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ON SINGULAR FIFTH-ORDER BOUNDARY VALUE PROBLEMS WITH DEFICIENCY INDICES $(5, 5)$

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Abstract. This paper is devoted to introduce a way of construction of the well-defined boundary conditions for the solutions of a singular fifth-order equation with deficiency indices $(5, 5)$. Imposing suitable separated and coupled boundary conditions some properties of the eigenvalues of the problems have been investigated.

1. Introduction

Formally symmetric third and fifth-order boundary value problems generated by well-defined boundary conditions have been studied in [3–6]. In the papers [3–5] the problems have been studied as regular problems. Then separated and coupled boundary conditions have been introduced for third and fifth-order equations in these papers. A singular third-order boundary value problem has been handled in [6] and well-defined boundary conditions have been imposed for the solutions of this third-order equations using auxiliary functions. In this paper our aim is to generalize these important results for the singular fifth-order equations.

Singular differential equations need to be studied in detail as the behaviour of solutions of the equation near the singular point may change depending on the nature of the coefficients of the equation. One of the challenges appears when one wants to describe the number of square-integrable linearly independent solutions with respect to some weight functions. In the literature, there are some tools to give an answer to this question. One of them is the deficiency indices theory (see, for example, [2]). This theory is about finding the dimensions of the following deficiency subspaces

$$N_\lambda = H \ominus (L - \lambda I)D(L), \quad N_{\overline{\lambda}} = H \ominus (L - \overline{\lambda} I)D(L),$$

where H is a Hilbert space, L is a symmetric operator in H, $D(L)$ is the domain of L and λ is a complex number. The dimensions of N_λ and $N_{\overline{\lambda}}$ can be found with the

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aid of the solutions of $L^*y = \overline{\lambda}y$ and $L^*y = \lambda y$, respectively. The deficiency indices (m, n) of L are defined by $m = \dim N_i$ and $n = \dim N_{-i}$.

In this paper we deal with singular fifth-order boundary value problems related with some separated and coupled boundary conditions and we introduce some results of these problems. This work will be the first work on singular fifth-order boundary value problems.

2. Basic results

The fifth-order equation will be considered on the interval $[a, b)$ as follows

$$i\left(q_0\left(q_0 f''\right)'\right)'' + (p_0 f'')'' + i\left[(q_2 f)' + q_2 f' - (q_1 f')'' - (q_1 f'')'\right] - (p_1 f')' + p_2 f = \lambda w f. \quad (1)$$

All the coefficients q_r, p_s, w, $r = 0, 1, 2$, $s = 0, 1$, are assumed to be real-valued functions such that $q_0^{-1}, p_1, p_2, q_2, q_1/q_0, p_0/q_0^2$ are integrable and the quasi-derivative $f^{[k]}$ [1] of f is absolutely continuous on each compact interval $[c, d] \subset [a, b)$, where $-\infty < a \le c < d < b \le \infty$, $k = 0, \ldots, 4$ and

$$f^{[0]} = f,$$
$$f^{[1]} = f', \qquad f^{[3]} = iq_0(q_0 f'')' + p_0 f'' - iq_1 f',$$
$$f^{[2]} = -\frac{1+i}{\sqrt{2}} q_0 f'', \quad f^{[4]} = -(iq_0(q_0 f'')' + p_0 f'' - iq_1 f')' + iq_1 f'' + p_1 f' - iq_2 f.$$

Moreover, we assume that $q_0 \ne 0$, $w > 0$ on $[a, b)$ and the only singularity for the equation (1) occurs at b.

We consider the Hilbert space L^2 as the standard Lebesgue space equipped with the inner product

$$(f, g) = \int_a^b f\overline{g}w\, dx.$$

To impose the well-defined boundary conditions for the solutions of (1) we shall consider the subspace D of L^2 covering the functions $f \in L^2$ with $\tau(f) \in L^2$, where

$$\tau(f) = \frac{1}{w}\left\{i\left(q_0\left(q_0 f''\right)'\right)'' + (p_0 f'')'' + i\left[(q_2 f)' + q_2 f' - (q_1 f')'' - (q_1 f'')'\right] - (p_1 f')' + p_2 f\right\}.$$

We define the maximal operator T on D as follows $Tf = \tau(f)$, where $f \in D$.

The Lagrange's formula can now be introduced as the following

$$(Tf, g) - (f, Tg) = [f; g], \quad (2)$$

where $[f; g] = [f, \overline{g}](b) - [f, \overline{g}](a)$, $[\cdot, \cdot](x) : D \times D \to \mathbb{C}$ and

$$[f, \overline{g}](x) := [f, \overline{g}] = f\overline{g}^{[4]} - f^{[4]}\overline{g} + f^{[1]}\overline{g}^{[3]} - f^{[3]}\overline{g}^{[1]} + if^{[2]}\overline{g}^{[2]}. \quad (3)$$

The minimal operator T_0 is defined as the restriction of T to the subspace $D_0 \subset D$ that consists of all $f \in D$ satisfying $f^{[r]}(a) = [f, \overline{g}](b) = 0$, where $r = 0, \ldots, 4$ and $g \in D$. Operator T_0 is a symmetric, densely defined, closed in L^2 and $T_0^* = T$ [1].

Constructing the vector F by the rule

$$F = \begin{bmatrix} f & f^{[1]} & f^{[2]} & f^{[3]} & f^{[4]} \end{bmatrix}^T, \tag{4}$$

we can introduce another representation of (3) as follows

$$[f, g] = G^* J F, \tag{5}$$

where G is constructed by g and the rule (4),

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where i denotes the 2×2 identity matrix.

Other representations of (3) can also be introduced as follows

$$[f, g] = \widehat{G}^* E \widehat{F} + i f^{[2]} \overline{g}^{[2]}, \tag{6}$$

where
$$\widehat{F} = \begin{bmatrix} f \\ f^{[1]} \\ f^{[3]} \\ f^{[4]} \end{bmatrix}, \quad \widehat{G} = \begin{bmatrix} g \\ g^{[1]} \\ g^{[3]} \\ g^{[4]} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and
$$[f, \overline{g}] = \begin{bmatrix} \overline{g} & \overline{g}^{[4]} \end{bmatrix} E_0 \begin{bmatrix} f \\ f^{[4]} \end{bmatrix} + \begin{bmatrix} \overline{g}^{[1]} & \overline{g}^{[3]} \end{bmatrix} E_0 \begin{bmatrix} f^{[1]} \\ f^{[3]} \end{bmatrix} + i f^{[2]} \overline{g}^{[2]}, \tag{7}$$

where
$$E_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Using these representations (5)-(7) we will share coupled boundary conditions for the solutions of (1).

Lemma 2.1. *The equation* (1) *has one and only one solution* $f(x, \lambda)$ *satisfying the initial conditions* $f^{[k]}(\xi, \lambda) = \zeta_k$, *where* $k = 0, \ldots, 4$, $\xi \in [a, b)$, $\zeta_k \in \mathbb{C}$. *Moreover,* $f(\cdot, \lambda)$ *is an entire function of* λ.

Proof. The equation (1) has the following representation

$$F' = [\lambda M + N] F, \quad x \in [a, b), \tag{8}$$

where F is the vector generated by f and (4), M and N are 5×5 matrices such that

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -w & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1-i}{\sqrt{2} q_0} & 0 & 0 \\ 0 & -\frac{1+i}{\sqrt{2}} \frac{q_1}{q_0} & i \frac{p_0}{q_0^2} & -\frac{1-i}{\sqrt{2} q_0} & 0 \\ -i q_2 & p_1 & -\frac{1+i}{\sqrt{2}} \frac{q_1}{q_0} & 0 & -1 \\ p_2 & i q_2 & 0 & 0 & 0 \end{bmatrix}.$$

Since the elements of N and M are locally integrable on $[a, b)$, (8) completes the proof. □

Now the direct calculation gives the following.

Lemma 2.2. $N^*J + JN = 0$.

We denote by $W\{f_1, \ldots, f_5\}$ the Wronskian of $f_1, \ldots, f_5$ defined as

$$W\{f_1, \ldots, f_5\} := \det \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} & f_5^{[1]} \\ f_1^{[2]} & f_2^{[2]} & f_3^{[2]} & f_4^{[2]} & f_5^{[2]} \\ f_1^{[3]} & f_2^{[3]} & f_3^{[3]} & f_4^{[3]} & f_5^{[3]} \\ f_1^{[4]} & f_2^{[4]} & f_3^{[4]} & f_4^{[4]} & f_5^{[4]} \end{bmatrix}.$$

This definition implies that the set of solutions $\{f_1, \ldots, f_5\}$ of (1) is linearly dependent provided that $W\{f_1, \ldots, f_5\}(x_0) = 0$, $x_0 \in [a, b)$ and $W\{f_1, \ldots, f_5\} \equiv 0$ on $[a, b)$ provided that $\{f_1, \ldots, f_5\}$ is linearly dependent (see [2, pp. 57-58]).

3. Boundary values at the singular point

In this section we will construct well-defined boundary values at the singular point. However, for this purpose, we need to know the deficiency indices of T_0.

It is known [7] that for $\operatorname{Im}\lambda > 0$ the deficiency index m of T_0 may get the values $m = 2, 3, 4, 5$ and for $\operatorname{Im}\lambda < 0$ the deficiency index n of T_0 may get the values $n = 3, 4, 5$. We consider the case $(m, n) = (5, 5)$ in this work.

We consider the solutions $z_1(x), \ldots, z_5(x)$ of

$$\tau(f) = 0, \quad x \in [a, b) \tag{9}$$

satisfying the initial conditions $z_k^{[s-1]}(a) = \delta_{ks}$, where $1 \leq k, s \leq 5$ and δ_{ks} is the Kronecker delta symbol.

Let us denote by $Z_1, \ldots, Z_5$ (5×1 vectors) generated by $z_1, \ldots, z_5$, respectively, by the rule (4). Since $W\{z_1, \ldots, z_5\}(a) = \det\{Z_1(a), \ldots, Z_5(a)\} = 1$, the set of solutions $\{z_1, \ldots, z_5\}$ is a linearly independent set. Using (9) and (2) we have the following equations

$$\begin{aligned} &[z_1, \overline{z}_2] = 0, \quad [z_1, \overline{z}_3] = 0, \quad [z_1, \overline{z}_4] = 0, \quad [z_1, \overline{z}_5] = 1, \\ &[z_2, \overline{z}_3] = 0, \quad [z_2, \overline{z}_4] = 1, \quad [z_2, \overline{z}_5] = 0, \\ &[z_3, \overline{z}_4] = 0, \quad [z_3, \overline{z}_5] = 0, \\ &[z_4, \overline{z}_5] = 0, \end{aligned} \tag{10}$$

and

$$[z_1, \overline{z}_1] = 0, \quad [z_2, \overline{z}_2] = 0, \quad [z_3, \overline{z}_3] = i, \quad [z_4, \overline{z}_4] = 0, \quad [z_5, \overline{z}_5] = 0, \tag{11}$$

Now we shall define the following 5×5 matrix $Z = \begin{bmatrix} Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \end{bmatrix}$, $x \in [a, b)$. Using (10) and (11) we obtain that for $x \in [a, b)$

$$Z^*JZ = \begin{bmatrix} [z_1, \overline{z}_1] & [z_2, \overline{z}_1] & [z_3, \overline{z}_1] & [z_4, \overline{z}_1] & [z_5, \overline{z}_1] \\ [z_1, \overline{z}_2] & [z_2, \overline{z}_2] & [z_3, \overline{z}_2] & [z_4, \overline{z}_2] & [z_5, \overline{z}_2] \\ [z_1, \overline{z}_3] & [z_2, \overline{z}_3] & [z_3, \overline{z}_3] & [z_4, \overline{z}_3] & [z_5, \overline{z}_3] \\ [z_1, \overline{z}_4] & [z_2, \overline{z}_4] & [z_3, \overline{z}_4] & [z_4, \overline{z}_4] & [z_5, \overline{z}_4] \\ [z_1, \overline{z}_5] & [z_2, \overline{z}_5] & [z_3, \overline{z}_5] & [z_4, \overline{z}_5] & [z_5, \overline{z}_5] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = J.$$

This shows that $(W\{z_1,\ldots,z_5\})^2=1$ for all $x\in[a,b)$. Since $W\{z_1,\ldots,z_5\}(a)=1$ we have $W\{z_1,\ldots,z_5\}\equiv 1$ for all $x\in[a,b)$.

We set $\Psi F=Z^{-1}F,\quad x\in[a,b)$, where the 5×1 vector F is given by (4). Since $Z\Psi F=F$, $x\in[a,b)$, we obtain that

$$\Psi F=\begin{bmatrix} W\{f,z_2,z_3,z_4,z_5\}\\ W\{z_1,f,z_3,z_4,z_5\}\\ W\{z_1,z_2,f,z_4,z_5\}\\ W\{z_1,z_2,z_3,f,z_5\}\\ W\{z_1,z_2,z_3,z_4,f\}\end{bmatrix},\quad x\in[a,b). \tag{12}$$

On the other hand we obtain that

$$\begin{bmatrix} [f,\overline{z}_1] & [z_2,\overline{z}_1] & [z_3,\overline{z}_1] & [z_4,\overline{z}_1] & [z_5,\overline{z}_1]\\ [f,\overline{z}_2] & [z_2,\overline{z}_2] & [z_3,\overline{z}_2] & [z_4,\overline{z}_2] & [z_5,\overline{z}_2]\\ [f,\overline{z}_3] & [z_2,\overline{z}_3] & [z_3,\overline{z}_3] & [z_4,\overline{z}_3] & [z_5,\overline{z}_3]\\ [f,\overline{z}_4] & [z_2,\overline{z}_4] & [z_3,\overline{z}_4] & [z_4,\overline{z}_4] & [z_5,\overline{z}_4]\\ [f,\overline{z}_5] & [z_2,\overline{z}_5] & [z_3,\overline{z}_5] & [z_4,\overline{z}_5] & [z_5,\overline{z}_5]\end{bmatrix}=\begin{bmatrix} [f,z_1] & 0 & 0 & 0 & -1\\ [f,z_2] & 0 & 0 & -1 & 0\\ [f,z_3] & 0 & i & 0 & 0\\ [f,z_4] & 1 & 0 & 0 & 0\\ [f,z_5] & 0 & 0 & 0 & 0\end{bmatrix}. \tag{13}$$

Left-hand side of (13) can also be considered as follows

$$\begin{bmatrix} \overline{z}_1 & \overline{z}_1^{[1]} & \overline{z}_1^{[2]} & \overline{z}_1^{[3]} & \overline{z}_1^{[4]}\\ \overline{z}_2 & \overline{z}_2^{[1]} & \overline{z}_2^{[2]} & \overline{z}_2^{[3]} & \overline{z}_2^{[4]}\\ \overline{z}_3 & \overline{z}_3^{[1]} & \overline{z}_3^{[2]} & \overline{z}_3^{[3]} & \overline{z}_3^{[4]}\\ \overline{z}_4 & \overline{z}_4^{[1]} & \overline{z}_4^{[2]} & \overline{z}_4^{[3]} & \overline{z}_4^{[4]}\\ \overline{z}_5 & \overline{z}_5^{[1]} & \overline{z}_5^{[2]} & \overline{z}_5^{[3]} & \overline{z}_5^{[4]}\end{bmatrix} J \begin{bmatrix} f & z_2 & z_3 & z_4 & z_5\\ f^{[1]} & z_2^{[1]} & z_3^{[1]} & z_4^{[1]} & z_5^{[1]}\\ f^{[2]} & z_2^{[2]} & z_3^{[2]} & z_4^{[2]} & z_5^{[2]}\\ f^{[3]} & z_2^{[3]} & z_3^{[3]} & z_4^{[3]} & z_5^{[3]}\\ f^{[4]} & z_2^{[4]} & z_3^{[4]} & z_4^{[4]} & z_5^{[4]}\end{bmatrix}. \tag{14}$$

Therefore (13) and (14) imply that the equation

$$i[f,\overline{z}_5]=iW\{f,z_2,z_3,z_4,z_5\},\quad x\in[a,b) \tag{15}$$

holds for all $x\in[a,b)$. With a similar discussion one obtains for all $x\in[a,b)$ that

$$\begin{aligned} i[f,\overline{z}_4] &= iW\{z_1,f,z_3,z_4,z_5\},\\ [f,\overline{z}_3] &= iW\{z_1,z_2,f,z_4,z_5\},\\ -i[f,\overline{z}_2] &= iW\{z_1,z_2,z_3,f,z_5\},\\ -i[f,\overline{z}_1] &= iW\{z_1,z_2,z_3,z_4,f\}.\end{aligned} \tag{16}$$

Using the equations (15) and (16) in (12) we get on $[a,b)$ that

$$\Psi F=\begin{bmatrix} [f,\overline{z}_5]\\ [f,\overline{z}_4]\\ -i[f,\overline{z}_3]\\ -[f,\overline{z}_2]\\ -[f,\overline{z}_1]\end{bmatrix},\quad x\in[a,b). \tag{17}$$

Since the following equation holds $G^*JF=(\Psi G)^*J(\Psi F)$, we obtain from (17) that (5) has the following representation

$$[f,\overline{g}]=[f,\overline{z}_1]\overline{[g,\overline{z}_5]}-[f,\overline{z}_5]\overline{[g,\overline{z}_1]}+[f,\overline{z}_2]\overline{[g,\overline{z}_4]}-[f,\overline{z}_4]\overline{[g,\overline{z}_2]}+i[f,\overline{z}_3]\overline{[g,\overline{z}_3]}. \tag{18}$$

Now we can construct other representations of (18) as follows

$$[f,\overline{g}]=(\psi g)^*E(\psi f)+i[f,\overline{z}_3]\overline{[g,\overline{z}_3]}, \tag{19}$$

where
$$\psi f = \begin{bmatrix}[f,\overline{z}_1]\\ [f,\overline{z}_2]\\ [f,\overline{z}_4]\\ [f,\overline{z}_5]\end{bmatrix}, \quad \psi g = \begin{bmatrix}[g,\overline{z}_1]\\ [g,\overline{z}_2]\\ [g,\overline{z}_4]\\ [g,\overline{z}_5]\end{bmatrix},$$

and
$$[f,\overline{g}] = \begin{bmatrix}\overline{[g,\overline{z}_1]} & \overline{[g,\overline{z}_5]}\end{bmatrix} E_0 \begin{bmatrix}[f,\overline{z}_1]\\ [f,\overline{z}_5]\end{bmatrix} + \begin{bmatrix}\overline{[g,\overline{z}_2]} & \overline{[g,\overline{z}_4]}\end{bmatrix} E_0 \begin{bmatrix}[f,\overline{z}_2]\\ [f,\overline{z}_4]\end{bmatrix} + i[f,\overline{z}_3]\overline{[g,\overline{z}_3]}. \tag{20}$$

These representations (18)-(20) will allow us to impose general well-defined boundary conditions for the solutions of the (1).

LEMMA 3.1. *The function* $[f(\cdot,\lambda),\overline{z}_k(\cdot)](x)$, $k=1,\dots,5$, *is an entire function of* λ *of growth at most* 1 *at each point* $x\in[a,b]$.

Proof. With the help of the solutions $f(x,\lambda)$ and $z_k(x)$ of (1) and (9), respectively, we shall construct the vectors F and Z_k by the rule (4). Now consider the representation $[f(\cdot,\lambda),\overline{z}_k(\cdot)](x) = Z_k^* JF$. Taking the derivatives of both sides with respect to x we obtain that $\frac{d}{dx}[f(\cdot,\lambda),\overline{z}_k(\cdot)](x) = Z_k^{*\prime} JF + Z_k^* JF'$. Since F' and Z_k' satisfy (8) for $\lambda\in\mathbb{C}$ and $\lambda=0$, respectively, we have $\frac{d}{dx}[f(\cdot,\lambda),\overline{z}_k(\cdot)](x) = Z_k^*(N^*J+JN)F + \lambda Z_k^* JMF$. Using Lemma 2.2 we obtain that
$$\frac{d}{dx}[f(.,\lambda),\overline{z}_k(.)](x) = \lambda w \overline{z}_k f. \tag{21}$$
The definition of ΨF implies that
$$f = [f,\overline{z}_5]\,z_1 + [f,\overline{z}_4]\,z_2 - i\,[f,\overline{z}_3]\,z_3 - [f,\overline{z}_2]\,z_4 - [f,\overline{z}_1]\,z_5. \tag{22}$$
The equations (21) and (22) show that
$$\frac{d}{dx}\begin{bmatrix}[f,\overline{z}_1]\\ [f,\overline{z}_2]\\ [f,\overline{z}_3]\\ [f,\overline{z}_4]\\ [f,\overline{z}_5]\end{bmatrix} = \lambda \begin{bmatrix} -\overline{z}_1 z_5 w & -\overline{z}_1 z_4 w & -i\overline{z}_1 z_3 w & \overline{z}_1 z_2 w & \overline{z}_1 z_1 w\\ -\overline{z}_2 z_5 w & -\overline{z}_2 z_4 w & -i\overline{z}_2 z_3 w & \overline{z}_2 z_2 w & \overline{z}_2 z_1 w\\ -\overline{z}_3 z_5 w & -\overline{z}_3 z_4 w & -i\overline{z}_3 z_3 w & \overline{z}_3 z_2 w & \overline{z}_3 z_1 w\\ -\overline{z}_4 z_5 w & -\overline{z}_4 z_4 w & -i\overline{z}_4 z_3 w & \overline{z}_4 z_2 w & \overline{z}_4 z_1 w\\ -\overline{z}_5 z_5 w & -\overline{z}_5 z_4 w & -i\overline{z}_5 z_3 w & \overline{z}_5 z_2 w & \overline{z}_5 z_1 w \end{bmatrix} \times \begin{bmatrix}[f,\overline{z}_1]\\ [f,\overline{z}_2]\\ [f,\overline{z}_3]\\ [f,\overline{z}_4]\\ [f,\overline{z}_5]\end{bmatrix}$$
i.e.
$$\frac{d}{dx}\mathcal{F}(x,\lambda) = \lambda A(x)\mathcal{F}(x,\lambda), \quad x\in[a,b). \tag{23}$$
We should note that $\|A(x)\|$ is integrable on $[a,b)$. Integrating both sides of (23) on $[a,x)\subseteq[a,b)$ we get that
$$\mathcal{F}(x,\lambda) = \mathcal{F}(a,\lambda) + \lambda\int_a^x A(t)\mathcal{F}(t,\lambda)dt. \tag{24}$$
Gronwall's inequality and (24) imply that
$$\|\mathcal{F}(x,\lambda)\| \le \|\mathcal{F}(a,\lambda)\| \exp\left(|\lambda|\int_a^x \|A(t)\|\,dt\right). \tag{25}$$
The equation (25) shows that
$$\|\mathcal{F}(b,\lambda) - \mathcal{F}(b',\lambda)\| \le |\lambda|\,\|\mathcal{F}(a,\lambda)\| \left(\int_{b'}^b \|A(t)\|\,dt\right) \exp\left(|\lambda|\int_a^b \|A(t)\|\,dt\right).$$

Therefore with the aid of Lemma 2.1 we get that $\mathcal{F}(b',\lambda)\to\mathcal{F}(b,\lambda)$ (uniformly) as $b'\to b$ in any compact subset of the complex plane. This proves that $[f(\cdot,\lambda),\overline{z}_k(\cdot)](b)$, $k=1,\ldots,5$, is entire in λ.

From (25) we have

$$\|\mathcal{F}(b,\lambda)\|\leq\|\mathcal{F}(a,\lambda)\|\exp\left(|\lambda|\int_a^b\|A(t)\|\,dt\right). \tag{26}$$

On the other side, equation (8) implies that $\mathcal{F}(\xi,\lambda)=O(\exp(const.\,|\lambda|))$, for each $\xi\in[a,b)$, which together with (26) completes the proof. □

4. Construction of the boundary conditions

In this section we shall share a way to impose separated and coupled boundary conditions for the solutions of (1). Firstly we shall consider the following boundary conditions

$$\begin{aligned}\sin\beta_1 f(a)+\cos\beta_1 f^{[4]}(a)&=0,\\ \sin\beta_2 f^{[1]}(a)+\cos\beta_2 f^{[3]}(a)&=0,\\ (i+\tan\beta_3)\,f^{[2]}(a)+(1+i\tan\beta_3)\,[f,\overline{z}_3](b)&=0,\\ \sin\beta_4[f,\overline{z}_1](b)+\cos\beta_4[f,\overline{z}_5](b)&=0,\\ \sin\beta_5[f,\overline{z}_2](b)+\cos\beta_5[f,\overline{z}_4](b)&=0,\end{aligned} \tag{27}$$

where $\beta_k\in\mathbb{R}$, and $k=1,\ldots,5$. For the solutions of (1), the conditions (27) are the separated boundary conditions.

Now we shall consider the following boundary conditions

$$\begin{bmatrix}[f,\overline{z}_1](b)\\ [f,\overline{z}_5](b)\end{bmatrix}=K_1\begin{bmatrix}f(a)\\ f^{[4]}(a)\end{bmatrix},\ \begin{bmatrix}[f,\overline{z}_2](b)\\ [f,\overline{z}_4](b)\end{bmatrix}=K_2\begin{bmatrix}f^{[1]}(a)\\ f^{[3]}(a)\end{bmatrix},\ [f,\overline{z}_3](b)=\frac{i+s}{1+is}f^{[2]}(a), \tag{28}$$

where K_1,K_2 are 2×2 real matrices satisfying

$$K_1^*E_0K_1=K_2^*E_0K_2=E_0,\quad \det K_1=\det K_2=1 \tag{29}$$

and s is a real number. We can call the boundary conditions (29) as the real-coupled boundary conditions. However, we can introduce another real-coupled boundary conditions as follows

$$(\psi f)(b)=K\widehat{F}(a),\quad [f,\overline{z}_3](b)=\frac{i+s}{1+is}f^{[2]}(a). \tag{30}$$

Here K is a 4×4 real matrix satisfying

$$K^*EK=E. \tag{31}$$

Other boundary conditions can be introduced as follows

$$\begin{aligned}&\begin{bmatrix}[f,\overline{z}_1](b)\\ [f,\overline{z}_5](b)\end{bmatrix}=e^{it_1}K_1\begin{bmatrix}f(a)\\ f^{[4]}(a)\end{bmatrix},\quad \begin{bmatrix}[f,\overline{z}_2](b)\\ [f,\overline{z}_4](b)\end{bmatrix}=e^{it_2}K_2\begin{bmatrix}f^{[1]}(a)\\ f^{[3]}(a)\end{bmatrix},\\ &[f,\overline{z}_3](b)=e^{it_3}\frac{i+s}{1+is}f^{[2]}(a).\end{aligned} \tag{32}$$

Here K_1 and K_2 are the matrices satisfying (29) and t_1, t_2, t_3 are some real numbers. We call the boundary conditions (32) as complex-coupled boundary conditions. Moreover we can also introduce the following complex-coupled boundary conditions

$$(\psi f)(b) = e^{il_1} K\widehat{F}(a), \quad [f, \overline{z}_3](b) = e^{il_2} \frac{i+s}{1+is} f^{[2]}(a). \tag{33}$$

Here K is the matrix satisfying (31) and l_1, l_2 are some real numbers.

We should note that all these boundary conditions (27), (28), (30), (32), (33) can be embedded into the following abstract boundary conditions:

$$(\Psi F)(b) = A\varsigma, \quad F(a) = B\varsigma. \tag{34}$$

Here A and B are complex 5×5 matrices such that the rank of the matrix constructed by A and B is 5 and ς is a 5×1 vector.

Theorem 4.1. *All the eigenvalues of the problems generated by the* (1) *and* (27), (28), (30), (32), (33) *are discrete with infinity as a possible accumulation point. Denoting them by* μ_0, μ_1, $\mu_2, \ldots$, *we can construct a convergent series as follows* $\sum_{\mu_n \neq 0} |\mu_n|^{-1-\varepsilon}$, *where* ε *is any positive number. Moreover, the order of each eigenvalue is at most* 5.

Proof. We will prove the first assertion using a 5×5 matrix solution $\Omega(x,\lambda)$ of (8) that satisfies the initial condition $\Omega(a,\lambda) = I$, where I is the 5×5 identity matrix. Now we can introduce the following equation on $[a,b)$ for an arbitrary solution $F(x,\lambda)$ of (8) $F(x,\lambda) = \Omega(x,\lambda)F(a,\lambda)$. Using the conditions (34) we obtain the following equation

$$[A - (\Psi\Omega)(b,\lambda)B]\,\varsigma = 0 \tag{35}$$

and hence

$$\upsilon(\lambda) := \det[A - (\Psi\Omega)(b,\lambda)B] = 0. \tag{36}$$

From (36) we can infer that the eigenvalues of each boundary value problem generated by (1) and (27), (28), (30), (32), (33) coincide with the eigenvalues of $\upsilon(\lambda)$. Using Lemma 2.1 and Lemma 3.1 we obtain that all eigenvalues are discrete with the possible point of accumulation at infinity.

From Lemma 3.1 we can infer that the order of (36) is at most 1 and hence the series is convergent for each $\varepsilon > 0$.

Finally the order of each eigenvalue is at most 5 because the number of linearly independent solutions ς of (35) is at most 5. This completes the proof. □

Theorem 4.2. *The problems generated by the* (1) *and* (27), (28), (30), (32), (33) *have all real eigenvalues.*

Proof. To prove this fact we will use the following equation

$$\begin{aligned}[f;g] &= [f,\overline{z}_1](b)\overline{[g,\overline{z}_5]}(b) - [f,\overline{z}_5](b)\overline{[g,\overline{z}_1]}(b) + [f,\overline{z}_2](b)\overline{[g,\overline{z}_4]}(b)\\ &- [f,\overline{z}_4](b)\overline{[g,\overline{z}_2]}(b) + i[f,\overline{z}_3](b)\overline{[g,\overline{z}_3]}(b)\\ &- \left(f(a)\overline{g^{[4]}}(a) - f^{[4]}(a)\overline{g}(a) f^{[1]}(a)\overline{g^{[3]}}(a) - f^{[3]}(a)\overline{g^{[1]}}(a) + if^{[2]}(a)\overline{g^{[2]}}(a)\right).\end{aligned} \tag{37}$$

If f, g satisfy the conditions (27) we get that

$$[f;g] = -\cot\beta_4[f,\overline{z}_5](b)\overline{[g,\overline{z}_5]}(b) + \cot\beta_4[f,\overline{z}_5](b)\overline{[g,\overline{z}_5]}(b) - \tan\beta_5[f,\overline{z}_4](b)\overline{[g,\overline{z}_4]}(b)$$

$$+\tan\beta_5[f,\overline{z}_4](b)\overline{[g,\overline{z}_4]}(b)+i\begin{pmatrix}\frac{i+\tan\beta_3}{1+i\tan\beta_3} & \frac{-i+\tan\beta_3}{1-i\tan\beta_3}\end{pmatrix}f^{[2]}(a)\overline{g^{[2]}}(a)$$

$$-\Big(-\cot\beta_1 f^{[4]}(a)\overline{g^{[4]}}(a)+\cot\beta_1 f^{[4]}(a)\overline{g^{[4]}}(a)$$

$$-\cot\beta_2 f^{[3]}(a)\overline{g^{[3]}}(a)+\cot\beta_2 f^{[3]}(a)\overline{g^{[3]}}(a)+if^{[2]}(a)\overline{g^{[2]}}(a)\Big)=0. \tag{38}$$

From (38) we obtain that $(\tau(f),g)=(f,\tau(g))$ and this shows that the eigenvalues of (1), (27) are all real. If f,g satisfy the conditions (28) we obtain from (37) that

$$[f;g]=\overline{\begin{bmatrix}[g,\overline{z}_1](b) & \overline{[g,\overline{z}_5]}(b)\end{bmatrix}}\,E_0\begin{bmatrix}[f,\overline{z}_1](b)\\ [f,\overline{z}_5](b)\end{bmatrix}$$

$$+\begin{bmatrix}\overline{[g,\overline{z}_2]}(b) & \overline{[g,\overline{z}_4]}(b)\end{bmatrix}E_0\begin{bmatrix}[f,\overline{z}_2](b)\\ [f,\overline{z}_4](b)\end{bmatrix}+i[f,\overline{z}_3](b)\overline{[g,\overline{z}_3]}(b)$$

$$-\left(\begin{bmatrix}\overline{g}(a) & \overline{g^{[4]}}(a)\end{bmatrix}E_0\begin{bmatrix}f(a)\\ f^{[4]}(a)\end{bmatrix}+\begin{bmatrix}\overline{g^{[1]}}(a) & \overline{g^{[3]}}(a)\end{bmatrix}E_0\begin{bmatrix}f^{[1]}(a)\\ f^{[3]}(a)\end{bmatrix}+if^{[2]}(a)\overline{g^{[2]}}(a)\right)$$

$$=\begin{bmatrix}\overline{g}(a) & \overline{g^{[4]}}(a)\end{bmatrix}K_1^*E_0K_1\begin{bmatrix}f(a)\\ f^{[4]}(a)\end{bmatrix}\begin{bmatrix}\overline{g^{[1]}}(a) & \overline{g^{[3]}}(a)\end{bmatrix}K_2^*E_0K_2\begin{bmatrix}f^{[1]}(a)\\ f^{[3]}(a)\end{bmatrix}$$

$$+i\begin{pmatrix}\frac{i+s}{1+is} & \frac{-i+s}{1-is}\end{pmatrix}f^{[2]}(a)\overline{g^{[2]}}(a)$$

$$-\left(\begin{bmatrix}\overline{g}(a) & \overline{g^{[4]}}(a)\end{bmatrix}E_0\begin{bmatrix}f(a)\\ f^{[4]}(a)\end{bmatrix}+\begin{bmatrix}\overline{g^{[1]}}(a) & \overline{g^{[3]}}(a)\end{bmatrix}E_0\begin{bmatrix}f^{[1]}(a)\\ f^{[3]}(a)\end{bmatrix}+if^{[2]}(a)\overline{g^{[2]}}(a)\right)=0.$$

Therefore, $(\tau(f),g)=(f,\tau(g))$ and hence all eigenvalues of (1), (28) are real.

If f,g satisfy the conditions (29) we obtain from (37) that

$$[f;g]=(\psi g)^*(b)E(\psi f)(b)+i[f,\overline{z}_3](b)\overline{[g,\overline{z}_3]}(b)-\Big(\widehat{G}^*(a)E\widehat{F}(a)+if^{[2]}(a)\overline{g^{[2]}}(a)\Big)$$

$$=\widehat{G}^*(a)K^*EK\widehat{F}(a)i\begin{pmatrix}\frac{i+s}{1+is} & \frac{-i+s}{1-is}\end{pmatrix}f^{[2]}(a)\overline{g^{[2]}}(a)-\Big(\widehat{G}^*(a)E\widehat{F}(a)+if^{[2]}(a)\overline{g^{[2]}}(a)\Big)=0.$$

Therefore, we have $(\tau(f),g)=(f,\tau(g))$ and hence all eigenvalues of (1), (29) are real.

The other assertions can be proved in a similar way. Therefore the proof is completed. □

5. Conclusion and remarks

In this paper we have considered a singular fifth-order differential equation in lim-5 case at the singular point together with some suitable well-defined boundary conditions. Then we have investigated some properties of the solutions and eigenvalues of the corresponding problems. This paper can be regarded as the generalization of the results of [6] to the fifth-order case. However, this generalization, as can be seen, is not a straightforward generalization. For instance, the representations of the bilinear concomitant have different and complicated forms that give rise to different boundary conditions. Now the future step is to generalize the obtained results to arbitrary odd-order singular boundary value problems.

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