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FRACTIONAL INTEGRAL INEQUALITIES OF VARIABLE ORDER ON SPHERICAL SHELL

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Abstract. Here left and right Riemann-Liouville generalized fractional radial integral operators of variable order over a spherical shell are introduced, as well as left and right weighted Caputo type generalized fractional radial derivatives of variable order over a spherical shell. After proving continuity of these operators, we establish a series of left and right fractional integral inequalities of variable order over the spherical shell of Opial and Hardy types. Extreme cases are met.

1. Background

We are inspired by [1–3].

Let $N \geq 2$, $S^{N-1} := \left\{x \in \mathbb{R}^N : |x| = 1\right\}$ the unit sphere on $\mathbb{R}^N$, where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Also denote the ball

$$B(0, R) := \left\{x \in \mathbb{R}^N : |x| < R\right\} \subseteq \mathbb{R}^N,\ R > 0,$$

and the spherical shell

$$A := B(0, R_2) - \overline{B(0, R_1)},\ 0 < R_1 < R_2.$$

For the following see [5, pp. 149-150] and [6, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$, and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Clearly here

$$\mathbb{R}^N - \{0\} = (0, \infty) \times S^{N-1}, \qquad \text{and} \qquad \overline{A} = [R_1, R_2] \times S^{N-1}.$$

In the sequel the following related theorem will be used:

Theorem 1.1 ([1, p. 458]). *Let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then*

$$\int_A f(x)\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(r\omega)\, r^{N-1} dr \right) d\omega.$$

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So we are able to write an integral in polar form using the polar coordinates (r,ω).
The area of S^{N-1} *is* $\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$.
The volume of $B(0,R)$, $\mathrm{Vol}(B(0,R)) = \int_{B(0,R)} dy = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$.
And the volume of A, $\mathrm{Vol}(A) = \frac{\pi^{\frac{N}{2}}\left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}$.

We need the following definition.

Definition 1.2. Let $[R_1, R_2] \subset \mathbb{R}$, $R_2 > R_1 > 0$, and $\alpha \in C([R_1, R_2])$, $\alpha > 1$. Let $\psi \in C^1([R_1, R_2])$ is increasing and $f \in C\left(\overline{A}\right)$. We define the left Riemann-Liouville generalized fractional radial integral operator for $f : A \to \mathbb{R}$ of variable order $\alpha(\cdot)$:

$$\left(I_{R_1+;\psi}^{\alpha(\cdot)} f\right)(x) = \left(I_{R_1+;\psi}^{\alpha(\cdot)} f\right)(r\omega) := \int_{R_1}^{r} \frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega) \, dz,$$

for all $x \in A$, where Γ is the gamma function.

We also define the right Riemann-Liouville generalized fractional radial integral operator for $f : A \to \mathbb{R}$ of variable order $\alpha(\cdot)$:

$$\left(I_{R_2-;\psi}^{\alpha(\cdot)} f\right)(x) = \left(I_{R_2-;\psi}^{\alpha(\cdot)} f\right)(r\omega) := \int_{r}^{R_2} \frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega) \, dz,$$

for all $x \in A$.

We give the next theorem.

Theorem 1.3 (related to Definition 1.2). *We have that* $I_{R_1+;\psi}^{\alpha(\cdot)} f$, $I_{R_2-;\psi}^{\alpha(\cdot)} f \in C\left(\overline{A}\right)$.

Proof. We will prove in details only that $I_{R_1+;\psi}^{\alpha(\cdot)} f \in C\left(\overline{A}\right)$, the proof for $I_{R_2-;\psi}^{\alpha(\cdot)} f \in C\left(\overline{A}\right)$ as similar is omitted. Indeed here $f(r\omega) \in C([R_1, R_2])$, for all $\omega \in S^{N-1}$. We can write

$$\left(I_{R_1+;\psi}^{\alpha(\cdot)} f\right)(r\omega) = \int_{R_1}^{R_2} \chi_{[R_1,r]}(z) \frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega) \, dz,$$

where χ is the characteristic function, for all $\omega \in S^{N-1}$.

Let $r_n \to r$, $\omega_n \to \omega$; then $\chi_{[R_1,r_n]}(z) \to \chi_{[R_1,r]}(z)$, a.e., $\frac{(\psi(r_n)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)$ $\to \frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)$, and $f(z\omega_n) \to f(z\omega)$. Furthermore it holds that

$$\chi_{[R_1,r_n]}(z) \frac{(\psi(r_n) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega_n)$$
$$\to \chi_{[R_1,r]}(z) \frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega),$$

a.e. on $[R_1, R_2]$.

Here it is $\alpha(z) - 1 > 0$ and $\alpha - 1 \in C([R_1, R_2])$. Thus, $\inf(\alpha - 1) \leq \alpha(z) - 1 \leq \sup(\alpha - 1)$, for all $z \in [R_1, R_2]$ and $\inf(\alpha - 1), \sup(\alpha - 1) > 0$.

If $|\psi(r_n) - \psi(z)| \leq 1$, then

$$|\psi(r_n) - \psi(z)|^{\alpha(z)-1} \leq |\psi(r_n) - \psi(z)|^{\inf(\alpha-1)},$$

and if $|\psi(r_n) - \psi(z)| > 1$, then

$$|\psi(r_n) - \psi(z)|^{\alpha(z)-1} \leq |\psi(r_n) - \psi(z)|^{\sup(\alpha-1)}.$$

That is

$$\begin{aligned} |\psi(r_n) - \psi(z)|^{\alpha(z)-1} &\leq |\psi(r_n) - \psi(z)|^{\inf(\alpha-1)} + |\psi(r_n) - \psi(z)|^{\sup(\alpha-1)} \\ &\leq (\psi(R_2) - \psi(R_1))^{\inf(\alpha-1)} + (\psi(R_2) - \psi(R_1))^{\sup(\alpha-1)} =: \lambda < \infty, \end{aligned}$$

true for all $r_n, z \in [R_1, R_2]$.

Therefore it holds

$$\begin{aligned} &\chi_{[R_1,r_n]}(z) \frac{|\psi(r_n) - \psi(z)|^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) |f(z\omega_n)| \\ &\leq \frac{\lambda}{\inf_{[R_1,R_2]} \Gamma(\alpha(z))} \|\psi'\|_{\infty,[R_1,R_2]} \|f\|_{\infty,\overline{A}} < \infty. \end{aligned}$$

Thus, by the denominated convergence theorem we obtain

$$\begin{aligned} &\int_{R_1}^{R_2} \chi_{[R_1,r_n]}(z) \frac{(\psi(r_n) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega_n) \, dz \\ \to &\int_{R_1}^{R_2} \chi_{[R_1,r]}(z) \frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z) f(z\omega) \, dz, \end{aligned}$$

proving the claim. □

Definition 1.4. Let $[R_1, R_2] \subset \mathbb{R}$, $R_2 > R_1 > 0$, and $\alpha \in C([R_1, R_2])$, such that $n-2 < \alpha(t) < n-1$, $n \in \mathbb{N} - \{1\}$, for all $t \in [R_1, R_2]$. Let $f \in C^n(\overline{A})$ and $\psi \in C^n([R_1, R_2])$ with ψ being increasing and $\psi'(t) \neq 0$, for all $t \in [R_1, R_2]$. The left ψ-Caputo generalized fractional radial derivative of $f : A \to \mathbb{R}$ of variable order $\alpha(\cdot)$ is given by:

$$^{C}D_{R_1+}^{\alpha(\cdot),\psi} f(x) = {}^{C}D_{R_1+}^{\alpha(\cdot),\psi} f(r\omega) := \int_{R_1}^{r} \frac{(\psi(r) - \psi(t))^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))} \psi'(t) f_{\psi}^{[n]}(t\omega) \, dt,$$

for all $x \in A$, where

$$f_{\psi}^{[n]}(x) = f_{\psi}^{[n]}(r\omega) := \left(\frac{1}{\psi'(r)} \frac{d}{dr} \right)^{n} f(r\omega),$$

with $f_{\psi}^{[0]}(r\omega) = f(r\omega)$, for all $\omega \in S^{N-1}$, for all $r \in [R_1, R_2]$.

And, the right ψ-Caputo generalized fractional radial derivative of $f : A \to \mathbb{R}$ of variable order $\alpha(\cdot)$ is given by:

$$\begin{aligned} &^{C}D_{R_2-}^{\alpha(\cdot),\psi} f(x) = {}^{C}D_{R_2-}^{\alpha(\cdot),\psi} f(r\omega) := \\ &(-1)^n \int_{r}^{R_2} \frac{(\psi(t) - \psi(r))^{n-\alpha(t)-1}}{\Gamma(n-\alpha(t))} \psi'(t) f_{\psi}^{[n]}(t\omega) \, dt, \end{aligned}$$

for all $x \in A$.

Remark 1.5 (to Definition 1.4). We observe that

$$ {}^{C}D_{R_1+}^{\alpha(\cdot),\psi} f(x) = I_{R_1+;\psi}^{n-\alpha(\cdot)} f_{\psi}^{[n]}(x), \tag{1} $$

and

$$ {}^{C}D_{R_2-}^{\alpha(\cdot),\psi} f(x) = (-1)^n I_{R_2-;\psi}^{n-\alpha(\cdot)} f_{\psi}^{[n]}(x), \tag{2} $$

for all $x \in A$.

By Theorem 1.3, both ${}^{C}D_{R_1+}^{\alpha(\cdot),\psi} f(x)$, ${}^{C}D_{R_2-}^{\alpha(\cdot),\psi} f \in C(A)$.

All the following statements in this section originates from [4] and they will be used in the main results.

Definition 1.6. Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ be a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b],X)$, $\nu \in C([a,b])$, $\nu > 1$. We define the left Riemann-Liouville generalized fractional Bochner integral operator of variable order $\nu(\cdot)$

$$ \left({}^{B}I_{a+;g}^{\nu(\cdot)} f\right)(x) := \int_a^x \frac{(g(x) - g(z))^{\nu(z)-1}}{\Gamma(\nu(z))} g'(z) f(z)\, dz, $$

for all $x \in [a,b]$. We assume ${}^{B}I_{a+;g}^{\nu(\cdot)} f \in C([a,b],X)$.

Definition 1.7. Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ be a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b],X)$, $\nu \in C([a,b])$, $\nu > 1$. We define the right Riemann-Liouville generalized fractional Bochner integral operator of variable order $\nu(\cdot)$

$$ \left({}^{B}I_{b-;g}^{\nu(\cdot)} f\right)(x) := \int_x^b \frac{(g(z) - g(x))^{\nu(z)-1}}{\Gamma(\nu(z))} g'(z) f(z)\, dz, $$

for all $x \in [a,b]$. We assume ${}^{B}I_{b-;g}^{\nu(\cdot)} f \in C([a,b],X)$.

We mention the following left fractional Opial type inequality of variable order (see also [1]).

Theorem 1.8. *All as in Definition 1.6. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ \int_a^x \left\| \left({}^{B}I_{a+;g}^{\nu(\cdot)} f\right)(w) \right\| \|f(w)\| g'(w)\, dw \leq $$

$$ 2^{-\frac{1}{q}} \left(\int_a^x \left(\int_a^w \left(\frac{(g(w) - g(z))^{\nu(z)-1}}{\Gamma(\nu(z))} \right)^p dz \right) dw \right)^{\frac{1}{p}} \left(\int_a^x \|f(w)\|^q (g'(w))^q\, dw \right)^{\frac{2}{q}}, \tag{3} $$

for all $x \in [a,b]$.

Next follows a right fractional Opial type inequality of variable order.

Theorem 1.9. *All as in Definition 1.7. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ \int_x^b \left\| \left({}^{B}I_{b-;g}^{\nu(\cdot)} f\right)(w) \right\| \|f(w)\| g'(w)\, dw \leq $$

$$ 2^{-\frac{1}{q}} \left(\int_x^b \left(\int_w^b \left(\frac{(g(z) - g(w))^{\nu(z)-1}}{\Gamma(\nu(z))} \right)^p dz \right) dw \right)^{\frac{1}{p}} \left(\int_x^b \|f(w)\|^q (g'(w))^q\, dw \right)^{\frac{2}{q}}, \tag{4} $$

for all $x \in [a,b]$.

We mention two extreme Opial inequalities ($p = 1$, $q = \infty$ case).

THEOREM 1.10. *All as in Definition 1.6. Then*

$$\int_a^x \left\| \left({}^B I_{a+;g}^{\nu(\cdot)} f \right)(w) \right\| \|f(w)\| \, dw$$

$$\leq \left(\int_a^x \left(\int_a^w \frac{(g(w) - g(z))^{\nu(z)-1}}{\Gamma(\nu(z))} g'(z) \, dz \right) dw \right) \|f\|_\infty^2, \tag{5}$$

for all $x \in [a, b]$.

THEOREM 1.11. *All as in Definition 1.7. Then*

$$\int_x^b \left\| \left({}^B I_{b-;g}^{\nu(\cdot)} f \right)(w) \right\| \|f(w)\| \, dw$$

$$\leq \left(\int_x^b \left(\int_w^b \frac{(g(z) - g(w))^{\nu(z)-1}}{\Gamma(\nu(z))} g'(z) \, dz \right) dw \right) \|f\|_\infty^2, \tag{6}$$

for all $x \in [a, b]$.

Next we mention two fractional Hardy's type inequalities of variable order.

THEOREM 1.12. *All as in Definition 1.6. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left\| \left\| {}^B I_{a+;g}^{\nu(\cdot)} f \right\| \right\|_q \leq \left(\int_a^b \left(\int_a^x \left(\frac{(g(x) - g(z))^{\nu(z)-1}}{\Gamma(\nu(z))} \right)^p dz \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|g' \|f\|\|_q. \tag{7}$$

THEOREM 1.13. *All as in Definition 1.7. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left\| \left\| {}^B I_{b-;g}^{\nu(\cdot)} f \right\| \right\|_q \leq \left(\int_a^b \left(\int_x^b \left(\frac{(g(z) - g(x))^{\nu(z)-1}}{\Gamma(\nu(z))} \right)^p dz \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|g' \|f\|\|_q. \tag{8}$$

The above mentioned results will be applied for $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$.

The purpose of this work is to derive interesting fractional radial integral inequalities of variable order.

2. Main results

We present the following left fractional Opial type inequality of variable order over the spherical shell.

THEOREM 2.1. *All as in Definition 1.2 with* ψ *strictly increasing. Then*

$$\int_A \left| \left(I_{R_1+;\psi}^{\alpha(\cdot)} f \right)(x) \right| |f(x)| \, dx \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf\limits_{[R_1,R_2]} (\psi')} \right)$$

$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)^{\frac{1}{2}}\left(\int_A f^2(x)\,dx\right).$$

Proof. Here we apply (3) for $q=2$ and for $f(\cdot\omega)\in C([R_1,R_2])$. We have that

$$\int_{R_1}^{R_2}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right||f(r\omega)|\,\psi'(r)\,dr\le$$

$$2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)^{\frac{1}{2}}\left(\int_{R_1}^{R_2}(f(r\omega))^2(\psi'(r))^2\,dr\right),$$

for all $\omega\in S^{N-1}$. Hence it holds

$$\inf_{[R_1,R_2]}(\psi')\int_{R_1}^{R_2}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right||f(r\omega)|\,dr\le 2^{-\frac{1}{2}}\|\psi'\|_\infty^2$$

$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)^{\frac{1}{2}}\left(\int_{R_1}^{R_2}(f(r\omega))^2\,dr\right),\tag{9}$$

for all $\omega\in S^{N-1}$.

By Theorem 1.3 we have that $\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(\cdot\omega)\in C([R_1,R_2])$, for all $\omega\in S^{N-1}$, where $\overline{A}=[R_1,R_2]\times S^{N-1}$. By $r^{N-1}r^{1-N}=1$, where $R_1\le r\le R_2$, and $R_2^{1-N}\le r^{1-N}\le R_1^{1-N}$, we obtain:

$$R_2^{1-N}\inf_{[R_1,R_2]}(\psi')\int_{R_1}^{R_2}r^{N-1}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right||f(r\omega)|\,dr$$

$$\le\inf_{[R_1,R_2]}(\psi')\int_{R_1}^{R_2}r^{N-1}r^{1-N}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right||f(r\omega)|\,dr$$

$$=\inf_{[R_1,R_2]}(\psi')\int_{R_1}^{R_2}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right||f(r\omega)|\,dr$$

$$\overset{(9)}{\le}2^{-\frac{1}{2}}\|\psi'\|_\infty^2\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)^{\frac{1}{2}}$$

$$\left(\int_{R_1}^{R_2}r^{N-1}r^{1-N}(f(r\omega))^2\,dr\right)\le$$

$$2^{-\frac{1}{2}}R_1^{1-N}\|\psi'\|_\infty^2\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)^{\frac{1}{2}}$$

$$\left(\int_{R_1}^{R_2}r^{N-1}(f(r\omega))^2\,dr\right),\quad\forall\,\omega\in S^{N-1}.$$

So far, for all $\omega \in S^{N-1}$, we have

$$\int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_1+;\psi}^{\alpha(\cdot)} f \right) (r\omega) \right| |f(r\omega)| \, dr \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf\limits_{[R_1,R_2]} (\psi')} \right)$$

$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 \, dr \right).$$

Hence it holds

$$\int_A \left| \left(I_{R_1+;\psi}^{\alpha(\cdot)} f \right) (x) \right| |f(x)| \, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_1+;\psi}^{\alpha(\cdot)} f \right) (r\omega) \right| |f(r\omega)| \, dr \right) d\omega$$

$$\leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf\limits_{[R_1,R_2]} (\psi')} \right) \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}}$$

$$\left(\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 \, dr \right) d\omega \right)$$

$$= 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf\limits_{[R_1,R_2]} (\psi')} \right) \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}}$$

$$\left(\int_A (f(x))^2 \, dx \right),$$

proving the claim. □

We give a right fractional Opial type inequality of variable order over the spherical shell.

THEOREM 2.2. *All as in Definition 1.2 with ψ strictly increasing. Then*

$$\int_A \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right) (x) \right| |f(x)| \, dx \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf\limits_{[R_1,R_2]} (\psi')} \right)$$

$$\left(\int_{R_1}^{R_2} \left(\int_{r}^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_A f^2(x) \, dx \right).$$

Proof. Here we apply (4) for $q = 2$ and for $f(\cdot\omega) \in C([R_1, R_2])$. We have that

$$\int_{R_1}^{R_2} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right) (r\omega) \right| |f(r\omega)| \, \psi'(r) \, dr$$

$$\leq 2^{-\frac{1}{2}} \left(\int_{R_1}^{R_2} \left(\int_{r}^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}}$$

$$\left(\int_{R_1}^{R_2} (f(r\omega))^2 (\psi'(r))^2 \, dr\right), \quad \forall \omega \in S^{N-1}.$$

Hence, for all $\omega \in S^{N-1}$, it holds

$$\inf_{[R_1,R_2]} (\psi') \int_{R_1}^{R_2} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)| \, dr \leq 2^{-\frac{1}{2}} \|\psi'\|_\infty^2$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_{R_1}^{R_2} (f(r\omega))^2 \, dr \right). \quad (10)$$

By Theorem 1.3 we have that $\left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(\cdot\omega) \in C([R_1, R_2])$, for all $\omega \in S^{N-1}$. We obtain

$$R_2^{1-N} \inf_{[R_1,R_2]} (\psi') \int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)| \, dr$$

$$\leq \inf_{[R_1,R_2]} (\psi') \int_{R_1}^{R_2} r^{N-1} r^{1-N} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)| \, dr$$

$$= \inf_{[R_1,R_2]} (\psi') \int_{R_1}^{R_2} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)| \, dr$$

$$\overset{(10)}{\leq} 2^{-\frac{1}{2}} \|\psi'\|_\infty^2 \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}}$$

$$\left(\int_{R_1}^{R_2} r^{N-1} r^{1-N} (f(r\omega))^2 \, dr \right)$$

$$\leq 2^{-\frac{1}{2}} R_1^{1-N} \|\psi'\|_\infty^2 \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}}$$

$$\left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 \, dr \right), \quad \forall \omega \in S^{N-1}. \quad (11)$$

So far we have, for all $\omega \in S^{N-1}$,

$$\int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)| \, dr \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|_\infty^2}{\inf_{[R_1,R_2]} (\psi')} \right)$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 \, dr \right).$$

Hence it holds

$$\int_A \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(x) \right| |f(x)| \, dx$$

$$=\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left|\left(I_{R_2-;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right|\left|f(r\omega)\right|dr\right)d\omega$$

$$\leq 2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\frac{\left\|\psi'\right\|_{\infty}^{2}}{\inf\limits_{[R_1,R_2]}(\psi')}\right)\left(\int_{R_1}^{R_2}\left(\int_{r}^{R_2}\left(\frac{(\psi(z)-\psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^{2}dz\right)dr\right)^{\frac{1}{2}}$$

$$\left(\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}(f(r\omega))^{2}dr\right)d\omega\right)$$

$$=2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\frac{\left\|\psi'\right\|_{\infty}^{2}}{\inf\limits_{[R_1,R_2]}(\psi')}\right)\left(\int_{R_1}^{R_2}\left(\int_{r}^{R_2}\left(\frac{(\psi(z)-\psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^{2}dz\right)dr\right)^{\frac{1}{2}}$$

$$\left(\int_{A}(f(x))^{2}dx\right),$$

proving the claim. □

We give two extreme Opial inequalities.

Theorem 2.3. *All as in Definition 1.2. Then*

$$\int_{A}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(x)\right|\left|f(x)\right|dx$$
$$\leq \frac{2\pi^{\frac{N}{2}}R_2^{N-1}}{\Gamma\left(\frac{N}{2}\right)}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)\,dz\right)dr\right)\left\|f\right\|_{\infty,\overline{A}}^{2}.$$

Proof. Here we apply (5). We have that, for all $\omega\in S^{N-1}$,

$$R_2^{1-N}\int_{R_1}^{R_2} r^{N-1}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right|\left|f(r\omega)\right|dr$$

$$\leq \int_{R_1}^{R_2} r^{N-1}r^{1-N}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right|\left|f(r\omega)\right|dr=\int_{R_1}^{R_2}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right|\left|f(r\omega)\right|dr$$

$$\overset{(5)}{\leq}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)\,dz\right)dr\right)\left\|f(\cdot\omega)\right\|_{\infty,[R_1,R_2]}^{2}$$

$$\leq\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)\,dz\right)dr\right)\left\|f\right\|_{\infty,\overline{A}}^{2}.$$

Hence it holds

$$\int_{A}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(x)\right|\left|f(x)\right|dx=\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left|\left(I_{R_1+;\psi}^{\alpha(\cdot)}f\right)(r\omega)\right|\left|f(r\omega)\right|dr\right)d\omega$$

$$\leq\frac{2R_2^{N-1}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\psi'(z)\,dz\right)dr\right)\left\|f\right\|_{\infty,\overline{A}}^{2},$$

proving the claim. □

THEOREM 2.4. *All as in Definition 1.2. Then*

$$\int_A \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(x) \right| |f(x)|\, dx$$

$$\leq \frac{2\pi^{\frac{N}{2}} R_2^{N-1}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z)\, dz \right) dr \right) \|f\|_{\infty,\overline{A}}^2 .$$

Proof. Here we apply (6). We have that

$$R_2^{1-N} \int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)|\, dr$$

$$\leq \int_{R_1}^{R_2} r^{N-1} r^{1-N} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)|\, dr = \int_{R_1}^{R_2} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)|\, dr$$

$$\overset{(6)}{\leq} \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z)\, dz \right) dr \right) \|f(\cdot\omega)\|_{\infty,[R_1,R_2]}^2$$

$$\leq \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z)\, dz \right) dr \right) \|f\|_{\infty,\overline{A}}^2 ,$$

for all $\omega \in S^{N-1}$. Hence it holds

$$\int_A \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(x) \right| |f(x)|\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left| \left(I_{R_2-;\psi}^{\alpha(\cdot)} f \right)(r\omega) \right| |f(r\omega)|\, dr \right) d\omega$$

$$\leq \frac{2R_2^{N-1}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \psi'(z)\, dz \right) dr \right) \|f\|_{\infty,\overline{A}}^2 ,$$

proving the claim. □

Two Hardy's type fractional inequalities of variable order on the spherical shell follow.

THEOREM 2.5. *All as in Definition 1.2. Then*

$$\int_A \left(I_{R_1+;\psi}^{\alpha(\cdot)} f(x) \right)^2 dx$$

$$\leq \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|_{\infty,[R_1,R_2]}^2 \left(\int_{R_1}^{R_2} \left(\int_{R_1}^r \left(\frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_A f^2(x)\, dx \right).$$

Proof. By (7) ($p = q = 2$ case), for all $\omega \in S^{N-1}$, we have

$$R_2^{1-N} \int_{R_1}^{R_2} r^{N-1} \left(I_{R_1+;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr$$

$$\leq \int_{R_1}^{R_2} r^{N-1} r^{1-N} \left(I_{R_1+;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr = \int_{R_1}^{R_2} \left(I_{R_1+;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr$$

$$\overset{(7)}{\leq} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^r \left(\frac{(\psi(r) - \psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{R_1}^{R_2} (\psi'(r))^2 (f(r\omega))^2\, dr \right)$$

$$\leq \|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)\left(\int_{R_1}^{R_2} r^{N-1}r^{1-N}(f(r\omega))^2 dr\right)$$
$$\leq R_1^{1-N}\|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)\left(\int_{R_1}^{R_2} r^{N-1}(f(r\omega))^2 dr\right).$$

So far we have proved that

$$\int_{R_1}^{R_2} r^{N-1}\left(I^{\alpha(\cdot)}_{R_1+;\psi}f(r\omega)\right)^2 dr \leq \left(\frac{R_2}{R_1}\right)^{N-1}\|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)\left(\int_{R_1}^{R_2} r^{N-1}(f(r\omega))^2 dr\right),$$

for all $\omega \in S^{N-1}$. Hence it holds

$$\int_A\left(I^{\alpha(\cdot)}_{R_1+;\psi}f(x)\right)^2 dx = \int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left(I^{\alpha(\cdot)}_{R_1+;\psi}f(r\omega)\right)^2 dr\right)d\omega$$
$$\leq \left(\frac{R_2}{R_1}\right)^{N-1}\|\psi'\|^2_{\infty,[R_1,R_2]}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)$$
$$\left(\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}(f(r\omega))^2 dr\right)d\omega\right)$$
$$= \left(\frac{R_2}{R_1}\right)^{N-1}\|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}\left(\frac{(\psi(r)-\psi(z))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)\left(\int_A (f(x))^2 dx\right),$$

proving the claim. □

Theorem 2.6. *All as in Definition 1.2. Then*

$$\int_A\left(I^{\alpha(\cdot)}_{R_2-;\psi}f(x)\right)^2 dx \leq \left(\frac{R_2}{R_1}\right)^{N-1}\|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2}\left(\int_{r}^{R_2}\left(\frac{(\psi(z)-\psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))}\right)^2 dz\right)dr\right)\left(\int_A f^2(x)\,dx\right).$$

Proof. By (8) ($p=q=2$ case) we have that, for all $\omega \in S^{N-1}$,

$$R_2^{1-N}\int_{R_1}^{R_2} r^{N-1}\left(I^{\alpha(\cdot)}_{R_2-;\psi}f(r\omega)\right)^2 dr$$

$$\leq \int_{R_1}^{R_2} r^{N-1} r^{1-N} \left(I_{R_2-;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr = \int_{R_1}^{R_2} \left(I_{R_2-;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr$$

$$\overset{(8)}{\leq} \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{R_1}^{R_2} (\psi'(r))^2 (f(r\omega))^2 dr \right)$$

$$\leq \|\psi'\|_{\infty,[R_1,R_2]}^2 \left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{R_1}^{R_2} r^{N-1} r^{1-N} (f(r\omega))^2 dr \right)$$

$$\leq R_1^{1-N} \|\psi'\|_{\infty,[R_1,R_2]}^2$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 dr \right).$$

So far we have proved that

$$\int_{R_1}^{R_2} r^{N-1} \left(I_{R_2-;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr \leq \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|_{\infty,[R_1,R_2]}^2$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 dr \right),$$

for all $\omega \in S^{N-1}$. Hence it holds

$$\int_A \left(I_{R_2-;\psi}^{\alpha(\cdot)} f(x) \right)^2 dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left(I_{R_2-;\psi}^{\alpha(\cdot)} f(r\omega) \right)^2 dr \right) d\omega$$

$$\leq \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|_{\infty,[R_1,R_2]}^2$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} (f(r\omega))^2 dr \right) d\omega \right)$$

$$= \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|_{\infty,[R_1,R_2]}^2$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{\alpha(z)-1}}{\Gamma(\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_A (f(x))^2 dx \right),$$

proving the claim. □

Fractional inequalities for the generalized fractional radial derivatives of variable order are considered in the sequel. First, we discuss Opial type inequalities. The proofs of the following two theorems will be omitted since they are similar to the proofs of Theorems 2.1 and 2.2, respectively (see also (1) and (2), respectively).

THEOREM 2.7. *All as in Definition 1.4. Then*

$$\int_A \left| \left({}^C D^{\alpha(\cdot),\psi}_{R_1+} f \right)(x) \right| \left| f^{[n]}_\psi (x) \right| dx \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|^2_\infty}{\inf\limits_{[R_1,R_2]} (\psi')} \right)$$
$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\frac{(\psi(r) - \psi(z))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_A \left(f^{[n]}_\psi (x) \right)^2 dx \right).$$

THEOREM 2.8. *All as in Definition 1.4. Then*

$$\int_A \left| \left({}^C D^{\alpha(\cdot),\psi}_{R_2-} f \right)(x) \right| \left| f^{[n]}_\psi (x) \right| dx \leq 2^{-\frac{1}{2}} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{\|\psi'\|^2_\infty}{\inf\limits_{[R_1,R_2]} (\psi')} \right)$$
$$\left(\int_{R_1}^{R_2} \left(\int_{r}^{R_2} \left(\frac{(\psi(z) - \psi(r))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \right)^2 dz \right) dr \right)^{\frac{1}{2}} \left(\int_A \left(f^{[n]}_\psi (x) \right)^2 dx \right).$$

We continue with extreme Opial type inequalities that are easily proved using (1) and Theorem 2.3, i.e. (2) and Theorem 2.4, respectively.

THEOREM 2.9. *All as in Definition 1.4. Then*

$$\int_A \left| \left({}^C D^{\alpha(\cdot),\psi}_{R_1+} f \right)(x) \right| \left| f^{[n]}_\psi (x) \right| dx \leq \frac{2\pi^{\frac{N}{2}} R_2^{N-1}}{\Gamma\left(\frac{N}{2}\right)}$$
$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \frac{(\psi(r) - \psi(z))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \psi'(z)\, dz \right) dr \right) \left\| f^{[n]}_\psi \right\|^2_{\infty,\overline{A}}.$$

THEOREM 2.10. *All as in Definition 1.4. Then*

$$\int_A \left| \left({}^C D^{\alpha(\cdot),\psi}_{R_2-} f \right)(x) \right| \left| f^{[n]}_\psi (x) \right| dx \leq \frac{2\pi^{\frac{N}{2}} R_2^{N-1}}{\Gamma\left(\frac{N}{2}\right)}$$
$$\left(\int_{R_1}^{R_2} \left(\int_{r}^{R_2} \frac{(\psi(z) - \psi(r))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \psi'(z)\, dz \right) dr \right) \left\| f^{[n]}_\psi \right\|^2_{\infty,\overline{A}}.$$

We finish with Hardy's type inequalities, again proved using (1) and (2) in Theorems 2.5 and 2.6.

THEOREM 2.11. *All as in Definition 1.4. Then*

$$\int_A \left({}^C D^{\alpha(\cdot),\psi}_{R_1+} f(x) \right)^2 dx \leq \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|^2_{\infty,[R_1,R_2]}$$
$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\frac{(\psi(r) - \psi(z))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_A \left(f^{[n]}_\psi (x) \right)^2 dx \right).$$

THEOREM 2.12. *All as in Definition 1.4. Then*

$$\int_A \left({}^C D^{\alpha(\cdot),\psi}_{R_2-} f(x) \right)^2 dx \leq \left(\frac{R_2}{R_1} \right)^{N-1} \|\psi'\|^2_{\infty,[R_1,R_2]}$$

$$\left(\int_{R_1}^{R_2} \left(\int_r^{R_2} \left(\frac{(\psi(z) - \psi(r))^{n-\alpha(z)-1}}{\Gamma(n-\alpha(z))} \right)^2 dz \right) dr \right) \left(\int_A \left(f_\psi^{[n]}(x) \right)^2 dx \right).$$

REFERENCES

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Springer, Heidelberg, New York, 2009.

[2] G.A. Anastassiou, *Intelligent: Fractional Inequalities and Approximations Expanded*, Springer, Heidelberg, New York, 2020.

[3] G.A. Anastassiou, *Generalized Fractional Calculus: New Advancements and Applications*, Springer, Heidelberg, New York, accepted for publication, 2020.

[4] G.A. Anastassiou, *Abstract Fractional Integral Inequalities of variable order*, submitted, 2020.

[5] W. Rudin *Real and Complex Analysis*, International Student Edition, Mc Graw Hill, London, New York, 1970.

[6] D. Stroock, *A Concise Introduction to the Theory of Integration*, Third Edition, Birkäuser, Boston, Basel, Berlin, 1999.

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