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THE BOREL MAPPING OVER SOME QUASIANALYTIC LOCAL RINGS

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Abstract. Let $M = (M_j)_j$ be an increasing sequence of positive real numbers with $M_0 = 1$ such that the sequence M_{j+1}/M_j increases and let $\mathcal{E}_n(M)$ be the Denjoy-Carleman class associated to this sequence. Let $\hat{\mathcal{E}}_n(M)$ denote the Taylor expansion at the origin of all elements that belong to the ring $\mathcal{E}_n(M)$. We say that $\hat{\mathcal{E}}_n(M)$ satisfies the splitting property if for each $f \in \hat{\mathcal{E}}_n(M)$ and $A \cup B = \mathbb{N}^n$ a partition of $\mathbb{N}^n$, when $G = \sum_{w \in A} a_w x^w$ and $H = \sum_{w \in B} a_w x^w$ are formal power series with $f = G + H$, then $G \in \hat{\mathcal{E}}_n(M)$ and $H \in \hat{\mathcal{E}}_n(M)$. Our first goal is to show that if the Borel mapping $\hat{}: \mathcal{E}_1(M) \to \mathbb{R}[[x_1]]$ is a homeomorphism onto its range for the inductive topologies, then the ring $\mathcal{E}_1(M)$ coincides with the ring of real analytic germs. Secondly, we will give a negative answer to the splitting property for the quasianalytic local rings $\mathcal{E}_n(M)$. In the last section, we will show that the ring of smooth germs that are definable in the polynomially bounded o-minimal structure of the real field expanded by all restricted functions in some Denjoy-Carleman rings does not satisfy the splitting property in general.

1. Introduction

Let $\mathcal{E}_n$ denote the ring of germs at the origin in $\mathbb{R}^n$ of smooth functions germs and $\mathbb{R}[[x_1, \ldots, x_n]]$ the ring of formal series with real coefficients. If $f \in \mathcal{E}_n$, we denote by $\hat{f} \in \mathbb{R}[[x_1, \ldots, x_n]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_n \ni f \mapsto \hat{f} \in \mathbb{R}[[x_1, \ldots, x_n]]$ is called the Borel mapping. A subring $\mathcal{C}_n \subseteq \mathcal{E}_n$ is called quasianalytic if the restriction of the Borel mapping to $\mathcal{C}_n$ is injective.

It is a classical result that is proved in [10] just by using techniques from Hilbert spaces that the Borel mapping restricted to the germs at 0 of functions in a quasianalytic Denjoy-Carleman class is never onto. We first prove that if the Borel mapping over the ring $\mathcal{E}_1(M)$ is a homeomorphism for the inductive topologies onto its range, then the ring $\mathcal{E}_1(M)$ coincides with the ring of real analytic germs. Secondly, we

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consider the property introduced by K. Nowak, see [6], that we call the splitting property by showing that there is no quasianalytic Denjoy-Carleman ring $\mathcal{E}_1(M)$ that satisfies this property and that for $n \geq 2$, the rings $\mathcal{E}_n(M)$ have in general a negative answer to this property.

The theory of o-minimal structures is a wide-ranging generalization of semi algebraic and subanalytic geometry. This theory has obtained a strong interest since 1991 when A. Wilkie [11] proved that a natural extension of the family of semi algebraic sets containing the exponential function $(\mathbb{R}, +, -, ., 0, 1, <, \exp)$ is an o-minimal structure. Another way to yield quasianalytic rings is to consider the germs of smooth functions definable in a polynomially bounded o-minimal expansion of the ordered field of real numbers.

Finally, we end this paper by giving a negative answer to the splitting property for the ring of smooth germs that are definable in the polynomially bounded o-minimal structure of the real field expanded by all restricted functions in some Denjoy-Carleman rings.

2. The Borel mapping over the quasianalytic Denjoy-Carleman rings

In this paper, we will not distinguish between notation of a function and its germ.

Let us recall some basic properties of the quasianalytic Denjoy-Carleman rings.

We use the following notation: for any multi-index $J = (j_1 \ldots, j_n)$ of $\mathbb{N}^n$, we denote the length $j_1 + \ldots + j_n$ of J by the corresponding lower case letter j. We put $D^J = \partial^j / \partial x_1^{j_1} \ldots \partial x_n^{j_n}$, $J! = j_1! \ldots j_n!$ and $x^J = x_1^{j_1} \ldots x_n^{j_n}$, where $x = (x_1, \ldots, x_n)$.

Let $M = (M_j)_j$ be an increasing sequence of positive real numbers, with $M_0 = 1$. We define the Denjoy-Carleman class $\mathcal{E}_n(M)$ to be the set of smooth germs f for which there exist a neighborhood U of 0 and positive constants C and σ such that $|D^J f(x)| \leq C\sigma^j j! M_j$ for any $J \in \mathbb{N}^n$ and $x \in U$.

Here, $C\sigma^j j!$ appears as "the analytic part" of the estimate, whereas M_j can be considered as a way to allow a defect of analyticity. If $\mathcal{O}_n$ denotes the ring of real-analytic function germs at the origin of $\mathbb{R}^n$, we clearly have $\mathcal{O}_n \subset \mathcal{E}_n(M) \subset \mathcal{E}_n$.

Frow now on, we shall always make the following assumption:

the sequence M is logarithmically convex.

This amounts to saying that the sequence M_{j+1}/M_j increases.

This assumption implies that the class $\mathcal{E}_n(M)$ is a local ring with maximal ideal $\{h \in \mathcal{E}_n(M) : h(0) = 0\}$, (see [10, Proposition 1]). By [10, Theorem 2], the local ring $\mathcal{E}_n(M)$ is quasianalytic if and only if $\sum_{j=0}^{+\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty$. It is well known, thanks to [10, Corollary 1], that $\mathcal{O}_n = \mathcal{E}_n(M)$ if and only if $\sup_{j \geq 1} (M_j)^{1/j} < \infty$. We know by [10, Corollary 2] that the ring $\mathcal{E}_n(M)$ is stable under derivation if and only if $\sup_{j \geq 1} (M_{j+1}/M_j)^{1/j} < \infty$.

In this section, we will consider just the case when $n = 1$ and that all the rings

$\mathcal{E}_1(M)$ are quasianalytic, unless otherwise stated. Therefore, we have that $\mathcal{E}_1(M) := \{f \in \mathcal{E}_1 : |f^{(k)}(x_1)| \le Ch^k M_k k!, \forall x_1 \in (-\epsilon, \epsilon) \forall k \in \mathbb{N}$ for some $\varepsilon, C, h > 0\}$. Set $\Lambda_M := \{f = \sum_{n=0}^{\infty} a_n x_1^n, |a_n| \le Ch^n M_n, \forall n$ for some $C, h > 0\}$. The Borel mapping is defined as $\hat{} : \mathcal{E}_1(M) \to \Lambda_M, f \mapsto \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x_1^n$. So the Borel mapping obviously satisfies $\hat{\mathcal{E}_1}(M) \subset \Lambda_M$.

It is easy to check that for any real $\tau > 0$, $\mathcal{E}_{\tau,M} := \{f \in \mathcal{E}_n, \|f\|_{\mathcal{E},\tau} < \infty\}$ is a Banach space with the norm $\|f\|_{\mathcal{E},\tau} := \sup_{x_1 \in (-\epsilon,\epsilon)} \sup_{n \in \mathbb{N}} \frac{|f^{(n)}(x_1)|}{M_n \tau^n n!}$, and that for every $h > 0$, $\Lambda_{M,h} := \{f = \sum_{n=0}^{\infty} a_n x_1^n \in \mathbb{R}[[x_1]], \|f\|_{\Lambda,h} < \infty\}$ is also a Banach space with the norm $\|f\|_{\Lambda,h} := \sup_{n \in \mathbb{N}} \frac{|a_n|}{M_n h^n}$.

Now, let us endow the spaces $\mathcal{E}_1(M)$ and Λ_M with the inductive topologies of the spaces $\mathcal{E}_{\tau,M}$ and $\Lambda_{M,h}$ respectively.

Let $\tau > 0$, we have that $\|\hat{f}\|_{\Lambda,\tau} = \sup_{n \in \mathbb{N}} \frac{|f^{(n)}(0)|}{M_n \tau^n n!} \le \|f\|_{\mathcal{E},\tau} \forall f \in \mathcal{E}_1(M)$. So the Borel mapping $\hat{} : \mathcal{E}_1(M) \to \Lambda_M$ is a continuous linear mapping.

It is well known that $\hat{}$ maps from $\mathcal{E}_1(M)$ to Λ_M, in general its range does not coincide with Λ_M especially if the ring of the real analytic germs is strictly contained in the quasianalytic ring $\mathcal{E}_1(M)$. The aim of this section is to show that if the Borel mapping $\hat{} : \mathcal{E}_1(M) \to \Lambda_M$ is a homeomorphism into its range, then the ring $\mathcal{E}_1(M)$ coincides with the ring of real analytic germs (i.e., $\sup_{j \ge 1} (M_j)^{1/j} < \infty$).

Proposition 2.1. *The range of the space $\mathcal{E}_1(M)$ by the Borel mapping $\hat{} : \mathcal{E}_1(M) \to \Lambda_M$ is dense in the space Λ_M for the inductive topology.*

Proof. By identifying a formal power series $f := \sum_{n=0}^{\infty} a_n x_1^n$ with the sequence of its coefficients $a := (a_n)_{n \in \mathbb{N}}$, for every $h > 0$, we have

$$\Lambda_{M,h} = \{(a_n)_{n \in \mathbb{N}} : \|a\|_{\Lambda,h} := \sup_n \frac{|a_n|}{h^n M_n} < \infty\}.$$

Then by [7, Section 2.10], Λ_M is a LB-space (inductive limit of an increasing sequence of Banach spaces) and is the union of the increasing sequence of Banach spaces: $\Lambda_{M,1} \subset \Lambda_{M,2} \subset \Lambda_{M,3} \subset \dots$

We endow the space Λ_M with the corresponding inductive topology. Let $a \in \Lambda_M$ and take $k \in \mathbb{N}$ such that $a \in \Lambda_{M,k}$: We now fix $\varepsilon > 0$ and take n_0 so that $2^{-n_0} \|a\|_{\Lambda,k} < \varepsilon$. Define $b_n = a_n$ for $n \le n_0$ and $b_n = 0$ for $n > n_0$. Then

$$\|a - b\|_{\Lambda,2k} = \sup_{n > n_0} \frac{|a_n|}{k^n M_n} \frac{1}{2^n} \le \|a\|_{\Lambda,k} \frac{1}{2^{n_0}} < \varepsilon.$$

Moreover, $b = (\widehat{\sum_{j=0}^{n_0} a_j x_1^j}) \in \hat{\mathcal{E}_1}(M)$, which proves the density. □

To prove the following lemma, let us recall the famous result of functional analysis: Let E and F be two arbitrary locally convex vector spaces which are linearly homeomorphic. If E is complete, then so is F.

Lemma 2.2. *The space $\hat{\mathcal{E}_1}(M)$ is closed in Λ_M if the Borel mapping $\hat{} : \mathcal{E}_1(M) \to \Lambda_M$ is a homeomorphism onto its range for the inductive topologies.*

Proof. By assumption, the Borel mapping $\wedge : \mathcal{E}_1(M) \to \hat{\mathcal{E}_1}(M)$ is a homeomorphism. We know that $\mathcal{E}_1(M)$ is complete [3, Proposition 1]. So, the space $\hat{\mathcal{E}_1}(M)$ which is a topological subspace of Λ_M is also complete. Consequently, the space $\hat{\mathcal{E}_1}(M)$ is closed in Λ_M. □

THEOREM 2.3. *If the Borel mapping* $\wedge : \mathcal{E}_1(M) \to \hat{\mathcal{E}_1}(M)$ *is a homeomorphism for the inductive topologies, then the ring* $\mathcal{E}_1(M)$ *coincides with the ring of real analytic germs* $\mathcal{O}_1$.

Proof. By Proposition 2.1 and Lemma 2.2 and under the assumptions of this theorem, the Borel mapping $\wedge : \mathcal{E}_1(M) \mapsto \Lambda_M$ is surjective. So, by [10, Theorem 3], if the quasianalytic ring $\mathcal{E}_1(M)$ contains strictly the ring of real analytic germs, then the mapping $\wedge$ is not surjective, which is a contradiction. □

We deduce the following criterion for the analyticity of the class $\mathcal{E}_1(M)$: If for every $\sigma > 0$, there are strictly positive C, τ and ϵ such that for all $f \in \mathcal{E}_1(M)$, we have $\sup_{|x_1|<\epsilon} \sup_n \frac{|f^{(n)}(x_1)|}{M_n \sigma^n n!} \leq C \sup_n \frac{|f^{(n)}(0)|}{M_n \tau^n n!}$. Furthemore, $\sup_{n\geq 1}(M_n)^{1/n} < \infty$.

We end this section by the following remark about the surjectivity of the Borel mapping over the Denjoy-Carleman rings $\mathcal{E}_1(M)$.

We recall that the local ring $\mathcal{E}_1(M)$ is strongly non-quasianalytic if there exists a constant C such that

$$\sum_{j=k}^{+\infty} \frac{M_j}{(j+1)M_{j+1}} \leq C \frac{M_k}{M_{k+1}} \text{ for any integer } k. \tag{1}$$

It is well known by [10, Theorem 4] that in the case where the ring $\mathcal{E}_1(M)$ is strongly non-quasianalytic, the Borel mapping $\wedge : \mathcal{E}_1(M) \to \Lambda_M$ is surjective but if we enlarge the space Λ_M to the space $\mathbb{R}[[x_1]]$, it becomes never surjective despite the fact that the sequence M satisfies the condition (1).

REMARK 2.4. The Borel mapping $\wedge : \mathcal{E}_1(M) \to \mathbb{R}[[x_1]]$ is never surjective for any sequence M.

If we take the formal power series $F(x_1) = \sum_{k=0}^{\infty} M_k k^k x_1^k$and the mapping $\wedge$ is surjective, then there exists $f \in \mathcal{E}_1(M)$ such that $\hat{f} = F$, and so there exist $C > 0$ and $A > 0$ such that $|f^{(k)}(0)| \leq CA^k k! M_k$ for all $k \in \mathbb{N}$. Consequently, we obtain $|M_k k^k k!| \leq CA^k k! M_k$ for all $k \in \mathbb{N}$. So $k \leq AC^{\frac{1}{k}}$ and for k sufficiently large, we get a contradiction.

3. Problem of splitting property over the quasianalytic local rings $\mathcal{E}_n(M)$

In this section, we will give a negative answer about the splitting property for the Denjoy-Carleman rings $\mathcal{E}_n(M)$, for all $n \in \mathbb{N}^*$.

Recall that we say that a subring $\mathcal{C}_n \subseteq \mathcal{E}_n$ is a quasianalytic ring if the Borel mapping $\mathcal{C}_n \ni f \mapsto \hat{f} \in \mathbb{R}[[x_1, \ldots, x_n]]$ is injective.

Definition 3.1. Let $\mathcal{C}_n \subseteq \mathcal{E}_n$ be a quasianalytic ring. We say that $\mathcal{C}_n$ has the splitting property, if for each $f \in \mathcal{C}_n$ such that $\hat{f} = \varphi_1 + \varphi_2$ where $\varphi_1 = \sum_{\omega \in A} a_\omega x^\omega$, $\varphi_2 = \sum_{\omega \in B} a_\omega x^\omega$ and $\mathbb{N}^n = A \cup B$, $A \cap B = \emptyset$, there exist $\psi_1, \psi_2 \in \mathcal{C}_n$ with $\hat{\psi}_1 = \varphi_1$, $\hat{\psi}_2 = \varphi_2$ and $f = \psi_1 + \psi_2$.

Example 3.2. The ring of real analytic germs $\mathcal{O}_n$ clearly satisfies the splitting property.

Lemma 3.3. *Let $f \in \mathcal{C}^\infty([0,1])$ be such that $f^{(k)}(0) > 0$ if k is even and $f^{(k)}(0) = 0$ if k is odd. Suppose that there exists $x_j \in (0,1]$ such that $f^{(j)}(x_j) = 0$ for some j. Then there is a sequence $x_j > x_{j+1} > \ldots > 0$ such that $f^{(k)}(x_k) = 0$ for all $k \geq j$.*

Proof. It suffices to show that there exists $x_{j+1} \in (0, x_j)$ with $f^{(j+1)}(x_{j+1}) = 0$. Then the lemma follows by iteration.

If j is odd, then $f^j(0) = f^{(j)}(x_j) = 0$ and so Rolle's theorem implies that there exists $x_{j+1} \in (0, x_j)$ with $f^{(j+1)}(x_{j+1}) = 0$.

If j is even, then $f^{(j)}(0) > 0$ and $f^{(j)}(x_j) = 0$. Setting $g := f^{(j)}$, we have $g(0) > 0, g'(0) = 0$ and $g''(0) > 0$. This implies that $g'(x) > 0$ for small $x > 0$ (in fact, $0 < g''(0) = \lim_{x\to 0} g'(x)/x$ and hence $g'(x)/x > 0$ for small $x > 0$). So g is monotone increasing for small $x \geq 0$. Then Rolle's theorem implies that there is $x_{j+1} \in (0, x_j)$ with $g'(x_{j+1}) = f^{(j+1)}(x_{j+1}) = 0$. □

Theorem 3.4. *Suppose that the quasianalytic local ring $\mathcal{O}_1$ is strictly contained in the ring $\mathcal{E}_1(M)$. Then for any sequence $M = (M_n)_n$, the ring $\mathcal{E}_1(M)$ does not satisfy the splitting property.*

Proof. Let $f(x) = \sum_{k=0}^{\infty} \frac{k! M_k}{(2m_k)^k} \cos(2m_k x)$, where $m_k = (k+1)M_{k+1}/M_k$. So, f is the real part of the function constructed in Theorem 1 in Thilliez's paper [10]. Then $|f^{(k)}(0)| \geq k! M_k$ for all even k, and hence $f \in \mathcal{E}_1(M) \backslash \mathcal{O}_1$. We have that $\hat{f} = \sum_{k \equiv 0,1(4)} a_k x^k - \sum_{k\equiv 2,3(4)} a_k x^k =: G - H$, where $a_k > 0$ if $k \equiv 0(2)$ and $a_k = 0$ if $k \equiv 1(2)$. Assume that the ring $\mathcal{E}_1(M)$ satisfies the splitting property. Then there exist germs $g, h \in \mathcal{E}_1(M)$ with $\hat{g} = G$, $\hat{h} = H$, and $f = g - h$. Theorem III in Bang's paper [1] combined with Lemma 3.3 implies that all derivatives of g and h are positive in the interval $(0, 1)$. Bernstein's theorem on absolutely monotone functions implies that g and h extend to analytic functions in a neighborhood of 0, a contradiction. □

Proposition 3.5. *Let $\mathcal{E}_n(M)$ be a quasianalytic and non-analytic Denjoy-Carleman ring such that $\sup_{j\geq 1}(M_{j+1}/M_j)^{1/j} < \infty$. Then the ring $\mathcal{E}_n(M)$ does not satisfy the splitting property for any $n \geq 2$.*

Proof. For $n \geq 2$, the quasianalytic ring $\mathcal{E}_n(M)$ contains the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$ and are closed under composition [10, Section 1.3] and under partial differentiation. So, if the ring $\mathcal{E}_n(M)$ satisfies the splitting property, we deduce by [9, Theorem 1.3] that $\mathcal{E}_n(M)$ is exactly the ring of real analytic germs $\mathcal{O}_n$, which contradicts our assumption. □

4. Problem of splitting property over the real field expanded by the restricted functions in $\mathcal{E}_{[-1,1]^n}(M)$

Let $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \cdot, <, 0, 1)$ be the ordered field of real numbers.

For each $n \in \mathbb{N}^*$, let $\mathcal{E}_{[-1,1]^n}(M)$ denote the ring of all functions $f : [-1,1]^n \mapsto \mathbb{R}$, for which there exist an open neighborhood U of $[-1,1]^n$, a smooth function $g : U \mapsto \mathbb{R}$ and positive constants C and σ such that $f = g|_{[-1,1]^n}$ and $|D^J g(x)| \leq C\sigma^j j! M_j$ for any $J \in \mathbb{N}^n$ and $x \in U$.

For each $n \in \mathbb{N}^*$, let $f \in \mathcal{E}_{[-1,1]^n}(M)$, and define $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ by $\widetilde{f}(x) = f(x)$ if $x \in [-1,1]^n$ and $\widetilde{f}(x) = 0$ otherwise. Assume that the ring $\mathcal{E}_{[-1,1]^n}(M)$ is closed under derivation (i.e., $\sup_{j\geq 1}(M_{j+1}/M_j)^{1/j} < \infty$). We let $\mathbb{R}_{\mathcal{E}_{[-1,1]^n}(M)} := (\overline{\mathbb{R}}, (\widetilde{f})_{f\in\mathcal{E}_{[-1,1]^n}(M)})$ be the expansion of the real field by all $\widetilde{f}$ for $f \in \mathcal{E}_{[-1,1]^n}(M)$.

In this section, we will give a negative answer for the splitting property for the ring of smooth germs that are definable in the o-minimal structures $(\mathbb{R}_{\mathcal{E}_{[-1,1]^n}(M)})_n$.

For each $n \in \mathbb{N}^*$, let $\mathcal{R}_{M,n}$ denote the ring of the smooth germs that are definable in the structure $\mathbb{R}_{\mathcal{E}_{[-1,1]^n}(M)}$. Put $\mathbb{R}_{\mathcal{E}(M)} := \bigcup_{n\in\mathbb{N}^*} \mathbb{R}_{\mathcal{E}_{[-1,1]^n}(M)}$; we know according to [8] that the structure $\mathbb{R}_{\mathcal{E}(M)}$ is o-minimal and polynomially bounded, so by [5], the rings $\mathcal{R}_{M,n}$ satisfy the quasianalyticity property for each $n \in \mathbb{N}^*$.

We know by [3, Proposition 2] that each $\mathcal{R}_{M,n}$ is a local ring whose maximal ideal is generated by the germ at zero of the coordinate functions $x \longmapsto x_i : \mathbb{R}^n \longmapsto \mathbb{R}$, $i = 1, \ldots, n$. Given an increasing sequence $M = (M_j)_j$ of a strictly positive real numbers and a positive integer p, let M^p denote the sequence $(M_{pj})_j$ and $\mathcal{E}_n(M^p)$ the Denjoy-Carleman class of the sequence $M^p := (M_{pj})_j$. Suppose furthermore that the sequence M is logarithmically convex. Clearly, $\mathcal{E}_n(M) \subseteq \mathcal{E}_n(M^p)$ for each $p \in \mathbb{N}$.

We know by [2, Proposition 2.3] that the ring of smooth germs of unary functions which are definable in the structure $\overline{\mathbb{R}}$ does not satisfy the splitting property. So, the aim of the following proposition is to give a necessary condition for satisfying this property for the ring of smooth germs that are definable in the structure $(\overline{\mathbb{R}}, (\widetilde{f})_{f\in\mathcal{E}_{[-1,1]}(M)})$.

Proposition 4.1. *Let $\mathcal{E}_1(M)$ be a quasianalytic and non-analytic ring such that $\sup_{j\geq 1}(M_{j+1}/M_j)^{1/j} < \infty$. If the ring $\mathcal{R}_{M,1}$ satisfies the splitting property, then there exists $p \in \mathbb{N}$ such that the ring $\mathcal{E}_1(M^p)$ is not quasianalytic (i.e., $\sum_{j=0}^{+\infty} \frac{M_{pj}}{(j+1)M_{pj+p}} < \infty$).*

Proof. Let $f(x) = \sum_{k=0}^{\infty} \frac{k! M_k}{(2m_k)^k} \cos(2m_k x)$, where $m_k = (k+1)M_{k+1}/M_k$. So, by the proof of Theorem 3.4, $f \in \mathcal{E}_1(M)$ and therefore $f \in \mathcal{R}_{M,1}$ and we have that $\hat{f} = \sum_{k\equiv 0,1(4)} a_k x^k - \sum_{k\equiv 2,3(4)} a_k x^k =: G - H$.

Assume that the ring $\mathcal{R}_{M,1}$ has the splitting property and that the rings $\mathcal{E}_1(M^p)$ are quasianalytic for all $p \in \mathbb{N}$. So, by [4, Theorem 1.6], there exist $g \in \mathcal{E}_1(M^p)$ and $h \in \mathcal{E}_1(M^q)$ (where p and q are positive integers) such that $\hat{g} = G$ and $\hat{h} = H$. Hence $g, h \in \mathcal{E}_1(M^{pq})$, and as $\mathcal{E}_1(M) \subseteq \mathcal{E}_1(M^{pq})$, the germ f also belongs to the quasianalytic ring $\mathcal{E}_1(M^{pq})$ and that $f = g - h$ thanks to the quasianalyticity. Since [1, Theorem III] combined with Lemma 3.3 implies that all derivatives of g and h are positive in the interval $(0,1)$, from Bernstein's theorem on absolutely monotone functions, it follows that g and h extend to analytic functions in a neighborhood of 0, and so does f, which is a contradiction as $f \notin \mathcal{O}_1$. □

Remark 4.2. Under the same assumptions of Proposition 4.1, for any $n \geq 2$, the ring $\mathcal{R}_{M,n}$ does not satisfy the splitting property.

The quasianalytic local rings $\mathcal{R}_{M,n}$ contain the ring of polynomials $\mathbb{R}[x_1,\ldots,x_n]$ and are closed under composition [10, Section 1.3] and under partial differentiation. So, if the ring $\mathcal{R}_{M,n}$ satisfies the splitting property, we deduce by [9, Theorem 1.3] that the ring $\mathcal{R}_{M,n}$ coincides with the ring of real analytic germs $\mathcal{O}_n$, which is absurd.

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