma}[0,\infty) : f(t) = O(t^{\gamma}), \gamma > 0\}$, Verma [15] defined the following generalization of Păltănea type operators based on certain parameter $\lambda > 0$ in the following way:

$$L^{\lambda}_{n,c}(f,x) = \sum_{m=1}^{\infty} \vartheta_{n,m}(x,c)\mathcal{H}_{n,m}(f,x), \tag{1}$$

where
$$\mathcal{H}_{n,m}(f,x) = \int_0^{\infty} \Phi^{\lambda}_{n,m}(t,c)f(t)\,dt + \vartheta_{n,0}(x,c)f(0), \tag{2}$$

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$$\vartheta_{n,m}(x,c) = \frac{(-x)^m}{m!}\varphi_{n,c}^{(m)}(x);$$

$$\Phi_{n,m}^{\lambda}(t,c) = \begin{cases} \dfrac{n\lambda}{\Gamma(m\lambda)} e^{-n\lambda t}(n\lambda t)^{m\lambda-1}, & c=0 \\ \dfrac{\Gamma\left(\frac{n\lambda}{c}+m\lambda\right)}{\Gamma(m\lambda)\Gamma\left(\dfrac{n\lambda}{c}\right)} \dfrac{c^{m\lambda}t^{m\lambda-1}}{(1+ct)^{\frac{n\lambda}{c}+m\lambda}}, & c\in\mathbb{N}, \end{cases}$$

and

$$\varphi_{n,c}(x) = \begin{cases} e^{-nx}, & c=0, \\ (1+cx)^{-\frac{n}{c}}, & c\in\mathbb{N}, \end{cases}$$

where $\{\varphi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of continuous functions defined on $[0,b]$, $b>0$ and for each $n\in\mathbb{N}$, $m\in\mathbb{N}^0 := \mathbb{N}\cup\{0\}$ satisfies that $\varphi_{n,c}$ is monotone with $(-1)^n\varphi_{n,c}(x)\geq 0$ for $x\in[0,\infty)$ and $\phi_{n,c}(0)=1$.

Lastly, there exists an integer c such that $\varphi_{n,c}^{(m+1)}(x) = -n\varphi_{n+c,c}^{(m)}(x)$, $n>\max\{0,-c\}$; $x\in[0,b]$. It can be verified that the operators $L_{n,c}^{\lambda}(f,x)$ are well defined for $f\in C^{\gamma}[0,\infty)$.

Moreover, for $\lambda=1$, the operators (1) reduce to the Srivastava-Gupta operators [14]. For $c=0$, $\lambda>0$, we obtain a modified operators due to Păltănea [12]. And, for $c=0$, $\lambda=1$ operators (1) reduce to Phillips operators [6,13].

It can be easily observed by simple computation of (2) that:

$$\int_0^{\infty}\Phi_{n,m}^{\lambda}(t,c)\,t^r\,dt = \frac{\Gamma(m\lambda+r)}{\Gamma(m\lambda)}\frac{1}{\prod_{i=1}^{r}(n\lambda-ic)}. \tag{3}$$

For $r=0$, we get $\displaystyle\int_0^{\infty}\Phi_{n,m}^{\lambda}(t,c)\,dt = 1.$

In 1988, M. Heilmann [10] considered a sequence of continuous functions defined on the interval $[0,\infty)$. This sequences is defined as follows:

$$M_{n,c}(f;x) = \sum_{m=0}^{\infty}\vartheta_{n,m}(x)\Upsilon_{n,m}(f,x), \tag{4}$$

where

$$\Upsilon_{n,m}(f,x) = (n-c)\int_0^{\infty}\vartheta_{n,m}(t)f(t)\,dt. \tag{5}$$

For $x\in[0,\infty)$, $\vartheta_{n,m}(x) = (-1)^m\frac{x^m}{m!}\varphi_n^m(x)$, where

$$\varphi_n(x) = \begin{cases} (1-x)^n & \text{for the interval } [0,1] \text{ with } c=1, \\ e^{-nx} & \text{for the interval } [0,\infty) \text{ with } c=0, \\ (1+cx)^{\frac{-n}{c}} & \text{for the interval } [0,\infty) \text{ with } c>0. \end{cases}$$

By simple calculation, we obtain

$$\Upsilon_{n,m}(f,x) = \frac{(m+r)!\Gamma(\frac{n}{c}-r-1)}{c^r m!\Gamma(\frac{n}{c}-1)}. \tag{6}$$

In this article we consider only two cases: $c>0$ and $c=0$.

2. Basic results

In this section, we discuss some lemmas which will be used in the main results.

LEMMA 2.1. *For* $e_r(t)=t^r$, $r\in\mathbb{N}\cup\{0\}$ *the moments of operators* (1) *are:*

$$L_{n,c}^{\lambda}(e_0,x)=1;$$
$$L_{n,c}^{\lambda}(e_1,x)=\frac{n\lambda}{n\lambda-c}x;$$
$$L_{n,c}^{\lambda}(e_2,x)=\frac{1}{(n\lambda-c)(n\lambda-2c)}[n(n+c)\lambda^2x^2+n\lambda(1+\lambda)x];$$
$$L_{n,c}^{\lambda}(e_3,x)=\frac{1}{(n\lambda-c)(n\lambda-2c)(n\lambda-3c)}[n\lambda^3(2c^2+3cn+n^2)x^3$$
$$+3n\lambda^2(c+n)(1+\lambda)x^2+n\lambda(1+3\lambda+\lambda^2x];$$
$$L_{n,c}^{\lambda}(e_4,x)=\frac{1}{(n\lambda-c)(n\lambda-2c)(n\lambda-3c)(n\lambda-4c)}[(c^2+nc)(2c+n)(3c+n)\lambda^4x^4$$
$$+6(2c^2+3nc+n^2)(n+c\lambda)\lambda^3x^3+(c+n)(7c\lambda^2+n(11+18\lambda))\lambda^2x^2$$
$$+(c\lambda^3+n(6+11\lambda+6\lambda^2))\lambda x].$$

Proof. From (2), we have

$$\mathcal{H}_{n,m}(e_0)=1;\quad \mathcal{H}_{n,m}(e_1)=\frac{m\lambda}{n\lambda-c},\qquad \mathcal{H}_{n,m}(e_2)=\frac{m\lambda(m\lambda+1)}{(n\lambda-c)(n\lambda-2c)};$$
$$\mathcal{H}_{n,m}(e_3)=\frac{m\lambda(m\lambda+1)(m\lambda+2)}{(n\lambda-c)(n\lambda-2c)(n\lambda-3c)};\quad \mathcal{H}_{n,m}(e_4)=\frac{m\lambda(m\lambda+1)(m\lambda+2)(m\lambda+3)}{(n\lambda-c)(n\lambda-2c)(n\lambda-3c)(n\lambda-4c)}.$$

In view of these equalities, we get the required result. □

LEMMA 2.2. *The moments of operators* (5) *with* $e_r(t)=t^r$, $r\in\mathbb{N}\cup\{0\}$ *are:*

$$M_{n,c}(e_0;x)=1;$$
$$M_{n,c}(e_1;x)=\frac{1}{n-2c}[nx+1];$$
$$M_{n,c}(e_2;x)=\frac{1}{(n^2-5cn+6c^2)}[(cn+n^2)x^2+4nx+2];$$
$$M_{n,c}(e_3;x)=\frac{1}{(n^2-5cn+6c^2)(n-4c)}[n(2c^2+3cn+n^2)x^3+9n(c+n)x^2+18nx+6];$$
$$M_{n,c}(e_4;x)=\frac{1}{(n^2-5cn+6c^2)(n-4c)(n-5c)}\Big[n(6c^3+11c^2n+6cn^2+n^3)x^4$$
$$+16n(2c^2+3cn+n^2)x^3+72n(c+n)x^2+96nx+24\Big].$$

Proof. From the equation (6), we obtain

$$\Upsilon_{n,m}(e_0,x)=1;\quad \Upsilon_{n,m}(e_1,x)=\frac{m+1}{(n-2c)};\quad \Upsilon_{n,m}(e_2,x)=\frac{(m+1)(m+2)}{(n-2c)(n-3c)};$$

$$\Upsilon_{n,m}(e_3,x)=\frac{(m+1)(m+2)(m+3)}{(n-2c)(n-3c)(n-4c)};\quad \Upsilon_{n,m}(e_4,x)=\frac{(m+1)(m+2)(m+3)(m+4)}{(n-2c)(n-3c)(n-4c)(n-5c)}.$$

Using these equalities, we get the required result. □

3. Hypergeometric form

The hypergeometric series and the confluent hypergeometric series are defined as:

$${}_2F_1(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}x^m \qquad \text{and} \qquad {}_1F_1(a,b;x)=\sum_{m=0}^{\infty}\frac{(a)_m}{(b)_m m!}x^m,$$

respectively, where the Pochhammer symbol $(n)_m$ is defined as:

$$(n)_m=n(n+1)(n+2)(n+3)\dots(n+m-1).$$

Lemma 3.1. *For $n>0$ and $r\geq 1$, we have*

$$M_{n,c}(e_r;x)=\frac{\Gamma(r+1)}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}{}_2F_1\left(\frac{n}{c},-r;1;-cx\right).$$

Proof. Using the equations (4) and (6) and the Pochhammer symbol $(n)_k$, we have

$$\begin{aligned}
M_{n,c}(e_r;x)=&(n-c)\sum_{m=0}^{\infty}\vartheta_{n,m}(x,c)\int_0^{\infty}\frac{(\frac{n}{c})_m}{m!}\frac{(tc)^m}{(1+ct)^{\frac{n}{c}+m}}t^r\,dt\\
=&(n-c)\sum_{m=0}^{\infty}\vartheta_{n,m}(x,c)\frac{(\frac{n}{c})_m}{m!}\frac{1}{c^r}\int_0^{\infty}\frac{(tc)^{m+r}}{(1+ct)^{\frac{n}{c}+m}}\,dt\\
=&(n-c)\sum_{m=0}^{\infty}\vartheta_{n,m}(x,c)\frac{(\frac{n}{c})_m}{m!}\frac{1}{c^{r+1}}\beta(r+m+1,\frac{n}{c}-r-1)\\
=&\frac{r!}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}(1+cx)^{\frac{-n}{c}}\sum_{m=0}^{\infty}\frac{(r+1)_m}{m!(1)_m}\left(\frac{cx}{1+cx}\right)^m\\
=&\frac{\Gamma(r+1)}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}{}_2F_1\left(\frac{n}{c},-r;1;-cx\right).
\end{aligned}$$

□

4. Difference of operators

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on $[0,\infty)$ with the norm $\|\cdot\|=\sup_{x\in[0,\infty)}|f(x)|<\infty$. In order to present the main results, we need some useful notation.

For $i\in\mathbb{N}$, let $e_i(x)=x^i$, $x\in[0,\infty)$. Let H be a positive linear functional defined on a subspace $\mathbb{B}$ of the closed space $C[0,\infty)$, such that $G(e_0)=1$, $b^H:=H(e_1)$ and $\mu_r^H=H(e_1-b^He_0)^r$, $r\in\mathbb{N}$.

Let us consider the operators (1) and (5). We have the following quantitative general results.

REMARK 4.1. From the generalised Păltănea type operators (1) we have

$$\mathcal{H}_{n,m}(f) = f\left(\frac{m}{n}\right), \quad \text{such that} \quad \mathcal{H}_{n,m}(e_0) = 1, \quad b^{\mathcal{H}_{n,m}} := \mathcal{H}_{n,m}(e_1)$$

and $\qquad \mu_r^{\mathcal{H}_{n,m}} = \mathcal{H}_{n,m}(e_1 - b^{\mathcal{H}_{n,m}} e_0)^r, \quad r \in \mathbb{N}.$

By simple calculations, we obtain

$$\begin{aligned}
\mathcal{H}_{n,m}(e_1) =& \frac{\Gamma(m\lambda+1)}{\Gamma(m\lambda)} \frac{1}{m\lambda-c} = \frac{m\lambda}{n\lambda-c};\\
\mu_2^{\mathcal{H}_{n,m}} =& \mathcal{H}_{n,m}\left(e_1 - b^{\mathcal{H}_{n,m}} e_0\right)^2\\
=& \mathcal{H}_{n,m}\left(e_2 + \left(\frac{m\lambda}{n\lambda-c}\right)^2 e_0^2 - 2e_1 \frac{m\lambda}{n\lambda-c}\right) = \frac{m^2\lambda^2 c + m\lambda(n\lambda-c)}{(n\lambda-c)^2(n\lambda-2c)};\\
\mu_4^{\mathcal{H}_{n,m}} =& \mathcal{H}_{n,m}\left(e_1 - b^{\mathcal{H}_{n,m}} e_0\right)^4\\
=& \frac{1}{(n\lambda-c)^4(n\lambda-2c)(n\lambda-3c)(n\lambda-4c)}\Big[-6m\lambda(c-n\lambda)^3 + 3m^4c^2\lambda^4(5c+n\lambda)\\
&+6k^3c\lambda^3(-5c^2+4cn\lambda+n^2\lambda^2) + 3k^2\lambda^2(c-n\lambda)^2(7c+n\rho)\Big].
\end{aligned}$$

REMARK 4.2. From the Hielmann type operators (5), we have $\Upsilon_{n,m}(e_0) = 1$, $b^{\Upsilon_{n,m}} = \Upsilon_{n,m}(e_1) = \frac{m+1}{n-2c}$. By simple calculations, we obtain

$$\begin{aligned}
\mu_2^{\Upsilon_{n,m}} =& \Upsilon_{n,m}\left(e_1 - b^{\Upsilon_{n,m}} e_0\right)^2 = \frac{cm^2+nm+n-c}{(n-2c)^2(n-3c)};\\
\mu_4^{\Upsilon_{n,m}} =& \frac{1}{(n-2c)^2(n-3c)(n-4c)(n-5c)}\Big[3c^2m^4(4c+n) + 6cm^3n(4c+n)\\
&+3m^2n(-2c^2+8cn+n^2) - 6m(c-2n)n^2 + 3(-4c^3+9c^2n-8cn^2+3n^3)\Big].
\end{aligned}$$

REMARK 4.3. For positive linear functionals $\mathcal{H}_{n,m}$ and $\Upsilon_{n,m}$, we estimate

$$\delta_2^2 = \sum_{m=0}^{\infty} \vartheta_{n,m}\left(b^{\mathcal{H}_{n,m}} - b^{\Upsilon_{n,m}}\right)^2.$$

Using Remarks 4.1 and 4.2, we get

$$\begin{aligned}
\delta_2^2 =& \sum_{m=0}^{\infty} \vartheta_{n,m}\left(\frac{m\lambda}{n\lambda-c} - \frac{m+1}{n-2c}\right)^2\\
=& \frac{1}{(n\lambda-c)^2(n-2c)^2} \sum_{m=0}^{\infty} \vartheta_{n,m}\left(m\lambda(n-2c) - (m+1)(n\lambda-c)\right)^2\\
=& \frac{1}{(n\lambda-c)^2(n-2c)^2} \sum_{m=0}^{\infty} \frac{(n/c)_m}{m!} \frac{(cx)^m}{(1+cx)^{n/c+m}}\\
&\times\left[c^2\lambda^2(1-2\lambda)^2 - 2ck(-1+2\lambda)(c-n\lambda) + (c-n\lambda)^2\right]
\end{aligned}$$

$$=\frac{nx(1+cx+nx)c^2(1-2\lambda)^2-2nxc(-1+2\lambda)(c-n\lambda)+(c-n\lambda)^2}{(n\lambda-c)^2(n-2c)^2}.$$

REMARK 4.4. For positive linear functionals $\mathcal{H}_{n,m}$ and $\Upsilon_{n,m}$, with the help of Remarks 4.1 and 4.2, we estimate

$$\begin{aligned}
\alpha(x) =&\frac{1}{2}\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(\mu_2^{\mathcal{H}_{n,m}}+\mu_2^{\Upsilon_{n,m}}\right)\\
=&\frac{1}{2}\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(\frac{m^2\lambda^2c+m\lambda(n\lambda-c)}{(n\lambda-c)^2(n\lambda-2c)}+\frac{cm^2+nm+n-c}{(n-2c)^2(n-3c)}\right)\\
=&\frac{1}{2(n\lambda-c)^2(n\lambda-2c)(n-2c)^2(n-3c)}\\
&\times\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(cm^2\left(-11cn^2\lambda^2+n^3\lambda^2(1+\lambda)+c^2n\lambda(5+16\lambda)-2c^3(1+6\lambda^2)\right)\right.\\
&+m\left(12c^4\lambda+n^4\lambda^2(1+\lambda)+4c^2n^2\lambda(3+4\lambda)-cn^3\lambda(1+11\lambda)-2c^3n(1+8\lambda+6\lambda^2)\right)\\
&\left.+(c-n)(c-n\lambda)^2(2c-n\lambda)\right)\\
=&\frac{1}{2(n-2c)^2(n-3c)(n\lambda-c)^2(n\lambda-2c)}\Big[(c-n)(c-n\lambda)^2(2c-n\lambda)\\
&+cnx(1+cx+nx)(-11cn^2\lambda^2+n^3\lambda^2(1+\lambda)+c^2n\lambda(5+16\lambda))\\
&-2c^4nx(1+cx+nx)(1+6\lambda^2))+nx(12c^4\lambda+n^4\lambda^2(1+\lambda)+4c^2n^2\lambda(3+4\lambda))\\
&-cn^4x\lambda((1+11\lambda)-2c^3n(1+8\lambda+6\lambda^2))\Big].
\end{aligned}$$

THEOREM 4.5. *For $n\in\mathbb{N}$, let $f^i\in C_B$, $\quad i\in\{0,1,2\}$ and $x\in[0,\infty)$. We have*

$$|(L_{n,c}^{\lambda}-M_{n,c})(f;x)|\le\|f''\|\alpha(x)+\omega(f'',\delta_1)(1+\alpha(x))+2\omega(f,\delta_2),$$

where $\alpha(x)=\frac{1}{2}\sum_{m=0}^{\infty}\vartheta_{n,m}\left(\mu_2^{\mathcal{H}_{n,m}}+\mu_2^{\Upsilon_{n,m}}\right)$,

$$\delta_1^2=\frac{1}{2}\sum_{m=0}^{\infty}\vartheta_{n,m}\left(\mu_4^{\mathcal{H}_{n,m}}+\mu_4^{\Upsilon_{n,m}}\right)\ \textit{and}\ \delta_2^2=\sum_{m=0}^{\infty}\vartheta_{n,m}\left(b^{\mathcal{H}_{n,m}}-b^{\Upsilon_{n,m}}\right)^2.$$

Now, we establish quantitative estimates for the difference of generalized Păltănea type operators (1) and Heilmann type operators (5) with the help of Theorem 4.5.

THEOREM 4.6. *Let $f^i\in C_B$, $\quad i\in\{0,1,2\}$ and $x\in[0,\infty)$. Then for $n\in\mathbb{N}$, we have*

$$|(L_{n,c}^{\lambda}-M_{n,c})(f;x)|\le\|f''\|\alpha(x)+\omega(f'',\delta_1)(1+\alpha(x))+2\omega(f,\delta_2),$$

where

$$\begin{aligned}
\alpha(x)=&\frac{1}{2(n-2c)^2(n-3c)(n\lambda-c)^2(n\lambda-2c)}\Big[(c-n)(c-n\lambda)^2(2c-n\lambda)\\
&+cnx(1+cx+nx)(-11cn^2\lambda^2+n^3\lambda^2(1+\lambda)+c^2n\lambda(5+16\lambda)-2c^3(1+6\lambda^2))\\
&+nx(12c^4\lambda+n^4\lambda^2(1+\lambda)+4c^2n^2\lambda(3+4\lambda)-cn^3\lambda(1+11\lambda)-2c^3n(1+8\lambda+6\lambda^2))\Big],
\end{aligned}$$

$$\delta_1^2 = \frac{1}{(n-2c)^4(n-3c)(n-4c)(n-5c)(n\lambda-c)^4(n\lambda-2c)(n\lambda-3c)(n\lambda-4c)}$$
$$\times \Big[s_1 n \left((n+c)(n+2c)(n+3c)x^4+6(n+c)(n+2c)x^3+7(n+c)x^2+x\right)$$
$$+s_2 nx\left(1+3cx+3nx+2c^2x^2+3cnx^2+n^2x^2\right)+s_3 nx(1+cx+nx)+s_4 nx+s_5\Big],$$
$$\delta_2^2 = \frac{nx(1+cx+nx)c^2(1-2\lambda)^2-2cnx(-1+2\lambda)(c-n\lambda)+(c-n\lambda)^2}{(n-2c)^2(n\lambda-c)^2},$$

and

$$s_1 = 3c^2\big((3c-n)(4c-n)(5c-n)(-2c+n)^4(5c\lambda^4+n\lambda^5)$$
$$+(c-n\lambda)^4(2c-n\lambda)(3c-n\lambda)(4c-n\lambda)(4c+n)\big);$$
$$s_2 = 6c\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(-5c^2\lambda^3+4cn\lambda^4+n^2\lambda^5)\right.$$
$$\left.+(c-n\lambda)^4(2c-n\lambda)(3c-n\lambda)(4c-n\lambda)(4cn+n^2)\right);$$
$$s_3 = 3\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(7c^3\lambda^2-13c^2n\lambda^3+5cn^2\lambda^4+n^3\lambda^5)\right.$$
$$\left.+(c-n\lambda)^4(2c-n\lambda)(3c-n\lambda)(4c-n\lambda)(-2c^2n+8cn^2+n^3)\right);$$
$$s_4 = \left((3c-n)(4c-n)(5c-n)(-2c+n)^4(-6c^3\lambda+18c^2n\lambda^2-18cn^2\lambda^3+6n^3\lambda^4)\right.$$
$$\left.+(c-n\lambda)^4(2c-n\lambda)(3c-n\lambda)(4c-n\lambda)(-6cn^2+12n^3)\right)$$
$$s_5 = 3(-4c^3+9c^2n-8cn^2+3n^3)(c-n\lambda)^4(2c-n\lambda)(3c-n\lambda)(4c-n\lambda).$$

Proof. By direct computation, using Remarks 4.1, 4.2 and 4.3. □

THEOREM 4.7 ([9]). *It $f \in C_B[0,\infty)$ with $f'' \in C_B[0,\infty)$, then*
$$|\left(L_{n,c}^{\lambda} - M_{n,c}\right)(f;x)| \le \alpha(x)\|f''\| + 2\omega\left(f;\beta(x)\right)$$
where
$$\alpha(x) = \frac{1}{2}\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(\mu_2^{\mathcal{H}_{n,m}} + \mu_2^{\Upsilon_{n,m}}\right), \quad \beta(x) = \left(\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(b^{\mathcal{H}_{n,m}} - b^{\Upsilon_{n,m}}\right)^2\right)^{\frac{1}{2}}.$$

The above theorem can be utilized to estimate the difference between operators (1) and (5).

THEOREM 4.8. *It $f \in C_B[0,\infty)$ with $f'' \in C_B[0,\infty)$, then*
$$|\left(L_{n,c}^{\lambda} - M_{n,c}\right)(f;x)| \le \alpha(x)\|f''\| + 2\omega\left(f;\beta(x)\right),$$
where
$$\alpha(x) = \frac{1}{2(n-2c)^2(n-3c)(n\lambda-c)^2(n\lambda-2c)}\Big[(c-n)(c-n\lambda)^2(2c-n\lambda)$$
$$+cnx(1+cx+nx)(-11cn^2\lambda^2+n^3\lambda^2(1+\lambda)+c^2n\lambda(5+16\lambda)-2c^3(1+6\lambda^2))$$
$$+nx(12c^4\lambda+n^4\lambda^2(1+\lambda)+4c^2n^2\lambda(3+4\lambda)-cn^3\lambda(1+11\lambda)-2c^3n(1+8\lambda+6\lambda^2))\Big],$$
$$\beta(x) = \frac{1}{(n\lambda-c)(n-2c)}\left(nx(1+cx+nx)c^2(1-2\lambda)^2-2nxc(-1+2\lambda)(c-n\lambda)+(c-n\lambda)^2\right)^{\frac{1}{2}}.$$

Proof. Follows directly from Remarks 4.3 and 4.4. □

5. Estimate with K-functional

In this section, we shall estimate the approximation of operators (1) and (5) in terms of K-functional. Let $\omega_2(f,\delta)$ denote the modulus of smoothness of f on the closed and bounded interval $[0,b]$, $b>0$ and defined as:

$$\omega_2(f,\delta)=\sup_{0<h\le\delta}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|,$$

and Peetre's K-functional is defined as:

$$K_2(f;\delta)=\inf_{g\in C_B[0,\infty)}\{\|f-g\|+\delta\|g''\|:g\in C_B[0,\infty)\}.$$

Let $C_B[0,\infty)=\{f:[0,\infty)\to\mathbb{R};\ f \text{ is bounded and continuous}\}$, with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$ and $W^2=\{f\in C_B[0,\infty):f',f''\in C_B[0,\infty)\}$.

A relation between the second order modulus of smoothness and Peetre's K-functional is given by: $K_2(f;\delta)\le C_0\omega_2(f;\delta)$, where C_0 is an absolute constant. Now, we suppose that $W^2\subset D$.

Theorem 5.1 ([4]). *Let $f\in D\cap C_B[0,\infty)$ and $f'\in C_B[0,\infty)$. Then*

$$|\left(L_{n,c}^{\lambda}-M_{n,c}\right)(f;x)|\le 4K_2\left(f:\frac{1}{8}\eta(x)\right)+\|f'\|\mu(x),$$

where
$$\eta(x)=\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(\mu_2^{\mathcal{H}_{n,m}}+\mu_2^{\Upsilon_{n,m}}\right),$$

and
$$\mu(x)=\sum_{m=0}^{\infty}\vartheta_{n,m}|b^{\mathcal{H}_{n,m}}-b^{\Upsilon_{n,m}}|\le\sum_{m=0}^{\infty}\vartheta_{n,m}|b^{\mathcal{H}_{n,m}}|+|b^{\Upsilon_{n,m}}|.$$

Now, we give an important result in terms of K-functional from Theorem 5.1.

Theorem 5.2. *Let $f\in D\cap C_B[0,\infty)$ and $f'\in C_B[0,\infty)$. For $n>\max\{\frac{c}{\lambda},2c\}$, we have*

$$\left|\left(L_{n,c}^{\lambda}-M_{n,c}\right)(f;x)\right|\le 4K_2\left(f:\frac{1}{4}\alpha(x)\right)+\|f'\|\left(\frac{n\lambda(1+2nx)-nxc(1+2\lambda)-c}{(n-2c)(n\lambda-c)}\right),$$

where

$$\begin{aligned}\alpha(x)=&\frac{1}{2(n-2c)^2(n-3c)(n\lambda-c)^2(n\lambda-2c)}\Big[(c-n)(c-n\lambda)^2(2c-n\lambda)\\&+cnx(1+cx+nx)(-11cn^2\lambda^2+n^3\lambda^2(1+\lambda)+c^2n\lambda(5+16\lambda)-2c^3(1+6\lambda^2))\\&+nx(12c^4\lambda+n^4\lambda^2(1+\lambda)+4c^2n^2\lambda(3+4\lambda)-cn^3\lambda(1+11\lambda)-2c^3n(1+8\lambda+6\lambda^2))\Big],\end{aligned}$$

Proof. For $f\in D\cap C_B[0,\infty)$ and $n>2c, n\lambda>c$, using operators (1) and (5), we have

$$\mu(x)\le\sum_{m=0}^{\infty}\vartheta_{n,m}(x)|b^{\mathcal{H}_{n,m}}|+|b^{\Upsilon_{n,m}}|\le\sum_{m=0}^{\infty}\vartheta_{n,m}(x)\left(\frac{m\lambda}{n\lambda-c}+\frac{m+1}{n-2c}\right)$$

$$\leq \frac{nx\lambda}{n\lambda - c} + \frac{1+nx}{n-2c} \leq \frac{n\lambda(1+2nx) - ncx(1+2\lambda) - c}{(n-2c)(n\lambda - c)}. \tag{7}$$

It is obvious that the required result follows from (7) and Remark 4.3. □

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