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OPEN-POINT AND BI-POINT OPEN TOPOLOGIES ON CONTINUOUS FUNCTIONS BETWEEN TOPOLOGICAL (SPACES) GROUPS

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Abstract. In this paper, we study the notions of point-open topology $C_p(X,H)$, open-point topology $C_h(X,H)$ [resp. $C_h(G,H)$] and bi-point-open topology $C_{ph}(X,H)$ [resp. $C_{ph}(G,H)$] on $C(X,H)$ [resp.$C(G,H)$], the set of all continuous functions from a topological space X (topological group G) to a topological group H. In this setting, we study the countability, separation axioms and metrizability. The equivalent conditions are given so that the space $C_h(G,H)$ is a zero-dimensional topological group. Further, if G is $H^{\star\star}$-regular, then $C_h(G,H)$ is Hausdorff if and only if G is discrete. It is shown that under certain conditions the topological groups $C_p(X,H)$, $C_h(X,H)$ and $C_{ph}(X,H)$ are ω-narrow. Sufficient conditions are given for the topological spaces $C_p(X,H)$, $C_h(X,H)$ and $C_{ph}(X,H)$ to be discretely selective and to have a disjoint shrinking.

1. Introduction

The space of real-valued continuous functions are studied extensively in the literature on topological spaces [1–3, 5, 6, 11]. Let X be any topological space and G, H be any topological groups. Then $C(X,H)$ denotes the group of all continuous functions from X to H, equipped with the " pointwise group operations". That is, the product of $f \in C(X,H)$ and $g \in C(X,H)$ is the function $fg \in C(X,H)$ defined by $fg(x) = f(x)g(x)$ for all $x \in X$, and the inverse element of f is the function $h \in C(X,H)$ defined by $h(x) = (f(x))^{-1}$ for all $x \in X$. The space $C(X,H)$ with the point-open (or pointwise convergence) topology is denoted by $C_p(X,H)$ and was studied by Shakhmatov and Spěvák [14]. It has a subbase consisting of sets of the form $[x,V]^+ = \{f \in C(X,H) : f(x) \in V\}$, where $x \in X$ and V is an open subset of H. We obtain further properties of $C_p(X,H)$.

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Jindal, McCoy and Kundu [8] introduced two new topologies on $C(X,\mathbb{R})$, namely the open-point topology and the bi-point-open topology. These two topologies on $C(X,\mathbb{R})$ have been studied in [9, 10, 13]. We study these two topologies on $C(X,H)$ [resp. $C(G,H)$], the set of all continuous functions from a topological space X (topological group G) to a topological group H. The space $C(X,H)$ with the open-point topology h is denoted by $C_h(X,H)$. It has a subbase consisting of sets of the form $[U,r]^- = \{f \in C(X,H) : f^{-1}(r) \cap U \neq \emptyset\}$, where $r \in H$ and U is an open subset of X. When H is the real line, the subbasis for the open-point topology in [8] turns out to be the same as the open-point topology on $C(X,H)$ above.

Now the bi-point-open topology on $C(X,H)$ is obtained by joining of point-open topology and the open-point topology on $C(X,H)$. In other words, it is the topology having subbasic open sets of both kinds: $[x,V]^+$ and $[U,r]^-$, where $x \in X$, V is an open subset of H, U is an open subset of X and $r \in H$. The bi-point-open topology on the space $C(X,H)$ is denoted by ph and the space $C(X,H)$ equipped with the bi-point-open topology ph is denoted by $C_{ph}(X,H)$. One can also view the bi-point-open topology on $C(X,H)$ as the weak topology on $C(X,H)$ generated by the identity maps $id_1 : C(X,H) \to C_p(X,H)$ and $id_2 : C(X,H) \to C_h(X,H)$.

The behaviors of the spaces $C_h(X,H)$ and $C_{ph}(X,H)$ may be quite different from the behaviors of $C_h(X,\mathbb{R})$ with open-point topology and $C_{ph}(X,\mathbb{R})$ with the bi-point-open topology, for instance, $C_h(X,\mathbb{R})$ is never Lindelöf nor second countable. In contrast to [8], it is shown that under some conditions $C_h(X,H)$ is neither Lindelöf nor second countable (see Theorem 4.8 and Corollary 4.9) and under certain other conditions it may be second countable space (see Theorem 4.11). In Section 3, we give a characterization for $C_h(G,H)$ to be regular and Hausdorff. It is shown that if G is $H^{\star\star}$-regular, then some equivalent conditions are given so that the space $C_h(G,H)$ is a zero-dimensional topological group and consequently, it follows that three types of regularity ($H^{\star\star}$-regularity, $H^{\star}$-regularity and H-regularity) coincide on the space $C_h(G,H)$. We show how the topological property, namely, zero-dimensionality of $C_p(X,H)$ depends on those of H. In Section 4, we study properties like countability and metrizability. Also it is found that if H is an ω-narrow topological group, then $C_p(X,H)$ is an ω-narrow topological group. Further, it is shown that if X is discrete and H is countable, the topological groups $C_h(X,H)$ and $C_{ph}(X,H)$ are ω-narrow. In Section 5, we give sufficient conditions for topological spaces $C_p(X,H)$, $C_h(X,H)$ and $C_{ph}(X,H)$ to be discretely selective and to have a disjoint shrinking. In the final Section 6, we give some properties of the restriction map.

In notation and the terminology, we follow [6] if not stated otherwise. All topological spaces are assumed to be Tychonoff (T_1+completely regular) and all topological groups are assumed to be Hausdorff. $\mathbb{N}$ denotes the set of all natural numbers and $\omega = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the additive group of reals with its usual topology. H_d denotes the group H with discrete topology. The identity elements of group G and H are denoted by e and $\tilde{e}$, respectively. A^c denotes the complement of A in a space. The letters i, j, k, l, m, n denote natural numbers. The symbols $\mathcal{V}_e$ and $\mathcal{V}_{\tilde{e}}$ denote the neighborhood basis at e and $\tilde{e}$ in G and H, respectively.

2. Preliminaries

In this section, a basis is obtained for each of the spaces discussed above. These bases are useful in establishing many properties of these spaces.

PROPOSITION 2.1 ([3]). *Let $\mathcal{B}$ be a basis of a topological group H. Then the collection $\mathcal{A} = \{[x_1, B_1]^+ \cap \ldots \cap [x_n, B_n]^+ : n \in \mathbb{N},\ x_i \in X,\ B_i \in \mathcal{B}\}$ is a basis of the space $C_p(X, H)$.*

PROPOSITION 2.2. *Let $\mathcal{B}$ be a basis of a space X. Then the collection $\mathcal{A} = \{[B_1, r_1]^- \cap \ldots \cap [B_n, r_n]^- : n \in \mathbb{N},\ r_i \in H,\ B_i \in \mathcal{B}\}$ is a basis of the space $C_h(X, H)$.*

PROPOSITION 2.3. *Let $\mathcal{B}_X$ and $\mathcal{B}_H$ be bases of space X and topological group H, respectively. Then the collection $\mathcal{A} = \{[x_1, B_1]^+ \cap \ldots \cap [x_n, B_n]^+ \cap [V_1, r_1]^- \cap \ldots \cap [V_m, r_m]^- : x_i \in X, r_j \in H,\ B_i \in \mathcal{B}_H$ and $V_j \in \mathcal{B}_X,\ 1 \le i \le n,\ 1 \le j \le m\}$ is a basis of the space $C_{ph}(X, H)$.*

The following two results are proved similarly to [8, Proposition 2.1 and 2.2].

THEOREM 2.4. *For each $f \in C(G, H)$ and any open set A in $C_h(G, H)$ containing f, there exist distinct points $y_1, \ldots, y_m$ in G, points $r_1, \ldots, r_m$ in H and $U \in \mathcal{V}_e$ such that $f \in [y_1U, r_1]^- \cap \ldots \cap [y_mU, r_m]^- \subseteq A$, where $r_i = f(y_i)$ for each $i = 1, 2, \ldots, m$ and for $i \neq j$, $\overline{y_iU} \cap \overline{y_jU} = \emptyset$.*

Proof. Let A be any open set in $C_h(G, H)$ containing f. Then there exists a basic open set $B = [V_1, r_1]^- \cap \ldots \cap [V_n, r_n]^-$ such that $f \in B \subseteq A$, where $r_i \in H$ and each V_i is an open set in G. So there exist $y_i \in V_i$ such that $f(y_i) = r_i$ for $1 \le i \le n$. If for some $1 \le i < j \le n$, $y_i = y_j$, then $r_i = r_j$ and $y_j \in V_i \cap V_j \neq \emptyset$. As $V_i \cap V_j$ is open in G containing y_j, there exists a $W \in \mathcal{V}_e$ such that $y_jW \subseteq V_i \cap V_j$. So $f \in [y_jW, r_j]^- \subseteq [V_i, r_i]^- \cap [V_j, r_j]^-$. Take $B_1 = [V_1, r_1]^- \cap \ldots \cap [V_{i-1}, r_{i-1}]^- \cap [V_{i+1}, r_{i+1}]^- \cap \ldots \cap [V_{j-1}, r_{j-1}]^- \cap [V_{j+1}, r_{j+1}]^- \cap \ldots \cap [V_n, r_n]^- \cap [y_jW, r_j]^-$. Clearly, $f \in B_1 \subseteq B$. By proceeding in this way, we get a basic open set $B_2 = [U_1, r_1]^- \cap \ldots \cap [U_m, r_m]^-$ such that $m \le n$, $f \in B_2 \subseteq A$, and for each $1 \le i \le m$, there exist $y_i \in U_i$ with $f(y_i) = r_i$ and these y_i are distinct. Since G is Hausdorff, there exist open sets D_i and D_j, $D_i \cap D_j = \emptyset$ for $i \neq j$ such that $y_i \in \tilde{D_i} = D_i \cap U_i$ for all i. Then $e \in y_1^{-1}\tilde{D_1} \cap \ldots \cap y_m^{-1}\tilde{D_m} = W_1$. Since G is regular, there exists a U in $\mathcal{V}_e$ such that $e \in U \subseteq \overline{U} \subseteq W_1$. Hence, $f \in [y_1U, r_1]^- \cap \ldots \cap [y_mU, r_m]^- \subseteq A$ such that $\overline{y_iU} \cap \overline{y_jU} = \emptyset$ as $\tilde{D_i} \cap \tilde{D_j} = \emptyset$. □

THEOREM 2.5. *For each $f \in C(X, H)$ and any open set A in $C_p(X, H)$ containing f, there exist distinct points $x_1, \ldots, x_m$ in X, points $z_1, \ldots, z_m$ in H and $W \in \mathcal{V}_{\tilde{e}}$ such that $f \in [x_1, z_1W]^+ \cap \ldots \cap [x_m, z_mW]^+ \subseteq A$, where $z_i = f(x_i)$ for each $i = 1, 2, \ldots, m$.*

Proof. Let A be any open set in $C_p(X, H)$ containing f. Then there exists a basic open set $B = [x_1, V_1]^+ \cap \ldots \cap [x_n, V_n]^+$ in $C_p(X, H)$ such that $f \in B \subseteq A$. So $f(x_i) \in V_i$. We know tha for any $x \in X$ and open sets U_1 and U_2 in H, $[x, U_1]^+ \cap [x, U_2]^+ = [x, U_1 \cap U_2]^+$. Therefore, if for some $1 \le i < j \le n$, $x_i = x_j$, then

$f \in [x_1, V_1]^+ \cap \ldots \cap [x_{i-1}, V_{i-1}]^+ \cap [x_{i+1}, V_{i+1}]^+ \cap \ldots \cap [x_{j-1}, V_{j-1}]^+ \cap [x_{j+1}, V_{j+1}]^+ \cap \ldots \cap [x_n, V_n]^+ \cap [x_i, V_i \cap V_j]^+ \subseteq B$. By proceeding in this way, we get a basic open set $B_1 = [x_1, V_1]^+ \cap \ldots \cap [x_m, V_m]^+$ such that $m \leq n$, $f \in B_1 \subseteq A$ and $x_i \in X$ are distinct. Now $f \in B_1$ implies that $f(x_i) = z_i \in V_i$ for each $1 \leq i \leq m$. As V_i is open in H containing z_i, there exist $W_i \in \mathcal{V}_{\tilde{e}}$ such that $z_i W_i \subseteq V_i$. Take $W \in \mathcal{V}_{\tilde{e}}$ such that $W \subseteq W_1 \cap \ldots \cap W_m$. Clearly, $z_i W \subseteq V_i$ for each $1 \leq i \leq m$ which implies that $[x_i, z_i W]^+ \subseteq [x_i, V_i]^+$. So $f \in [x_1, z_1 W]^+ \cap \ldots \cap [x_m, z_m W]^+ \subseteq A$, where $x_i \in X$ are distinct, $z_i \in H$ and $W \in \mathcal{V}_{\tilde{e}}$. □

Theorem 2.6. *For each $f \in C(G, H)$ and any open set A in $C_{ph}(G, H)$ containing f, there exist distinct points $x_1, \ldots, x_m$ in G, distinct points $y_1, \ldots, y_n$ in G, points $r_1, \ldots, r_n, z_1, \ldots, z_m$ in H, $U \in \mathcal{V}_e$ and $W \in \mathcal{V}_{\tilde{e}}$ such that $f \in [x_1, z_1 W]^+ \cap \ldots \cap [x_m, z_m W]^+ \cap [y_1 U, r_1]^- \cap \ldots \cap [y_n U, r_n]^- \subseteq A$, where $z_i = f(x_i)$ for each $1 \leq i \leq m$, $r_j = f(y_j)$ for each $1 \leq j \leq n$ and whenever $i \neq k$, $\overline{y_i U} \cap \overline{y_k U} = \emptyset$.*

3. A characterization of zero-dimensional topological group

First, we show that the space $C_h(X, H)$ is always T_1, which generalizes [8, Proposition 3.1].

Theorem 3.1. *For any space X and topological group H, $C_h(X, H)$ is a T_1 space.*

Proof. Let $f \in C(X, H)$ be arbitrary and $g \in C(X, H) \setminus \{f\}$. Then there exists a point $x \in X$ such that $f(x) \neq g(x)$. Since $\{g(x)\}$ is closed in H, the set $F = f^{-1}\{g(x)\}$ is closed in X. Therefore, $U = F^c$ is open in X. This implies that $[U, g(x)]^-$ is an open set in $C_h(X, H)$ such that $g \in [U, g(x)]^- \subseteq C(X, H) \setminus \{f\}$. Thus, $C(X, H) \setminus \{f\}$ is open in $C_h(X, H)$ for any $f \in C(X, H)$. Therefore, $\{f\}$ is closed in $C_h(X, H)$ for any $f \in C(X, H)$. Hence, $C_h(X, H)$ is T_1. □

Since arbitrary product of Tychonoff spaces is Tychonoff [6, Theorem 2.3.11] and a subspace of a Tychonoff space is Tychonoff [6, Theorem 2.1.6], the proof of the following theorem is immediate.

Theorem 3.2. *For any space X and topological group H, $C_p(X, H)$ is a Tychonoff space.*

Theorem 3.3. *For any space X and topological group H, $C_{ph}(X, H)$ is a T_2 space.*

Now we recall some definitions and prove a lemma. Then we show that under some conditions, $C_h(G, H)$ is a T_2 space if and only if G is discrete.

Definition 3.4 ([14]). Given a non-trivial topological group H, a topological space X is called
(i) H-regular if for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, there exist an $f \in C(X, H)$ and a point $g \in H \setminus \{\tilde{e}\}$ such that $f(x) = g$ and $f(F) \subseteq \{\tilde{e}\}$.

(ii) $H^{\star}$-regular if there exists a point $g \in H \setminus \{\tilde{e}\}$ such that for every closed set $F \subseteq X$ and each point $x \in X \setminus F$, there exists an $f \in C(X, H)$ such that $f(x) = g$ and $f(F) \subseteq \{\tilde{e}\}$.

(iii) $H^{\star\star}$-regular provided that, whenever F is a closed subset of X, $x \in X \setminus F$ and $g \in H$, there exists an $f \in C(X, H)$ such that $f(x) = g$ and $f(F) \subseteq \{\tilde{e}\}$.

It is clear that X is $H^{\star\star}$-regular $\implies$ X is $H^{\star}$-regular $\implies$ X is H-regular. When X is a topological group, the terms from Definition 3.4 remain the same as for topological space X.

Theorem 3.5. *Every H-regular topological space X is completely regular.*

Proof. Let X be an H-regular topological space. Let $x \in X$ be arbitrary and F be any closed set in X not containing x. Then there exist an $f \in C(X, H)$ and a $g \in H \setminus \{\tilde{e}\}$ such that $f(x) = g$ and $f(F) \subseteq \{\tilde{e}\}$. Since $g \notin \{\tilde{e}\}$, there exists an $h \in C(H, \mathbb{R})$ such that $h(g) = 1$ and $h\{\tilde{e}\} = \{0\}$. Then $h \circ f : X \to \mathbb{R}$ is a continuous function such that $(h \circ f)(x) = 1$ and $(h \circ f)(F) = \{0\}$. Hence, X is completely regular. □

Theorem 3.6 ([14, Proposition 2.3]). *Let X be a topological space and H be a non-trivial topological group. Then the following statements hold.*
(i) If H is pathwise connected, then X is $H^{\star\star}$-regular.

(ii) If H contains a homeomorphic copy of the unit interval $[0, 1]$, then X is $H^{\star}$-regular.

(iii) If X is zero-dimensional in the sense of ind, then X is $H^{\star\star}$-regular.

In particular, in all three cases, X is H-regular.

Lemma 3.7. *For any $H^{\star\star}$-regular space X, given distinct points $g_1, \ldots, g_n \in X$ and (not necessarily distinct) points $h_1, \ldots, h_n \in H$, there exists a function $f \in C(X, H)$ such that $f(g_i) = h_i$ for all $i = 1, 2, \ldots, n$.*

Proof. If $n = 1$, then a constant function serves the purpose. So let $n \in \mathbb{N}$ and $n \geq 2$. If $Y = \{g_1, \ldots, g_n\}$, then for every $i \leq n$, the set $F_i = Y \setminus \{g_i\}$ is closed in X and does not contain g_i. Since X is $H^{\star\star}$-regular, there exists $f_i \in C(X, H)$ such that $f_i(g_i) = h_i$ and $f_i(F_i) = \{\tilde{e}\}$. Clearly, $f = f_1 f_2 \ldots f_n$ is a continuous function from X to H such that $f(g_i) = h_i$ for all $i = 1, 2, \ldots, n$. □

Theorem 3.8. *Let G be a $H^{\star\star}$-regular topological group and K be any non-trivial topological group. Then the following statements are equivalent:*
(i) $\{e\}$ is open in G.

(ii) $C_h(G, H)$ is a zero-dimensional topological group.

(iii) $C_h(G, H)$ is $K^{\star\star}$-regular.

(iv) $C_h(G, H)$ is $K^{\star}$-regular.

(v) $C_h(G, H)$ is K-regular.

(vi) $C_h(G, H)$ is completely regular.

(vii) $C_h(G, H)$ is regular.

(viii) $C_h(G, H)$ is Hausdorff.

Proof. $(i) \Rightarrow (ii)$ First we will prove that for a discrete topological group G and an arbitrary topological group H, the space $C_h(G,H) = H_d^G$, where H_d is the group H endowed with discrete topology. Since G is discrete, every H-valued function on G is continuous, so $C(G,H) = H^G$. For arbitrary $g \in G$, $h \in H$, we have $[\{g\},h]^- = \{f \in C(G,H) = H^G : \{g\} \cap f^{-1}(h) \neq \emptyset\} = \{f \in C(G,H) = H^G : f(g) = h\}$. Since $[\{g\},h]^-$ is a subbasic open set in $C_h(G,H)$ and $\{f \in C(G,H) = H^G : f(g) = h\}$ is a subbasic open set in H_d^G. Thus $C_h(G,H) = H_d^G$. Now since H_d^G is always zero-dimensional and arbitrary product of topological groups is a topological group, so $C_h(G,H)$ is a zero-dimensional topological group.
$(ii) \Rightarrow (iii)$ Since $C_h(G,H)$ is zero-dimensional, Theorem 3.6 (iii) implies that $C_h(G,H)$ is $K^{\star\star}$-regular.
$(iii) \Rightarrow (iv) \Rightarrow (v)$ Every $K^{\star\star}$-regular ($K^\star$-regular) space is $K^\star$-regular (K-regular).
$(v) \Rightarrow (vi)$ By Theorem 3.5.
$(vi) \Rightarrow (vii) \Rightarrow (viii)$ Obvious.
$(viii) \Rightarrow (i)$ Suppose that $\{e\}$ is not open in G. This implies that no finite subset of G is open in G. Also, $\overline{G \setminus \{e\}} = G$. Let U be an open set in G containing e and $y \in U$ be such that $y \neq e$. Then $F = G \setminus U$ is a closed subset of G not containing y. Since G is $H^{\star\star}$-regular, there exist $f, g \in C(G,H)$ such that $f(x) = g(x)$ for all $x \in F$ and $f(y) \neq g(y)$. Since $C_h(G,H)$ is a Hausdorff space, there exist disjoint basic open sets $A = [x_1W_1, t_1]^- \cap \ldots \cap [x_lW_l, t_l]^-$ and $B = [y_1V_1, r_1]^- \cap \ldots \cap [y_kV_k, r_k]^-$ in $C_h(G,H)$ containing f and g, respectively. There exist $a_i \in x_iW_i$ and $b_i \in y_iV_i$ such that $f(a_i) = t_i$ and $g(b_i) = r_i$, respectively. By Theorem 2.4, there exist $W_1, W_2 \in \mathcal{V}_e$ and $n \leq l$, $m \leq k$ such that $f \in A_1 = [a_1W_1, t_1]^- \cap \ldots \cap [a_nW_1, t_n]^- \subseteq A$ and $g \in B_1 = [b_1W_2, r_1]^- \cap \ldots \cap [b_mW_2, r_m]^- \subseteq B$. Take $W = W_1 \cap W_2$ so that $f \in A_2 = [a_1W, t_1]^- \cap \ldots \cap [a_nW, t_n]^- \subseteq A$ and $g \in B_2 = [b_1W, r_1]^- \cap \ldots \cap [b_mW, r_m]^- \subseteq B$. Since W is an infinite set, we can choose distinct points $w_i \in a_iW$ and $z_j \in b_jW$ such that $\{w_i : i = 1, \ldots, n\} \cap \{z_j : j = 1, \ldots, m\} = \emptyset$. By Lemma 3.7, there exists $h \in C(G,H)$ such that $h(w_i) = t_i$ and $h(z_j) = r_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. So we arrived at a contradiction. Hence, $\{e\}$ is open in G. □

In the above theorem, the implication "$(i) \Rightarrow (ii)$" in fact contains a generalized version of [10, Lemma 3.7].

Corollary 3.9. *If a topological space X is discrete, then $C_{ph}(X,H)$ is a topological group.*

Proof. Since X is discrete, $C_h(X,H)$ is a topological group. But $C_p(X,H)$ is always a topological group, so $C_{ph}(X,H)$ is a topological group. □

Theorem 3.10. *If H is a zero-dimensional topological group, then the space $C_p(X,H)$ is zero-dimensional.*

Proof. Since H is zero-dimensional, Tychonoff product H^X is zero-dimensional. This implies that $C_p(X,H)$ is zero-dimensional being a subspace of the zero-dimensional space H^X. □

Corollary 3.11. *Let $\{H_i : i \in I\}$, where I is any index set, be a family of zero-dimensional spaces, then the space $\prod_{i \in I} C_p(X, H_i)$ is zero-dimensional.*

Proof. Since arbitrary product of zero-dimensional spaces is zero-dimensional, the proof follows from Theorem 3.10. □

Corollary 3.12. *$C_p(X, H)$ is zero-dimensional for any countable topological group H.*

Proof. Since every countable regular space is zero-dimensional, the proof follows from Theorem 3.10. □

Corollary 3.13. *If H is discrete, then $C_p(X, H/K)$ is zero-dimensional for any space X and any subgroup K of H.*

Proof. By [4, Theorem 3.1.14], if H is a locally compact totally disconnected topological group and K is a closed subgroup of H, then the quotient space H/K is zero-dimensional, the proof now follows from Theorem 3.10. □

Definition 3.14. A T_1 topological space X is said to be S-normal if for any two non-empty sets A and B with $\overline{A} \cap B = \emptyset$ or $A \cap \overline{B} = \emptyset$ there exist disjoint open sets U and V containing A and B, respectively.

Example 3.15. Every discrete space is S-normal.

Theorem 3.16. *If the topological group H is S-normal, then the space $C_p(X, H)$ is zero-dimensional.*

Proof. The collection $\mathcal{A} = \{[x_1, V_1]^+ \cap \ldots \cap [x_m, V_m]^+ : m \in \mathbb{N},\ x_i \in X$, each V_i is an open subset of $H\}$ is a base for the space $C_p(X, H)$. Let $f \in ([x_i, V_i]^+)^c$. Then $f(x_i) = z_i \in V_i^c$. Therefore, $\overline{\{z_i\}} \cap V_i = \emptyset$. As H is S-normal, there exist disjoint open sets A_i and B_i containing z_i and V_i, respectively. Then $f \in [x_i, A_i]^+ \subseteq ([x_i, B_i]^+)^c \subseteq ([x_i, V_i]^+)^c$. Hence, the space $C_p(X, H)$ is zero-dimensional. □

4. Metrizability and countability

In [12], it is proved that the space $C_p(X, Y)$ is first countable if and only if X is a countable and Y is a first countable space (see [12, Corollary 1.5(a)]), where X is a completely regular space and Y contains a non trivial path. In the following theorem, we give sufficient conditions for space $C_h(G, H)$ to be first countable, where G and H are topological groups.

Theorem 4.1. *Let the countable topological group G be such that G is first countable. Then $C_h(G, H)$ is also first countable.*

Proof. Let $f \in C(G, H)$ be arbitrary and A be any open set in $C_h(G, H)$ containing f. Then there exist $y_i \in H$ and open sets W_i in G such that $f \in [W_1, y_1]^- \cap \ldots \cap [W_m, y_m]^- \subseteq A$. So there exist $z_i \in W_i$ such that $f(z_i) = y_i$. By Theorem 2.4, there exist $W \in \mathcal{V}_e$ and $n \leq m$ such that $f \in [z_1W, y_1]^- \cap \ldots \cap [z_nW, y_n]^- \subseteq A$. Without loss of generality, we assume $\mathcal{V}_e$ to be countable as G to be first countable. Clearly, the collection $\mathcal{A} = \{[z_1W, y_1]^- \cap \ldots \cap [z_nW, y_n]^- : z_i \in G,\ y_i = f(z_i),\ W \in \mathcal{V}_e\}$ is a countable neighborhood basis for f. Hence, $C_h(G, H)$ is first countable. □

The proof of the following theorem is similar to [8, Theorem 5.2].

THEOREM 4.2. *If $C_h(X,H)$ is separable and H is uncountable, then every non-empty open subset of X is uncountable.*

COROLLARY 4.3. *If $C_h(X,H)$ is separable and H is uncountable, then X is dense in itself.*

COROLLARY 4.4. *If $C_{ph}(X,H)$ is separable and H is uncountable, then every non-empty open subset of X is uncountable.*

Proof. Since the space $C_{ph}(X,H)$ is finer than the space $C_h(X,H)$, the proof follows from Theorem 4.2. □

COROLLARY 4.5. *If $C_{ph}(X,H)$ is separable and H is uncountable, then X is dense in itself.*

COROLLARY 4.6. *If G is $H^{\star\star}$-regular, $C_h(G,H)$ is separable and H is uncountable, then $C_h(G,H)$ is not metrizable.*

Proof. By Corollary 4.3 and Theorem 3.8, $C_h(G,H)$ is not a Hausdorff space. Hence, $C_h(G,H)$ is not metrizable. □

COROLLARY 4.7. *If X is countable and H is uncountable, then $C_h(X,H)$ is not separable.*

Proof. If $C_h(X,H)$ is separable, then, by Theorem 4.2, every non-empty open subset of X is uncountable. But X is countable. Hence, $C_h(X,H)$ is not separable. □

In [8], it is shown that spaces $C_h(X)$ and $C_{ph}(X)$ are never Lindelöf or second countable. In the following results we show that this is in fact true for spaces $C_h(X,H)$ and $C_{ph}(X,H)$ for any uncountable topological group H.

THEOREM 4.8. *Let H be an uncountable topological group. Then the space $C_h(X,H)$ is not Lindelöf.*

Proof. It is enough to prove that there exists an uncountable closed and discrete subset of $C_h(X,H)$. Let $\mathcal{A}$ be the set of all constant functions in $C(X,H)$. Let $f \in C(X,H)$ be arbitrary and $x \in X$. Then $U_f = [X, f(x)]^-$ is an open neighborhood of f whose intersection with $\mathcal{A}$ is finite. Thus, $\mathcal{A}$ is an uncountable closed and discrete subset of $C_h(X,H)$. Hence, $C_h(X,H)$ is not Lindelöf. □

COROLLARY 4.9. *Let H be an uncountable topological group. Then the space $C_h(X,H)$ is not second countable.*

COROLLARY 4.10. *Let H be an uncountable topological group. Then the space $C_{ph}(X,H)$ is neither Lindelöf nor second countable.*

Note that Corollary 4.9 gives a necessary condition for a second countable space $C_h(X,H)$. Now, in the next theorem, we give sufficient conditions for $C_h(X,H)$ to be second countable.

Theorem 4.11. *Let X be a countable discrete space and H be countable. Then $C_h(X,H)$ is second countable.*

Proof. Since X is discrete, $C_h(X,H) = H_d^X$, where H_d is the group H with discrete topology. Since H_d is countable and discrete, H_d is second countable. Also X is countable, so $C_h(X,H) = H_d^X$ is second countable. □

The proof of the first of next two theorems is similar to [16, Theorem S.171.], while the second is similar to [8, Proposition 4.5].

Theorem 4.12. *Let X be a $H^{\star\star}$-regular space such that $C_p(X,H)$ has a countable π-base at some of its points, then X is countable.*

Theorem 4.13. *Let f be the identity element of $C_h(X,H)$. If f has countable pseudocharacter in $C_h(X,H)$ and X is H-regular. Then X has a countable π-base.*

Theorem 4.14. *For any discrete space X, the following statements are equivalent:*

(i) $C_h(X,H)$ is metrizable.

(ii) $C_h(X,H)$ is first countable.

(iii) X has a countable π-base.

(iv) X is countable.

Proof. $(i) \Leftrightarrow (ii)$ Since X is discrete, $C_h(X,H) = H_d^X$ is a topological group. In a topological group, first countability is equivalent to metrizability.
$(ii) \Leftrightarrow (iii)$ Since $C_h(X,H)$ is first countable and T_1 space, $C_h(X,H)$ has a countable pseudocharacter. So X has a countable π-base by Theorem 4.13.
$(iii) \Leftrightarrow (iv)$ The proof is obvious.
$(iv) \Leftrightarrow (i)$ Since X is discrete, $C_h(X,H) = H_d^X$, where H_d is the group H with discrete topology. Since H_d is metrizable and X is countable, H_d^X is metrizable. □

Corollary 4.15. *For $H^{\star\star}$-regular topological group G, $C_h(G,H)$ is metrizable if and only if G is countable and discrete.*

Proof. Since $C_h(G,H)$ is metrizable, $C_h(G,H)$ is regular. So by Theorem 3.8, G is discrete. The proof follows by Theorem 4.14. □

Recall that a semitopological group G is called *ω-narrow* if for every open neighborhood V of the neutral element e in G, there exists a countable subset A of G such that $AV = VA = G$ (see [4, Section 2.3]).

Theorem 4.16. *For an ω-narrow topological group H, $C_p(X,H)$ is ω-narrow.*

Proof. Since the Tychonoff product of an arbitrary family of ω-narrow topological groups is an ω-narrow topological group [4, Proposition 3.4.3], H^X is ω-narrow. Also $C_p(X,H)$ is a topological subgroup of H^X. So $C_p(X,H)$ is ω-narrow as subgroup of an ω-narrow topological group [4, Theorem 3.4.4]. □

The next three corollaries follow directly from Theorem 4.16 and Propositions 3.4.5, 3.4.8 and 3.4.3 from [4], respectively.

Corollary 4.17. *For an ω-narrow topological group H, $C_p(X,H)$ is first countable if and only if it is second countable.*

Corollary 4.18. *For a separable topological group H, $C_p(X,H)$ is ω-narrow.*

Corollary 4.19. *Let $\{H_i : i \in I\}$, where I is any index set, be a family of ω-narrow topological groups. Then $\prod_{i\in I} C_p(X,H_i)$ is ω-narrow.*

Theorem 4.20. *If X is discrete and H is countable, then $C_h(X,H)$ is an ω-narrow topological group.*

Proof. Since X is discrete, $C_h(X,H) = H_d^X$ is a topological group, where H_d is the group H with discrete topology. We know that every countable topological group is ω-narrow and arbitrary product of ω-narrow topological group is ω-narrow. So, $C_h(X,H) = H_d^X$ is an ω-narrow topological group. □

Theorem 4.21. *If X is a discrete space, then $C_h(X,H) = C_{ph}(X,H)$.*

Proof. The proof is similar to [10, Proposition 3.6]. □

Corollary 4.22. *If X is discrete and H is countable, then $C_{ph}(X,H)$ is an ω-narrow topological group.*

Proof. The proof follows from Theorem 4.20 and Theorem 4.21. □

5. Discrete selection and disjoint shrinking

Definition 5.1 ([7]). A topological space X is called a P-space if every G_δ set of X is open.

Definition 5.2 ([15]). A space X is discretely selective if for any sequence $\mu = \{U_n : n \in \omega\}$ of non-empty open subsets of X, there exists a closed discrete set $D = \{x_n : n \in \omega\}$ such that $x_n \in U_n$ for each $n \in \omega$. The set D will be called a selection for the family μ.

In [15], it is proved that $C_p(X)$ is discretely selective if and only if the space X is uncountable. The following theorems give sufficient conditions for spaces $C_p(X,H)$, $C_h(X,H)$ and $C_{ph}(X,H)$ to be discretely selective.

Theorem 5.3. *Let H be an uncountable P-group and X be an uncountable $H^{\star\star}$-regular space. Then $C_p(X,H)$ is discretely selective.*

Proof. Let $\{U_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_p(X,H)$. For each $n \in \omega$, let $f_n \in U_n$. Then for each $n \in \omega$, there exists a non-empty basic open set $A_n = [y_1^n, V_1^n]^+ \cap \ldots \cap [y_{l_n}^n, V_{l_n}^n]^+$ such that $f_n \in A_n \subseteq U_n$, where $y_1^n, \ldots, y_{l_n}^n \in X$, $V_1^n, \ldots, V_{l_n}^n$ are non-empty open subsets of H. By the construction of the proof of Theorem 2.5, we can assume that $y_1^n, \ldots, y_{l_n}^n$ are distinct points in X. For each $n \in \omega$, let $f_n(y_i^n) = r_i^n$ for each $i = 1, \ldots, l_n$. Clearly, $Y = \{y_i^n : i = 1, \ldots, l_n, n \in \omega\}$ is a countable subset of X. So let $p \in X \setminus Y$. Since X is $H^{\star\star}$-regular, for each $n \in \omega$ there exists a function $g_n \in C(X,H)$ such that $g_n(y_i^n) = r_i^n$ and $g_n(p) = h_n$ for all $i = 1, \dots, l_n$, where all h_n are distinct points in H. Clearly, $g_n \in U_n$ for each $n \in \omega$. Since H is a P-group, there exists a sequence, say $\langle H_n \rangle$, of non-empty open subsets of H such that $h_n \in H_n$ for all $n \in \omega$ and whenever $m \neq n$, $H_m \cap H_n = \emptyset$. Therefore, the set $D = \{h_n : n \in \omega\}$ is closed and discrete in H. To prove that $\tilde{D} = \{g_n : n \in \omega\}$ is a closed and discrete set in $C_p(X, H)$, let $f \in C(X, H)$ and let $f(p) = h \in H$. Since D is closed and discrete in H, there exists an open neighborhood U_h of h whose intersection with D is finite. This gives us an open neighborhood $U_f = [p, U_h]^+$ of f in $C_p(X, H)$ whose intersection with $\tilde{D}$ is finite. Hence, $\tilde{D}$ is closed and discrete in $C_p(X, H)$. □

Theorem 5.4. *Let X be a $H^{\star\star}$-regular space such that X°, the set of isolated points, is uncountable and $|H| \geq \omega$. Then $C_h(X, H)$ is discretely selective.*

Proof. Let $\{U_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_h(X, H)$. For each $n \in \omega$, let $f_n \in U_n$. Then for each $n \in \omega$, there exists non-empty basic open set $A_n = [V_1^n, r_1^n]^- \cap \dots \cap [V_{l_n}^n, r_{l_n}^n]^-$ such that $f_n \in A_n \subseteq U_n$, where $r_1^n, \dots, r_{l_n}^n \in H$ and $V_1^n, \dots, V_{l_n}^n$ are non-empty open subsets of X. For each $n \in \omega$ there exist points $y_1^n, \dots, y_{l_n}^n$ in X such that $r_i^n = f_n(y_i^n)$ for each $i = 1, \dots, l_n$. By the construction of the proof of [8, Proposition 2.1], we can assume that $y_1^n, \dots, y_{l_n}^n$ are distinct points of X. Let $Y_n = \{y_1^n, \dots, y_{l_n}^n\}$ for each $n \in \omega$. Clearly, $Y = \bigcup_{n\in\omega} Y_n$ is a countable subset of X. So let $p \in X^\circ \setminus Y$. Since X is $H^{\star\star}$-regular and $|H| \geq \omega$, for each $n \in \omega$, there exists a function $g_n \in C(X, H)$ such that $g_n(y_i^n) = r_i^n$ and $g_n(p) = h_n$ for all $i = 1, \dots, l_n$, where we can assume that the elements $h_n \in H$ are distinct. Clearly $g_n \in U_n$ for each $n \in \omega$. Consider the set $D = \{g_n : n \in \omega\}$. To see that D is discrete and closed in $C_h(X, H)$, let $f \in C(X, H)$, Then $[\{p\}, f(p)]^-$ is an open neighborhood of f whose intersection with D is finite. Hence, D is closed and discrete. □

Theorem 5.5. *Let G be a countable metric group. Then $C_h(G, H)$ is discretely selective if and only if $C_h(G, H)$ is discrete.*

Proof. Since G is a countable metric group, Theorem 4.1 implies that $C_h(G, H)$ is first countable. Thus, proof follows from [15, 3.2(b)]. □

Theorem 5.6. *Let X be a $H^{\star\star}$-regular space such that X° is uncountable and $|H| \geq \omega$. Then $C_{ph}(X, H)$ is discretely selective.*

Proof. Let $\{O_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_{ph}(X, H)$. For each $n \in \omega$, let $f_n \in O_n$. Then for each $n \in \omega$, there exists a non-empty basic open set $A_n = [x_1^n, U_1^n]^+ \cap \dots \cap [x_{t_n}^n, U_{t_n}^n]^+ \cap [V_1^n, r_1^n]^- \cap \dots \cap [V_{l_n}^n, r_{l_n}^n]^-$ such that $f_n \in A_n \subseteq O_n$, where $x_1^n, \dots, x_{t_n}^n \in X$, $r_1^n, \dots, r_{l_n}^n \in H$ and $V_1^n, \dots, V_{l_n}^n$ are non-empty open subsets of X, $U_1^n, \dots, U_{t_n}^n$ are non-empty open subsets of H. For each $n \in \omega$, there exist points $y_1^n, \dots, y_{l_n}^n \in X$ and points $z_1^n, \dots, z_{t_n}^n \in H$ such that $f_n(y_i^n) = r_i^n$ and $f_n(x_j^n) = z_j^n$ for each $i = 1, \dots, l_n$, $j = 1, \dots, t_n$. Without loss of generality, assume that $y_1^n, \dots, y_{l_n}^n$ are distinct points in X and $x_1^n, \dots, x_{t_n}^n$ are distinct points in X. Let $Y_n = \{y_1^n, \dots, y_{l_n}^n\} \cup \{x_1^n, \dots, x_{t_n}^n\}$ for each $n \in \omega$. Clearly,

$Y = \bigcup_{n\in\omega} Y_n$ is a countable subset of X. So let $p \in X^\circ \setminus Y$. Since X is $H^{\star\star}$-regular and $|H| \geq \omega$, for each $n \in \omega$, there exists a function $g_n \in C(X, H)$ such that $g_n(y_i^n) = r_i^n$, $g_n(x_j^n) = z_j^n$ and $g_n(p) = h_n$ for all $i = 1, \ldots, l_n$ and for all $j = 1, \ldots, t_n$, where we can assume that the elements $h_n \in H$ are distinct. Clearly $g_n \in O_n$ for each $n \in \omega$. Consider the set $D = \{g_n : n \in \omega\}$. To see that D is discrete and closed in $C_{ph}(X, H)$, let $f \in C(X, H)$. Then $[\{p\}, f(p)]^-$ is an open neighborhood of f in $C_{ph}(X, H)$ whose intersection with D is finite. Hence, D is closed and discrete. □

Definition 5.7. [15] Given a space X, a sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of X is said to have a disjoint shrinking if for every $n \in \omega$, there exists a non-empty open set $V_n \subseteq U_n$ such that $V_m \cap V_n = \emptyset$ for $m \neq n$.

In [15], it is proved that every sequence of non-empty open sets in $C_p(X)$ has a disjoint shrinking if and only if the space X is uncountable. The following theorems give sufficient conditions for such sequence to have a disjoint shrinking in spaces $C_p(X, H)$, $C_h(X, H)$ and $C_{ph}(X, H)$.

Theorem 5.8. *Let H be an uncountable P-group and X be an uncountable $H^{\star\star}$-regular space. Then $C_p(X, H)$ has a disjoint shrinking.*

Proof. Let $\{U_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_p(X, H)$. For each $n \in \omega$, let $f_n \in U_n$. As in Theorem 5.3, we can construct a countable subset Y of X. Let $p \in X \setminus Y$. Since H is an uncountable P-group, there exist a sequence, say $\langle h_n \rangle$, of distinct points of H and a sequence, say $\langle W_n \rangle$, of non-empty open subsets of H such that $h_n \in W_n$ for all $n \in \omega$ and whenever $m \neq n$, $W_m \cap W_n = \emptyset$. Since X is $H^{\star\star}$-regular, $O_n = U_n \cap [p, W_n]^+ \subseteq U_n$ is a non-empty open subset of $C_p(X, H)$ and $O_n \cap O_m = \emptyset$ for all $n \neq m$, as $f \in O_n \cap O_m$ implies that $f(p) \in W_n \cap W_m$, a contradiction. Hence, every sequence of non-empty open sets in $C_p(X, H)$ has a disjoint shrinking. □

Theorem 5.9. *Let X be a $H^{\star\star}$-regular space such that X° is uncountable and H be an uncountable topological group. Then every sequence of non-empty open sets in $C_h(X, H)$ has a disjoint shrinking.*

Proof. Let $\{U_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_h(X, H)$. For each $n \in \omega$, let $f_n \in U_n$. As in Theorem 5.4, we can construct a countable subset Y of X, where $Y = \bigcup_{n\in\omega} Y_n = \bigcup_{n\in\omega}\{y_1^n, \ldots, y_{l_n}^n\}$ and $f_n(y_i^n) = r_i^n$ for all $i = 1, \ldots, l_n$. Let $p \in X^\circ \setminus Y$. Consider the set $B = \{r_i^n : i = 1, \ldots, l_n, n \in \omega\}$. Since B is a countable subset of H, choose distinct points $s_n \in H \setminus B$ for each $n \in \omega$. Since X is $H^{\star\star}$-regular, for each $n \in \omega$, there exist functions $g_n \in C(X, H)$ such that $g_n(y_i^n) = r_i^n$ and $g_n(p) = s_n$ for all $i = 1, \ldots, l_n$. For every $n \in \omega$, $O_n = U_n \cap [\{p\}, s_n]^- \subseteq U_n$ is a non-empty open subset of $C_h(X, H)$ and $O_n \cap O_m = \emptyset$ for all $n \neq m$, since $f \in O_n \cap O_m$ implies that $f(p) = s_n = s_m$, a contradiction. Hence, every sequence of non-empty open sets in $C_h(X, H)$ has a disjoint shrinking. □

Theorem 5.10. *Let X be a $H^{\star\star}$-regular space such that X° is uncountable and H be an uncountable topological group. Then every sequence of non-empty open sets in $C_{ph}(X, H)$ has a disjoint shrinking.*

Proof. Let $\{O_n : n \in \omega\}$ be a sequence of non-empty open subsets of the space $C_{ph}(X,H)$. For each $n \in \omega$, let $f_n \in O_n$. As in Theorem 5.6, we can construct a countable subset Y of X, where $Y = \bigcup_{n\in\omega} Y_n = \bigcup_{n\in\omega}(\{y_1^n,\dots,y_{l_n}^n\} \cup \{x_1^n,\dots,x_{t_n}^n\})$ and $f_n(y_i^n) = r_i^n$, $f_n(x_j^n) = z_j^n$ for all $i = 1,\dots,l_n$, $j = 1,\dots,t_n$. Let $p \in X^\circ \setminus Y$. Consider the set $B = \{r_i^n : i = 1,\dots,l_n,\ n \in \omega\}$. Since B is a countable subset of H, choose distinct points $s_n \in H \setminus B$ for each $n \in \omega$. Since X is $H^{\star\star}$-regular, for each $n \in \omega$, there exist functions $g_n \in C(X,H)$ such that $g_n(y_i^n) = r_i^n$, $g_n(x_j^n) = z_j^n$ and $g_n(p) = s_n$ for all $i = 1,\dots,l_n$ and for all $j = 1,\dots,t_n$. For every $n \in \omega$, $D_n = O_n \cap [\{p\}, s_n]^- \subseteq O_n$ is a non-empty open subset of $C_{ph}(X,H)$ and $D_n \cap D_m = \emptyset$ for all $n \neq m$, since $f \in D_n \cap D_m$ implies that $f(p) = s_n = s_m$, a contradiction. Hence, every sequence of non-empty open sets in $C_{ph}(X,H)$ has a disjoint shrinking. □

6. Restriction maps

The following theorem is easy to prove (for instance, see [16, p. 96]).

THEOREM 6.1. *If X is an $H^{\star\star}$-regular topological space, then the space $C_p(X,H)$ is dense in H^X.*

A map $f : X \to Y$, where X is any non-empty set and Y is a topological space, is called *almost onto* if $f(X)$ is dense in Y. If A is a subspace of X, then the restriction map $\pi_A : C(X,H) \to C(A,H)$ is defined by $\pi_A(f) = f|A$ for all $f \in C(X,H)$ and studied by many authors (see [9,11]).

THEOREM 6.2. *Let X be an $H^{\star\star}$-regular space and A be a subspace of X. Then for the map $\pi_A : C_p(X,H) \to C_p(A,H)$, the following statements hold.*
(i) The map π_A is continuous.

(ii) The map π_A is almost onto.

(iii) The map π_A is an injection if and only if A is dense in X.

(iv) The map π_A is a homeomorphism if and only if $A = X$.

Proof. (i) Let $[x,U]^+$ be any subbasic open set in $C_p(A,H)$, where $x \in A$ and U is open in H. If $\pi_A^{-1}[x,U]^+ = \emptyset$, then it is open. So let $\pi_A^{-1}[x,U]^+ \neq \emptyset$. In this case, $\pi_A^{-1}[x,U]^+ = [x,U]^+$ which is an open set in $C_p(X,H)$. Thus, the map π_A is continuous.
(ii) Let $U = [x_1,V_1]^+ \cap \dots \cap [x_n,V_n]^+$ be any non-empty basic open set in $C_p(A,H)$, where $x_i \in A$ and V_i is open in H, whenever $1 \le i \le n$. Without loss of generality, assume that x_i are distinct. Let $f \in U$. Then $f(x_i) \in V_i$. Let $f(x_i) = r_i$ for all $i = 1,\dots,n$. Since X is $H^{\star\star}$-regular, there exists a function $h \in C(X,H)$ such that $h(x_i) = r_i$ for all $i = 1,\dots,n$. Therefore, $\pi_A(h) \in U$ and hence, π_A is almost onto.
(iii) Suppose that A is not dense in X. Take any $x \in X \setminus \overline{A}$. Since X is H-regular, there exist $f \in C(X,H)$ and $g_1 \in H \setminus \{\tilde{e}\}$ such that $f(x) = g_1$ and $f(\overline{A}) \subseteq \{\tilde{e}\}$. Thus, $\pi_A(f) = \pi_A(E)$, where $E(x) = \tilde{e}$ for all $x \in X$. But $f \neq E$, a contradiction.

Conversely, let A be dense in X and f, g be any two distinct members of $C_p(X,H)$. Then $h = fg^{-1}$ is a continuous function. Consider the non-empty open set $U = h^{-1}(H \setminus \{\tilde{e}\})$. Since A is dense in X, let $y \in A \cap U$. Clearly, $f(y) \neq g(y)$ which means that $\pi_A(f) \neq \pi_A(g)$ and thus, π_A is an injection.
(iv) Let π_A be a homeomorphism. So A is dense in X by (iii). If $A \neq X$, then there exists $x \in X \setminus A$. The set $B = \{f \in C_p(X,H) : f(x) = g_1\}$ is not dense in $C_p(X,H)$ because it does not intersect the non-empty open set $[x, H \setminus \{g_1\}]^+$. But $\pi_A(B)$ is dense in $C_p(A,H)$ as follows: let $g \in C_p(A,H)$ and $V = [y_1, U_1]^+ \cap \ldots \cap [y_n, U_n]^+$ be any non-empty basic open set in $C_p(A,H)$ containing g, where y_i are distinct members of A and U_i are non-empty open sets in H. Then there exists $f \in C_p(X,H)$ such that $f(x) = g_1$ and $f(y_i) = g(y_i)$ for all $i \leq n$. Thus $\pi_A(f) \in V \cap \pi_A(B)$ which proves that $\pi_A(B)$ is dense in $C_p(A,H)$. Hence, ${\pi_A}^{-1}$ is not continuous as B is not dense in $C_p(X,H)$. Hence, if π_A is a homeomorphism, then $A = X$. Conversely, if $X = A$, then π_A is the identity map and hence, a homeomorphism. □

Theorem 6.3. *Let A be an open subspace of a $H^{\star\star}$-regular space X and $\pi_A : C_h(X, H) \to C_h(A,H)$ be the restriction map. Then the following statements hold.*
(i) The map π_A is continuous.

(ii) The map π_A is almost onto.

(iii) The map π_A is an injection if and only if A is dense in X.

(iv) If X is discrete, then the map π_A is a homeomorphism if and only if $A = X$.

Proof. (i) Let $[U,r]^-$ be any subbasic open set in $C_h(A,H)$, where $r \in H$ and U is open in A. If $\pi_A^{-1}[U,r]^- = \emptyset$, then it is open. So let $\pi_A^{-1}[U,r]^- \neq \emptyset$. In this case, $\pi_A^{-1}[U,r]^- = [U,r]^-$ which is an open set in $C_h(X,H)$ as A is an open subset of X. Thus, the map π_A is continuous.
(ii) Let $U = [V_1, r_1]^- \cap \ldots \cap [V_n, r_n]^-$ be any non-empty basic open set in $C_h(A,H)$, where $r_i \in H$ and V_i is open in A, whenever $1 \leq i \leq n$. Let $f \in U$; then there exist $y_i \in V_i$ such that $f(y_i) = r_i$ for all $i = 1, \ldots, n$. Without loss of generality, assume that y_i are distinct. Since X is $H^{\star\star}$-regular, there exists a function $h \in C(X,H)$ such that $h(y_i) = r_i$ for all $i = 1, \ldots, n$. Therefore, $\pi_A(h) \in U$ and hence, π_A is almost onto.
(iii) The proof is on the similar lines of Theorem 6.2 (iii).
(iv) Let π_A be a homeomorphism. So A is dense in X by (iii). Since X is discrete, we have $A = X$. Conversely, if $A = X$, then π_A is the identity map and hence, a homeomorphism. □

Theorem 6.4. *Let A be an open subspace of an $H^{\star\star}$-regular space X and $\pi_A : C_{ph}(X, H) \to C_{ph}(A,H)$ be the restriction map. Then the following statements hold.*
(i) The map π_A is continuous.

(ii) The map π_A is almost onto.

(iii) The map π_A is an injection if and only if A is dense in X.

(iv) If X is discrete, then the map π_A is a homeomorphism if and only if $A = X$.

Proof. The proof follows from Theorem 6.2 and Theorem 6.3. □

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