MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 1 (2022), 26–34
March 2022

research paper
оригинални научни рад

NEARNESS STRUCTURE ON TEXTURE SPACES

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Abstract. Textures are point-set setting for fuzzy sets, and they provide a framework for the complement-free mathematical concepts. This paper is the first of a series of two papers on the theory of nearness spaces. This paper aims to give a new perspective for nearness structure from the textural point of view. It is proved that nearness spaces are embeddable into texture space which is connected with nearness structure.

1. Introduction

In mathematics, one of the ways to deal with problems that are topological by nature is the conceptualization approach. Here, the aim of the approach is to find topological concepts that can be expressed with basic topological arguments. A distinctive example of this is the "near/far away" concept which is a natural extension of geometry and has an important place in topology and the fields related to topology. Topology characterizes the nearness between a point and a set by using the concept of closure concept. For a metric space (X, d), if $D(A, B) = 0$, then A is near to B where $D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$, for all $A, B \subseteq X$. In this case, the closure of A is $\mathrm{cl}(A) = \{x \in X \mid D(A, \{x\}) = 0\}$. Proximity is achieved through the axiomatic of the nearness between two sets. Further, the notion of nearness spaces were introduced by Herrlich in [13] attempts to characterize of an arbitrary collection of sets, and can be used as a unifying framework for various topological structures such as uniformity and proximity [8, 9, 14, 17].

Texture theory is point-set setting for fuzzy sets and hence, some properties of fuzzy lattices (i.e. Hutton algebra) can be discussed based on textures [1–3, 5]. Ditopologies on textures unify the fuzzy topologies and classical topologies and bitopologies without the set complementation [6, 7]. Further, the notions of uniformity and metric which are related to topology in the complement free textural context was introduced in [11, 16]. On the other hand, the notion of proximity to the point free setting of textures was defined in [12, 18].

2020 Mathematics Subject Classification: 54E17, 54E05A15, 06D72, 54E15.
Keywords and phrases: nearness; symmetric topological space; cover; texture; fuzzy sets.

The aim of this first paper is to introduce nearness structure for texture spaces, and to prove that it is a more general structure than the nearness in the sense of Herrlich. It is believed that the adaptation of this concept will have important consequences for concepts such as uniformity and proximity with related textural context which are planned to be discussed in some future article.

2. Texture spaces

This section is devoted to some fundamental definitions and results of the texture theory from [1–3, 5, 6].

Definition 2.1. Let U be a set and $\mathcal{U} \subseteq \mathcal{P}(U)$. Then $\mathcal{U}$ is called *a texturing* of U if
(T1) $\emptyset \in \mathcal{U}$ and $U \in \mathcal{U}$,

(T2) $\mathcal{U}$ is a complete and completely distributive lattice such that arbitrary meets coincide with intersections, and finite joins with unions,

(T3) $\mathcal{U}$ is point-separating.
Then the pair $(U, \mathcal{U})$ is called a *texture space* or *texture*.

For $u \in U$, the *p-sets* and the *q-sets* are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}, \quad \text{respectively.}$$

Note that p-sets and q-sets form a powerful duality in the texture theory, so that many concepts in this theory are defined based on these sets.

Complementation

A mapping $\sigma_U : \mathcal{U} \to \mathcal{U}$ is called a *complementation* on $(U, \mathcal{U})$ if it satisfies the conditions $\sigma_U(\sigma_U(A)) = A$ for all $A \in \mathcal{U}$ and $A \subseteq B \Longrightarrow \sigma_U(B) \subseteq \sigma_U(A)$ for all $A, B \in \mathcal{U}$.

Example 2.2. (i) For any set X, $(X, \mathcal{P}(X), \pi)$, $\pi(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete* texture representing the usual set structure of X. Clearly, $P_x = \{x\}$, $Q_x = X \setminus \{x\}$ for all $x \in X$.

(ii) Let $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$ and $\lambda((0, r]) = (0, 1 - r]$, $r \in [0, 1]$. Clearly $(L, \mathcal{L}, \lambda)$ is the Hutton texture of $(\mathbb{I}, \prime)$, where $\mathbb{I} = [0, 1]$ with its usual order and $r' = 1 - r$ for $r \in \mathbb{I}$. Here $P_r = Q_r = (0, r]$ for all $r \in L$.

(iii) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1 - t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in \mathbb{I}$.

(iv) For textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, $\mathcal{U} \otimes \mathcal{V}$ is product texturing of $U \times V$ [5]. Note that the product texturing $\mathcal{U} \otimes \mathcal{V}$ of $U \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (U \times B)$, $A \in \mathcal{U}$ and $B \in \mathcal{V}$. Here, for $(u, v) \in U \times V$, $P_{(u,v)} = P_u \times P_v$ and $Q_{(u,v)} = (Q_u \times V) \cup (U \times Q_v)$.

Ditopology

A pair (τ, κ) of subsets of $\mathcal{U}$ is called a *ditopology* on a texture $(U, \mathcal{U})$ where the *open sets* family τ and the *closed sets* family κ satisfy

$$\begin{array}{ll} U, \emptyset \in \tau, & U, \emptyset \in \kappa \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, & K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa \\ G_i \in \tau, i \in I \implies \bigvee_{i \in I} G_i \in \tau, & K_i \in \kappa, i \in I \implies \bigcap_{i \in I} K_i \in \kappa. \end{array}$$

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. For $A \in \mathcal{U}$ we define the *closure* $\mathrm{cl}(A)$ and the *interior* $\mathrm{int}(A)$ of A under (τ, κ) by the equalities

$$\mathrm{cl}(A) = \bigcap \{K \in \kappa \mid A \subseteq K\}, \text{ and } \mathrm{int}(A) = \bigvee \{G \in \tau \mid G \subseteq A\}.$$

Usually the family τ is called a *topology*, and the family κ is called a *cotopology*.

If σ is a complementation on $(U, \mathcal{U})$ and $\kappa = \sigma(\tau)$, then (τ, κ) is called a complemented ditopology on $(U, \mathcal{U}, \sigma)$.

Dicovers

Let $(U, \mathcal{U})$ be a texture space. A difamily $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ of elements of $\mathcal{U} \times \mathcal{U}$ which satisfies $\bigcap_{j \in J_1} B_j \subseteq \bigvee_{j \in J_2} A_j$ for all partitions (J_1, J_2) of J, including the trivial partitions, is called a *dicover* of $(U, \mathcal{U})$. An important example is the family $\mathcal{P} = \{(P_u, Q_u) \mid U \not\subseteq Q_u\}$ which is a dicover for any texture $(U, \mathcal{U})$. If $\mathcal{C}$ is a dicover, then we often write $L\mathcal{D}M$ in place of $(L, M) \in \mathcal{D}$. We recall the following definitions for dicovers.

(i) $\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $j \in J$ we have $L \, \mathcal{D} \, M$ so that $A_j \subseteq L$ and $M \subseteq B_j$. In this case we write $\mathcal{C} \prec \mathcal{D}$.

(ii) If $\mathcal{C}$, $\mathcal{D}$ are dicovers then $\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A \, \mathcal{C} \, B, \, C \, \mathcal{D} \, D\}$ is the greatest lower bound (meet) of $\mathcal{C}$, $\mathcal{D}$ with respect to refinement.

(iii) The *star* and *co-star* of $C \in \mathcal{U}$ with respect to $\mathcal{C}$ are respectively the sets $\mathrm{St}(\mathcal{C}, C) = \bigvee \{A_j \mid J \in j, C \not\subseteq B_j\} \in \mathcal{U}$, and $\mathrm{CSt}(\mathcal{C}, C) = \bigcap \{B_j \mid j \in j, A_j \not\subseteq C\} \in \mathcal{U}$.

3. Nearness in texture spaces

In this section we will introduce the notion of nearness structure in the texture space theory. Firstly we recall that the definition of nearness space in the sense of Herrlich [13].

Definition 3.1. Let X be a set and $\eta = \{\mathcal{A} \subseteq \mathcal{P}(X) \mid \mathcal{A} \neq \emptyset, \, X = \bigcup \mathcal{A}\}$.

A nearness space is a pair (X, η) if η satisfies the following axioms: (1) $\mathcal{A} \in \eta$ and $\mathcal{A} \prec \mathcal{B}$ imply $\mathcal{B} \in \eta$. (2) $\mathcal{A}, \mathcal{B} \in \eta$ imply $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \in \eta$. (3) $\mathcal{A} \in \eta$ imply $\{\mathrm{int}(A) \mid A \in \mathcal{A}\} \in \eta$ where $\mathrm{int}(A) = \{x \in X \mid \{A, X \setminus \{x\} \in \eta\}$.

This definition leads to the following concepts for texture spaces.

Definition 3.2. Let $(U, \mathcal{U})$ be a texture space.
(a) Let μ be a non-empty set of non-empty dicovers of $(U, \mathcal{U})$. Then μ is called dinearnes structure if it satisfies the following conditions:

(N1) If $\mathcal{C} \prec \mathcal{D}$ and $\mathcal{C} \in \mu$, then $\mathcal{D} \in \mu$.

(N2) If $\mathcal{C} \in \mu$ and $\mathcal{D} \in \mu$, then $\mathcal{C} \wedge \mathcal{D} \in \mu$, where

$$\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A\mathcal{C}B, C\mathcal{D}D\}.$$

(N3) If $\mathcal{C} \in \mu$, then

$$(\mathrm{int}_\mu(A), \mathrm{cl}_\mu(B)) \mid A\mathcal{C}B\} \in \mu$$

where $A \in \mathcal{U}$,

$$\mathrm{int}_\mu A = \bigvee \{P_u \mid \forall P_u \not\subseteq Q_v,\ \{(A, \emptyset), (\emptyset, P_v)\} \in \mu\}$$

$$\mathrm{cl}_\mu A = \bigcap \{Q_u \mid \forall P_v \not\subseteq Q_u,\ \{(\emptyset, A), (Q_v, \emptyset)\} \in \mu\}$$

(b) A triple $(U, \mathcal{U}, \mu)$, where μ is a dinearness structure on $(U, \mathcal{U})$, is called dinearness space.

Note that $(\mathrm{int}(A), \mathrm{cl}(A))$ will be used instead of $(\mathrm{int}_\mu(A), \mathrm{cl}_\mu(B))$, unless it causes confusion.

Lemma 3.3. *Let $(U, \mathcal{U}, \mu)$ be a dinearness space and $A, B \in \mathcal{U}$. If $A \subseteq B$, then we have:* $\mathrm{int}(A) \subseteq \mathrm{int}(B)$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

Proof. We prove $\mathrm{int}(A) \subseteq \mathrm{int}(B)$, leaving the dual proof of $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ to the reader. We suppose $\mathrm{int}(A) \not\subseteq \mathrm{int}(B)$ Then there exists $r \in U$ such that

$$\mathrm{int}(A) \not\subseteq Q_r, \quad P_r \not\subseteq \mathrm{int}(B).$$

Since $\mathrm{int}(A) \not\subseteq Q_r$, there exists $u \in U$ such that $P_u \not\subseteq Q_r$ and $\{(A, \emptyset), (\emptyset, P_v)\} \in \mu$ for all $P_u \not\subseteq Q_v$. Now let $P_r \not\subseteq Q_m$ for some $m \in U$. Then $P_u \not\subseteq Q_m$ and $\{(A, \emptyset), (\emptyset, P_m)\} \in \mu$. Since $A \subseteq B$, $\{(A, \emptyset), (\emptyset, P_m)\} \prec \{(B, \emptyset), (\emptyset, P_m)\} \in \mu$ and so we have a contradiction $P_r \subseteq \mathrm{int}(B)$. □

Theorem 3.4. *Let $(U, \mathcal{U}, \mu)$ be a dinearness space. Then the mappings*

$$\mathrm{int} : \mathcal{U} \to \mathcal{U}, \quad A \mapsto \mathrm{int}(A) \quad \textit{and} \quad \mathrm{cl} : \mathcal{U} \to \mathcal{U}, \quad A \mapsto \mathrm{cl}(A)$$

are interior and closure operators, respectively.

Proof. We prove that cl is a closure operator, leaving the dual proof that int is a interior operator to the reader.
(i) Firstly, we observe that $\{(\emptyset, \emptyset), (Q_r, \emptyset)\} \notin \mu$ for some $r \in U$. Then $cl(\emptyset) = \emptyset$.

(ii) Now we show that $A \subseteq \mathrm{cl}(A)$. Suppose $A \not\subseteq \mathrm{cl}(A)$. Then we can write $A \not\subseteq Q_r$, $P_r \not\subseteq \mathrm{cl}(A)$ for some $r \in U$. Since $P_r \not\subseteq \mathrm{cl}(A)$, $P_r \not\subseteq Q_u$ for some $u \in U$ such that $\{(\emptyset, A), (Q_v, \emptyset)\} \in \mu$ where $P_v \not\subseteq Q_u$ for all $v \in U$. From the definition of dicover, $A \not\subseteq Q_r$ is obtained, which is a contradiction.

(iii) Now, we must prove that $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$. From (ii), $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$. Let $\mathrm{cl}(\mathrm{cl}(A)) \not\subseteq \mathrm{cl}(A)$. Then $\mathrm{cl}(\mathrm{cl}(A)) \not\subseteq Q_r$, $P_r \not\subseteq \mathrm{cl}(A)$ for some $r \in U$. So there exists

$u \in U$ such that $P_r \not\subseteq \mathrm{cl}(A)$ and $(\emptyset, A), (Q_v, \emptyset) \in \mu$ where $P_v \not\subseteq Q_u$ for all $v \in U$. Then we can write $(\emptyset, A), (Q_r, \emptyset) \in \mu$. Now let $P_m \not\subseteq Q_r$, $m \in U$. Since $P_m \not\subseteq Q_u$, we have $\{(\emptyset, A), (Q_m, \emptyset)\} \in \mu$, and from (N3) condition, $\{(\emptyset, \mathrm{cl}(A)), (\mathrm{int}(Q_m), \emptyset)\} \in \mu$. So, $\{(\emptyset, \mathrm{cl}(A)), (\mathrm{int}(Q_m), \emptyset)\} \prec \{(\emptyset, \mathrm{cl}(A)), (Q_m, \emptyset)\} \in \mu$ and we have $\mathrm{cl}(\mathrm{cl}(A)) \subseteq Q_r$ which is a contradiction.

(iv) Finally we prove that $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$ by Lemma 3.3. Conversely, we suppose $\mathrm{cl}(A \cup B) \not\subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Then $\mathrm{cl}(A \cup B) \not\subseteq Q_r$, $P_r \not\subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for some $r \in U$. Hence, $P_r \not\subseteq \mathrm{cl}(A)$ and $P_r \not\subseteq \mathrm{cl}(B)$. Because of $P_r \not\subseteq \mathrm{cl}(A)$, we have $\{(\emptyset, A), (Q_v, \emptyset)\} \in \mu$ such that $P_r \not\subseteq Q_u$ for some $u \in U$ where $P_v \not\subseteq Q_u$ for all $v \in U$.

Likewise, since $P_r \not\subseteq \mathrm{cl}(B)$, we have $\{(\emptyset, B), (Q_v, \emptyset)\} \in \mu$ such that $P_r \not\subseteq Q_{u'}$ where $P_v \not\subseteq Q_{u'}$ for all $v \in U$. Hence, $P_m \not\subseteq Q_u$ and $P_m \not\subseteq Q_{u'}$, and so $\mathcal{C} = \{(\emptyset, A), (Q_m, \emptyset)\} \in \mu$ and $\mathcal{D} = \{(\emptyset, B), (Q_m, \emptyset)\} \in \mu$. From (N2) condition,

$$\mathcal{C} \wedge \mathcal{D} = \{(\emptyset, A \cup B), (\emptyset, A), (\emptyset, B), (Q_m, \emptyset)\} \in \mu$$

and so

$$\mathcal{C} \wedge \mathcal{D} \prec \{(\emptyset, A \cup B), (Q_m, \emptyset)\} \in \mu.$$

Consequently, we have $cl(A \cup B) \subseteq Q_r$ which is a contradiction.

□

THEOREM 3.5. *Let $(U, \mathcal{U}, \mu)$ be a dinearness space. Then the pair (τ_μ, κ_μ) is a ditopology on $(U, \mathcal{U})$ where*

$$\tau_\mu = \{G \in \mathcal{U} \mid \mathrm{int}(G) = G\}, \quad \kappa_\mu = \{F \in \mathcal{U} \mid \mathrm{cl}(F) = F\}.$$

Proof. We prove that the family κ_μ is a cotopology on $(U, \mathcal{U})$, leaving the dual proof that the family int_μ is a topology to the reader. Clearly $U \in \kappa_\mu$. Further, $\emptyset \in \kappa_\mu$, since $\mathrm{cl}(\emptyset) = \emptyset$ by Theorem 3.4.

Let $A, B \in \kappa_\mu$. Then $\mathrm{cl}(A) = A$ and $\mathrm{cl}(B) = B$. Thus $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) = A \cup B$ by Teorem 3.4, and we have $A \cup B \in \kappa_\mu$.

Let $\{K_j\}_{j \in J} \subseteq \kappa_\mu$. By Theorem 3.4, we can write $\mathrm{cl}(\bigcap_{j \in J}(K_j)) \subseteq \bigcap_{j \in J} K_j$. Because of $\mathrm{cl}(K_j) = K_j$ for all $j \in J$, it is obtained that

$$\bigcap_{j \in J} K_j \subseteq K_j \Longrightarrow \mathrm{cl}(K_j) = K_j \subseteq \mathrm{cl}(\bigcap_{j \in J} K_j) \Longrightarrow \mathrm{cl}(\bigcap_{j \in J} K_j) \subseteq \bigcap K_j.$$

Consequently, $\bigcap_{j \in J} K_j \in \kappa_\mu$. □

Now we give the relation between the notion of nearness in the sense of Herrlich and dinearness structure. Firstly, we recall some useful results from [10, Proposition 11.1].

REMARK 3.6. Let U be a non-empty set. Then
(a) Let $\mathcal{C} = \{A_j \mid j \in J\} \subseteq \mathcal{P}(U)$. Then $\mathcal{C}$ is a cover of U if and only if $\{(A, X \setminus A) \mid A \in \mathcal{C}\}$ is a dicover of the discrete texture space $(U, \mathcal{P}(U))$.

(b) If a family $\mathcal{D} = \big\{(A_i, B_i) \mid i \in I\big\}$ is a dicover of $(U, \mathcal{P}(U))$ then the families $\{A_i\}_{i \in I}$ and $\{X \setminus B_i\}_{i \in I}$ are covers of U.

Now let $(U, \mathcal{U})$ be a texture. The family of dicovers of $(U, \mathcal{U})$ will be denoted by $\mathcal{DC}$.

THEOREM 3.7. *Let (X, η) be a nearness space and $\mathcal{DC}$ be the dicover family of $(U, \mathcal{P}(U))$. That is, $\mathcal{DC} = \big\{\mathcal{C} = \{(A_i, B_i) \mid i \in I\} \mid \mathcal{C}$ is a dicover of $(U, \mathcal{U})$ $\big\}$. Then the family,*

$$\mu = \big\{ \mathcal{C} \in \mathcal{DC} \mid \{A_i\}_{i\in I} \in \eta \quad ve \quad \{X \setminus B_i\}_{i\in I} \in \eta \big\}$$

is a dinearness structure on the discrete space $(X, \mathcal{P}(X))$.

Proof. We observe that $\mu \neq \emptyset$, since $\{(X, \emptyset)\} \in \mu$.

(N1) Let $\mathcal{C} \in \mu$ and $\mathcal{D} = \big\{(C_i, D_i) \mid i \in I\big\} \in \mathcal{DC}$. Suppose that $\mathcal{C} \prec \mathcal{D}$ and $(A, B) \in \mathcal{C}$. Then there exists $(C, D) \in \mathcal{D}$ such that $A \subseteq C$ and $D \subseteq B$. Here, the families $\{C_i\}_{i\in I}$ and $\{X \setminus D_i\}_{i\in I}$ are cover of X by Remark 3.6.

So, $\{A_i\}_{i\in I} \in \eta$ and $\{X \setminus B_i\}_{i\in I} \in \eta$, and we have $\mathcal{D} \in \mu$.

(N2) Let $\mathcal{C} = \{(A_i, B_i) \mid i \in I\} \in \mu$ and $\mathcal{D} = \{(C_i, D_i) \mid i \in I\} \in \mu$. Then $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \in \eta$ where $\mathcal{A}_1 = \{A_i\}_{i\in I}$, $\mathcal{B}_1 = \{X \setminus B_i\}_{i\in I}$, $\mathcal{A}_2 = \{C_j\}_{j\in J}$ and $\mathcal{B}_2 = \{X \setminus D_j\}_{j\in J}$.

Thus, $\{A \cap C \mid A \in \mathcal{A}_1, C \in \mathcal{A}_2\} \in \eta$ and $\{(X \setminus B) \cap (X \setminus D) \mid B \in \mathcal{B}_1, D \in \mathcal{B}_2\} = \{X \setminus (B \cup D) \mid B \in \mathcal{B}_1, D \in \mathcal{B}_2\} \in \eta$ and so we have

$$\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid (A, B) \in \mathcal{C},\ (C, D) \in \mathcal{D}\} \in \mu.$$

Let $\mathcal{C} \in \mu$. Since $\mathcal{A} = \{A_i\}_{i\in I} \in \eta$ and $\mathcal{B} = \{X \setminus B_i\}_{i\in I} \in \eta$, $\{\mathrm{int}_\eta(A) \mid A \in \mathcal{A}\} \in \eta$ and $\{\mathrm{int}_\eta(X \setminus B) \mid X \setminus B \in \mathcal{B}\} \in \eta$. For some $(A, B) \in \mathcal{C}$, we have

$$\begin{aligned}
\mathrm{int}_\mu A &= \bigvee \{P_u \mid \forall P_u \not\subseteq Q_v, \{(A, \emptyset), (\emptyset, P_v)\} \in \mu\} \\
&= \bigcup \{\{u\} \mid \forall u = v, \{(A, \emptyset), (\emptyset, P_v)\} \in \mu\} \\
&= \{u \mid \{(A, \emptyset), (\emptyset, \{u\})\} \in \mu\} = \{u \mid \{A, X - \{u\}\} \in \eta\} = \mathrm{int}_\eta(A)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{cl}_\mu B &= \bigcap \{Q_u \mid \forall P_v \not\subseteq Q_u, \{(\emptyset, B), (Q_v, \emptyset)\} \in \mu\} \\
&= \bigcap \{X \setminus \{u\} \mid \forall u = v, \{(\emptyset, B), (Q_v, \emptyset)\} \in \mu\} \\
&= X \setminus \bigcup \{u \mid \{(\emptyset, B), (Q_u, \emptyset)\} \in \mu\} \\
&= X \setminus \{u \mid \{X \setminus B, X \setminus \{u\}\} \in \eta\} = X \setminus \mathrm{int}_\eta(X \setminus B) = \mathrm{cl}_\eta(B).
\end{aligned}$$

Hence $X \setminus \mathrm{cl}_\mu(B) = X \setminus cl_\eta(B) = \mathrm{int}_\eta(X \setminus B)$ and $\{\mathrm{int}_\mu(A_i) \mid i \in I\} \in \eta$ ve $\{X \setminus \mathrm{cl}_\mu(B_i) \mid i \in I\} \in \eta$. Consequently, $\big\{(\mathrm{int}_\mu(A), \mathrm{cl}_\mu(B)) \mid A\mathcal{B}\big\} \in \mu$. □

Now, we will give dinearness structure by using bi-R_0 ditopology. Firstly, we recall [15] that a topological space X is called a R_0, or symmetric, topological space if $G \subseteq X$ is open set and $x \in G$, then $\mathrm{cl}\,\{x\} \subseteq G$. On the other hand, a ditopology (τ, κ) on complemented texture $(U, \mathcal{U})$ is bi-R_0 if $G \in \tau$ and $G \not\subseteq Q_u$, then $\mathrm{cl}(P_u) \subseteq G$, or equivalently, if $F \in \kappa$ and $P_u \not\subseteq F$, then $F \subseteq \mathrm{int}(Q_u)$ [7].

Note that a topological space $(X, \mathcal{T})$ is R_0 if and only if the corresponding ditopology $(\mathcal{T}, \mathcal{T}^c)$ on the complemented discrete texture $(X, \mathcal{P}(X), \pi)$ which by Example 2.2 (i) is bi-R_0 [7, Example 3.3].

Let $(X, \mathcal{T})$ be a R_0 topological space. Then (X, η) is a nearness space [13] where $\eta = \{\mathcal{A} \subseteq \mathcal{P}(X) \mid X = \bigcup_{A\in\mathcal{A}} \mathrm{int}(A)\}$. Then we have the following corollary.

Corollary 3.8. *Let $(X,\mathfrak{T})$ be an R_0 topological space and the pair (X,η) be corresponding nearness space. Now let $(\mathfrak{T},\mathfrak{T}^c)$ be the corresponding ditopology on the discrete texture $(X,\mathcal{P}(X))$. Then the family*

$$\begin{aligned}\mu &= \big\{\,\mathcal{C}\in\mathcal{DC}\mid \{A_i\}_{i\in I}\in\eta \text{ and } \{X\setminus B_i\}_{i\in I}\in\eta\big\}\\ &= \big\{\,\mathcal{C}\in\mathcal{DC}\mid X=\bigcup_{(A_i,B_i)\in\mathcal{C}}\operatorname{int}(A_i) \quad \text{and} \quad X=\bigcup_{(A_i,B_i)\in\mathcal{C}}\operatorname{int}(X\setminus B_i)\big\}\\ &= \big\{\,\mathcal{C}\in\mathcal{DC}\mid X=\bigcup_{(A_i,B_i)\in\mathcal{C}}\operatorname{int}(A_i) \quad \text{and} \quad X=\bigcup_{(A_i,B_i)\in\mathcal{C}}X\setminus\operatorname{cl}(B_i)\big\}\\ &= \big\{\,\mathcal{C}\in\mathcal{DC}\mid X=\bigcup_{(A_i,B_i)\in\mathcal{C}}\operatorname{int}(A_i) \quad \text{and} \quad \emptyset=\bigcap_{(A_i,B_i)\in\mathcal{C}}\operatorname{cl}(B_i)\big\}\end{aligned}$$

is a dinearness structure on $(X,\mathcal{P}(X))$ by Teorem 3.7.

An approach for complemented texture spaces will be given in the last section.

Lemma 3.9. *Let $(U,\mathcal{U},\sigma)$ be a complemented texture space and $\mathcal{C},\mathcal{D}\in\mathcal{DC}$. The following are satisfied.*
(i) The family $\sigma(\mathcal{C})=\{(\sigma(B),\sigma(A))\mid A\mathcal{C}B\}$ is a dicover of $(U,\mathcal{U})$.

(ii) $\sigma(\mathcal{C}\wedge\mathcal{D})=\sigma(\mathcal{C})\wedge\sigma(\mathcal{D})$.

(iii) $\mathcal{C}\prec\mathcal{D}\Longleftrightarrow\sigma(\mathcal{C})\prec\sigma(\mathcal{D})$.

Proof. (i) Let (I_1,I_2) be a partition of an index set I. Since $\mathcal{C}=\{(A_i,B_i)\mid\in I\}$ is a dicover of $(U,\mathcal{U})$, we can write $\bigcap_{i\in I_1}B_i\subseteq\bigvee_{i\in I_2}A_i$ and $\sigma(\bigvee_{i\in I_2}A_i)\subseteq\sigma(\bigcap_{i\in I_1}B_i)\Longrightarrow\bigcap_{i\in I_2}\sigma(A_i)\subseteq\bigvee_{i\in I_1}\sigma(B_i)$. Thus $\sigma(\mathcal{C})=\{(\sigma(B),\sigma(A))\mid A\mathcal{C}B)\}$ is a dicover of $(U,\mathcal{U})$.

(ii) We observe that $\mathcal{C}\wedge\mathcal{D}=\{(A\cap C,B\cup D)\mid(A,B)\in\mathcal{C},(C,D)\in\mathcal{D}\}$, and then we have

$$\begin{aligned}\sigma(\mathcal{C}\wedge\mathcal{D})&=\{(\sigma(B\cup D),\sigma(A\cap C))\mid(\sigma(B),\sigma(A))\in\sigma(\mathcal{C}),(\sigma(D),\sigma(C))\in\sigma(\mathcal{D})\}\\&=\{(\sigma(B)\cap\sigma(D),\sigma(A)\cup\sigma(C))\mid(\sigma(B),\sigma(A))\in\sigma(\mathcal{C}),(\sigma(D),\sigma(C))\in\sigma(\mathcal{D})\}\\&=\sigma(\mathcal{C})\wedge\sigma(\mathcal{D}).\end{aligned}$$

(iii) Let $\mathcal{C}\prec\mathcal{D}$. Then it is obtained that

$$\begin{aligned}\mathcal{C}\prec\mathcal{D}&\Longrightarrow\forall(A,B)\in\mathcal{C},\exists(C,D)\in\mathcal{D}\quad\text{such that}\quad A\subseteq C,\ D\subseteq B\\&\Longrightarrow\forall(\sigma(B),\sigma(A))\in\sigma(\mathcal{C}),\exists(\sigma(D),\sigma(C))\in\sigma(\mathcal{D})\\&\qquad\text{such that}\quad\sigma(C)\subseteq\sigma(A),\ \sigma(B)\subseteq\sigma(D)\\&\Longrightarrow\sigma(\mathcal{C})\prec\sigma(\mathcal{D}).\end{aligned}$$

□

Theorem 3.10. *Let $(U,\mathcal{U},\sigma)$ be a complemented texture space and μ be a dinearness structure on $(U,\mathcal{U})$. Set*

$$\operatorname{int}_{\sigma(\mu)}A=\sigma(\operatorname{cl}_\mu\sigma(A))\quad\text{and}\quad\operatorname{cl}_{\sigma(\mu)}A=\sigma(\operatorname{int}_\mu\sigma(A)),\quad\forall A\in\mathcal{U}.$$

Then we have:

(i) If (τ_μ, κ_μ) is the obtaining ditopology by μ, then $\sigma(\tau_\mu) = \kappa_{\sigma(\mu)}$.

(ii) The family $\sigma(\mu) = \{\sigma(\mathcal{C}) \mid \mathcal{C} \in \mu\}$ is a dinearness structure on $(U, \mathcal{U})$.

Proof. (i) We observe that

$$\begin{aligned} F \in \sigma(\tau_\mu) &\Longrightarrow \sigma(F) \in \tau_\mu \\ &\Longrightarrow \operatorname{int}_\mu(\sigma(F)) = \sigma(F) \Longrightarrow \sigma(\operatorname{int}_\mu(\sigma(F))) = \sigma(\sigma(F)) \\ &\Longrightarrow \operatorname{cl}_{\sigma(\mu)} F = F \Longrightarrow F \in \kappa_{\sigma(\mu)} \end{aligned}$$

and

$$\begin{aligned} F \in \kappa_{\sigma(\mu)} &\Longrightarrow \operatorname{cl}_{\sigma(\mu)} F = F \\ &\Longrightarrow \sigma(\operatorname{cl}_{\sigma(\mu)} F) = \sigma(F) \Longrightarrow \operatorname{int}_\mu(\sigma(F))) = \sigma(F) \\ &\Longrightarrow \sigma(F) \in \tau_\mu \Longrightarrow F \in \sigma(\tau_\mu). \end{aligned}$$

(ii) We show that $\sigma(\mu)$ is a dinearness structure.

(N1) Let $\sigma(\mathcal{C}) \in \sigma(\mu)$ and $\mathcal{D} \in \mathcal{DC}_U$. Suppose $\sigma(\mathcal{C}) \prec \mathcal{D}$. Then we have $\mathcal{D} \in \sigma(\mu)$, since $\mathcal{C} \prec \sigma(\mathcal{D})$ and $\sigma(\mathcal{D}) \in \mu$.

(N2) It is clear from Lemma 3.9.

Let $\mathcal{C} = \{(A_i, B_i) \mid i \in I\} \in \mu$ and $\sigma(\mathcal{C}) = \{(\sigma(B_i), \sigma(A_i)) \mid i \in I, (A_i, B_i) \in \mathcal{C}\}$. Now we prove that

$$\{(\operatorname{int}_{\sigma(\mu)}(\sigma(B_i)), \operatorname{cl}_{\sigma(\mu)}(\sigma(A_i))) \mid (\sigma(B_i), \sigma(A_i)) \in \sigma(\mathcal{C}))\} \in \sigma(\mu).$$

Since $\mathcal{C} \in \mu$, we can write

$$\{(\operatorname{int}_\mu(A_i), \operatorname{cl}_\mu(B_i)) \mid i \in I, (A_i, B_i) \in \mathcal{C}\} \in \mu$$

and, thus,

$$\{(\sigma(\operatorname{cl}_\mu(B_i)), \sigma(\operatorname{int}_\mu(A_i))) \mid i \in I, (A_i, B_i) \in \mathcal{C}\} \in \sigma(\mu).$$

Finnaly, by assumption, we have

$$\sigma(\operatorname{cl}_\mu B_i) = \operatorname{int}_{\sigma(\mu)} \sigma(B_i) \quad \text{and} \quad \sigma(\operatorname{int}_\mu A_i) = \operatorname{cl}_{\sigma(\mu)} \sigma(A_i) \in \sigma(\mu). \qquad \square$$

Definition 3.11. Let $(U, \mathcal{U}, \sigma)$ be a complemented texture space and μ be a dinearness structure on $(U, \mathcal{U})$. Then the quadruple $(U, \mathcal{U}, \sigma, \mu)$ is called complemented dinearness texture space if $\mu = \sigma(\mu)$.

Corollary 3.12. *Let $(U, \mathcal{U}, \sigma, \mu)$ be a complemented dinearness texture space. Then the pair (τ_μ, κ_μ) is a complemented ditopology which is obtained by μ.*

Acknowledgement. The author sincerely thanks the reviewers for their valuable comments that improve the presentation of the paper.

This work has been supported by the Turkish Scientific and Technological Research Council under the project 116F434.

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(received 27.03.2020; in revised form 18.01.2021; available online 26.12.2021)

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