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A NEW CLASS OF FINSLER METRICS

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Abstract. In this paper, we construct a new class of Finsler metrics which are not always (α, β)-metrics. We obtain the spray coefficients and Cartan connection of these metrics. We have also found a necessary and sufficient condition for them to be projective. Finally, under some suitable conditions, we obtain many new Douglas metrics from the given one.

1. Introduction

(α, β)-metrics form a rich class of Finsler metrics. They are computable and the patterns offer references for more study in Finsler spaces. Then, introducing new Finsler metrics which are not (α, β)-metrics helps us to evaluate the patterns. There are some classes of Finsler metrics which are not always (α, β)-metrics such as generalized (α, β)-metrics [15] or spherically symmetric Finsler metrics [17].

Here we are going to consider the Finsler metrics given by

$$\bar{F} = F\phi(s), \tag{1}$$

where F is a Finsler metric, $s = \frac{\beta}{F}$, $\beta = b_i y^i$, $\|\beta\|_F < b_0$ and $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$. We call them (F, β)-metrics. These metrics are not always (α, β)-metrics even if F is an (α, β)-metric.

Let $F = \alpha + \gamma$ be a Randers metric, where α is a Riemannian metric and γ is a 1-form on M. Put

$$\bar{F} = \frac{(F+\beta)^2}{F} = \frac{(\alpha+\gamma+\beta)^2}{\alpha+\beta} = \alpha\frac{(1+s+\bar{s})^2}{1+s},$$

where $s = \frac{\beta}{\alpha}$ and $\bar{s} = \frac{\gamma}{\alpha} \neq s$. Here $\bar{F} = \alpha\Psi(s, \bar{s})$ is a Finsler metric but not (α, β)-metric. Whereas, for any 1-form β on M, $\bar{F} = F + \beta = \alpha + \beta + \gamma$ is a Randers metric.

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Let F be a projectively flat Finsler metric such as the generalized Berwald's metric

$$F=\frac{\big((1+\langle a,x\rangle)(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)+(1-|x|^2)\langle a,y\rangle\big)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}},$$

where $a\in\mathbb{R}^n$ is a constant vector. It is locally projectively flat with constant flag curvature $K=0$ [11]. For any closed 1-form β such that $\beta=\frac{\langle a,y\rangle}{1+\langle a,x\rangle}$, metric $\bar{F}=F+\beta$ is also a projectively flat Finsler metric and β is closed form (see Theorem 1.1).

One could consider the above metrics as a change of a given Finsler metric. Various Finsler changes have been extensively studied and they have numerous applications.

In 1971, Matsumoto introduced the transformation of Finsler metric $\bar{F}(x,y)=F(x,y)+\beta(x,y)$, where $\beta(x,y)=b_i(x)y^i$ is a 1-form [9]. In 1984, Shibata [12] introduced the transformation of Finsler metric $\bar{F}(x,y)=f(F,\beta)$, where $\beta(y)=b_i(x)y^i$, $b_i(x)$ are components of a covariant vector in (M^n,F) and f is positively homogenous function of degree 1 in F and β. This change of metric is called a $\beta-$change.

In 1980, while studying the conformal transformation of Finsler spaces, H. Izumi [8] introduced the concept of an h-vector b_i. The h-vector b_i, as well as the function of coordinates x^i itself, are also dependent on y^i. The h-vector b_i is v-covariant constant with respect to the Cartan connection and satisfies $FC^h_{ij}b_h=\rho h_{ij}$, where ρ is a non-zero scalar function, C^h_{ij} are components of Cartan tensor and h_{ij} are components of angular metric tensor. We will prove the following theorem.

Theorem 1.1. *An (F,β)-metric $\bar{F}=F\phi(s)$, where $s=\frac{\beta}{F}$, $\beta(x,y)=b_i(x,y)y^i$ with h-vector b_i, is projectively flat if and only if*

$$2(\phi-s\phi'+\rho\phi')h_{ir}G^r+2F\phi' s_{i0}+\phi''\frac{\Theta}{\lambda}m_i=0, \tag{2}$$

where $r_{ij}:=\frac{1}{2}\big(b_{i|j}+b_{j|i}\big)$, $s_{ij}:=\frac{1}{2}\big(b_{i|j}-b_{j|i}\big)$, $s_{i0}=s_{ij}y^j$, $\Theta=(\phi-s\phi'+\rho\phi')r_{00}-2F\phi' s_0$, $\lambda=\phi-s\phi'+\rho\phi'+(b^2-s^2)\phi''$, $h_{ij}=g_{ij}-\ell_i\ell_j$ and $m_i=b_i-s\ell_i$.

One could easily show that the above theorem is satisfied for every (F,β)-metric with $\beta(y)=b_i(x)y^i$ just by putting $\rho=0$, with β not being necessarily an h-vector.

In this paper, we study the (F,β)-metric with F being an m-root Finsler metric. Let (M,F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i,y^i) the coordinates in a local chart on TM. Let F be a scalar function on TM defined by $F=\sqrt[m]{A}$, where A is given by $A:=a_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ with $a_{i_1\dots i_m}$ symmetric in all its indices. Then F is called an m-root Finsler metric.

Theorem 1.1 includes all known results about projective changes of Finsler metrics [10, 13, 14]. For instance, we get the following two corollaries which were stated as theorems in the respected papers.

Corollary 1.2 ([2, 10]). *Let $F=\sqrt[m]{A}$ $(m>3)$ be an m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$, where A is irreducible. Then Randers change $\bar{F}=F+\beta$ with $\beta=b_i(x)y^i$ is locally projectively flat if and only if it is locally Minkowski.*

Corollary 1.3 ([13]). *Let $F = \sqrt[m]{A}$ ($m > 3$) be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Then Matsumoto change $\bar{F} = \frac{F^2}{F-\beta}$ with $\beta = b_i(x)y^i$ is locally projectively flat if and only if $\frac{\partial A}{\partial x^l} = 0$ and $b_i = constant$.*

Finally, one could easily conclude that the following also holds.

Corollary 1.4. *Let $F = \sqrt[m]{A}$ ($m > 3$) be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Then (F,β)-metric $\bar{F} = F\phi(\frac{\beta}{F})$ with $\beta = b_i(x)y^i$ is locally projectively flat if and only if*

$$-m(m-1)\lambda(\phi - s\phi')y_i A_0 A^{1-\frac{4}{m}} + m\lambda(\phi - s\phi')(A_{0i} - A_{x^i})A^{1-\frac{2}{m}} + 2\phi'(\lambda s_{i0} - \phi'' s_0 m_i)A^{\frac{1}{m}} + \phi''(\phi - s\phi')r_{00}m_i = 0,$$

where $\lambda = \phi - s\phi' + (b^2 - s^2)\phi''$, r_{00}, s_{i0} and s_0 are represented as (20), A_{0i}, A_{x^i} and A_0 are defined in (46).

A Finsler metric is called Douglas metric if the Douglas tensor $D = 0$. The Douglas curvature was introduced by J. Douglas in 1927 [4]. In the same paper it was proven that Douglas and Weyl tensors are invariant under projective changes. Roughly speaking, a Douglas metric is a Finsler metric having the same geodesics as a Riemannian metric. Hence, in this paper we are going to obtain the conditions under which the change $\bar{F} = F\phi(s)$ of Douglas space becomes a Douglas space. Then we will prove the following.

Theorem 1.5. *Let (M,F) be a Douglas space. An (F,β)-metric $\bar{F} = F\phi(\frac{\beta}{F})$ with h-vector b_i is Douglas if and only if*

$$H^{ij} := \frac{F\phi'}{\phi - s\phi' + \rho\phi'}(s^i_0 y^j - s^j_0 y^i) + \frac{\phi''\left[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\right]}{2(\phi - s\phi' + \rho\phi')\lambda}(b^i y^j - b^j y^i), \qquad (3)$$

where r_{ij}, s^i_0 and s_0 are represented as (20), are homogeneous polynomial in y^i of degree 3.

By the above theorem one could obtain many Douglas metrics from a given one. For example, the following corollary introduces some new Douglas metrics from a given m-root Finsler metric of Douglas type.

Corollary 1.6. *(i) Let $F = \sqrt[m]{A}$ ($m > 3$) be an m-root Finsler metric of Douglas type. Then Randers change $\bar{F} = F + \beta$ with $\beta = b_i(x)y^i$ is of Douglas type if and only if $s_{ij} = 0$.*

(ii) Let $F = \sqrt[m]{A}$ ($m > 3$) be an m-root Finsler metric of Douglas type. Then Matsumoto change $\bar{F} = \frac{F^2}{F-\beta}$ with $\beta = b_i(x)y^i$ is of Douglas type if and only if $b_{i|j} = 0$.

2. Preliminaries

Let M be a smooth manifold and $TM := \bigcup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. A Finsler metric on M is a function

$F : TM \longrightarrow [0, +\infty)$ with the following properties
(i) F is C^∞ on $TM\backslash\{0\}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $x \in M$, the following quadratic form $\boldsymbol{g}_y$ on T_xM is positive definite, $\boldsymbol{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\big[F^2(y + su + tv)\big]|_{t,s=0}$, $u, v \in T_xM$.
Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\boldsymbol{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\boldsymbol{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\big[\boldsymbol{g}_{y+tw}(u, v)\big]|_{t=0}$, $u, v, w \in T_xM$.

The family $\boldsymbol{C} := \{\boldsymbol{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\boldsymbol{C} = 0$ if and only if F is Riemannian.

Given a Finsler manifold (M, F), then a global vector field $\boldsymbol{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\boldsymbol{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on TM_0 given by $G^i = \frac{1}{4}g^{il}\Big\{\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\Big\}y^j y^k$. $\boldsymbol{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\boldsymbol{G}$ is called a geodesic in M.

The Cartan connection in M is given as $C\Gamma = (\Gamma^i_{jk}, N^i_j, C^i_{jk})$, where

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\Big\{\frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\Big\},\ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^m_i\frac{\partial}{\partial y^m},\ N^i_j = \dot{\partial}_j G^i,\ N^i_j y^j = 2G^i.$$

Note that ∂_i and $\dot{\partial}_i$ denote the derivations with respect to x^i and y^i respectively. For the Cartan connection, we define $X^i_{j|k} := \frac{\delta X^i_j}{\delta x^k} + X^r_j\Gamma^i_{rk} - X^i_r\Gamma^r_{jk}$, $X^i_{j;k} := \dot{\partial}_k X^i_j + X^r_j C^i_{rk} - X^i_r C^r_{jk}$, where "$|$" and "$;$" denote the horizontal and vertical covariant derivative of X^i_j. Also, the axioms $g_{ij|k} = 0$ and $g_{ij;k} = 0$ hold.

The h-vector b_i is v-covariant constant with respect to the Cartan connection and satisfies $FC^h_{ij}b_h = \rho h_{ij}$, where ρ is a non-zero scalar function, $C^h_{ij} = g^{mh}C_{ijm}$ and h_{ij} are components of angular metric tensor. Thus if b_i is an h-vector then (i) $b_{i;j} = 0$, (ii) $FC^h_{ij}b_h = \rho h_{ij}$. Put $c^h = g^{ij}C^h_{ij}$. Hence we obtain

$$\rho = \frac{F}{n-1}c^h b_h, \tag{4}$$

$$\dot{\partial}_j b_i = \frac{\rho}{F}h_{ij}. \tag{5}$$

Since $\rho \neq 0$ and $h_{ij} \neq 0$, the h-vector b_i depends not only on positional coordinates but also on directional arguments. Izumi [8] showed that ρ is independent of directional arguments and that if b_i is an h-vector then ${}^*b_i := b_i - \rho\ell_i$ and $b := \|\beta\|_F$ are independent of y.

3. (F, β)-metrics

Throughout the paper we shall use the notations $\ell_i := \dot{\partial}_i F$, $\ell_{ij} := \dot{\partial}_i\dot{\partial}_j F$, $\ell_{ijk} := \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F$. Let b_i be an h-vector in the Finsler space (M, F). Since $h_{ij}y^j = 0$, we have $\dot{\partial}_i\beta = b_i$. Contracting with y^j will be denoted by the subscript 0. For example, we write $b_{i|0}$ for $b_{i|j}y^j$.

Using (5) and the fact that $\ell_{i|j} = \ell_{ij|k} = 0$ we have the following relations

$$\partial_j b_i = b_{i|j} + \rho N^r_j \ell_{ir} + b_r \Gamma^r_{ij}, \tag{6}$$

$$\partial_j \ell_i = N^r_j \ell_{ir} + \ell_r \Gamma^r_{ij}, \tag{7}$$

$$\partial_k \ell_{ij} = N^r_k \ell_{ijr} + \ell_{rj}\Gamma^r_{ik} + \ell_{ir}\Gamma^r_{jk}. \tag{8}$$

For $s = \beta/F$, by (6) and the fact that $\partial_k F = \ell_r N^r_k$ we have

$$\dot{\partial}_i s = \frac{1}{F} m_i, \quad \partial_i s = \frac{1}{F}\big(b_{0|i} + m_r N^r_i\big), \tag{9}$$

where $m_i := b_i - s\ell_i$. Using (6) and (7) we get

$$\begin{aligned} \dot{\partial}_k m_i &= (\rho - s)\ell_{ik} - \frac{1}{F} m_k \ell_i, \\ \partial_k m_i &= b_{i|k} + (\rho - s)\ell_{ir}N^r_k - \frac{1}{F} m_r N^r_k \ell_i + m_r \Gamma^r_{ik} - \frac{1}{F} b_{0|k}\ell_i. \end{aligned} \tag{10}$$

Differentiating equation (1) with respect to y^i, y^j, y^k and using the first equations in (9) and (10) imply that

$$\bar{\ell}_i = \phi \ell_i + \phi' m_i, \tag{11}$$

$$\bar{\ell}_{ij} = \Big(\phi - s\phi' + \rho\phi'\Big)\ell_{ij} + \frac{\phi''}{F} m_i m_j, \tag{12}$$

$$\begin{aligned} \bar{\ell}_{ijk} =& \big[\phi - s\phi' + \rho\phi'\big]\ell_{ijk} + \frac{\phi''}{F}(\rho - s)\big[m_k\ell_{ij} + m_j\ell_{ik} + m_i\ell_{jk}\big] + \frac{\phi'''}{F^2} m_i m_j m_k \\ & - \frac{\phi''}{F^2}\big[m_i m_j \ell_k + m_i m_k \ell_j + m_j m_k \ell_i\big]. \end{aligned} \tag{13}$$

Definition 3.1. A Finsler metric $\bar{F}$ is called (F,β)-metric if it has the following form $\bar{F} = F\phi(s)$, $s := \frac{\beta}{F}$, where F is a Finsler metric and $\beta = b_i y^i$ is a 1-form on an n-dimensional manifold M, $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ and $\|\beta\|_F < b_0$.

(F,β)-metric $\bar{F}$ is called (F,β)-metric with h-vector if $b_i := b_i(x,y)$ be an h-vector on (M,F).

Lemma 3.2. *For any Finsler metric F and 1-form $\beta = b_i y^i$ with h-vector b_i on manifold M with $\|\beta\|_F < b_0$, $\bar{F} = F\phi(s)$ is a Finsler metric if and only if the positive C^∞ function $\phi = \phi(s)$ satisfies*

$$\phi(s) - s\phi'(s) + \rho\phi'(s) > 0, \quad \phi(s) - s\phi'(s) + \rho\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \tag{14}$$

when $n \geq 3$ or $\phi(s) - s\phi'(s) + \rho\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, when $n = 2$, where $s = \frac{\beta}{F}$ and b are arbitrary numbers with $|s| \leq b < b_0$ and ρ is given by (4).

Proof. The case $n = 2$ is similar to $n \geq 3$, so we only prove the proposition for $n \geq 3$. It is easy to verify that $\bar{F}$ is a function with regularity and positive homogeneity. In the following we will verify strong convexity. Direct computations yield the fundamental tensor $\bar{g}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j\bar{F}^2$ as follows

$$\bar{g}_{ij} = \eta g_{ij} + \eta_0 b_i b_j + \eta_1(\ell_i b_j + \ell_j b_i) + \eta_2 \ell_i \ell_j, \tag{15}$$

where $\eta := \phi(\phi - s\phi' + \rho\phi')$, $\eta_0 := \phi\phi'' + \phi'\phi'$, $\eta_1 := \phi\phi' - s\eta_0$, $\eta_2 := -s\eta_1 - \rho\phi\phi'$. Using [11, Lemma 1.1.1], we obtain

$$\det(\bar{g}_{ij}) = \phi^{n+1}(\phi - s\phi' + \rho\phi')^{n-2}\big(\phi - s\phi' + \rho\phi' + (b^2 - s^2)\phi''\big)\det(g_{ij}). \tag{16}$$

Assume that (14) is satisfied. Using (14) and (16), we get $\det(\bar{g}_{ij}) > 0$. The rest of the proof is similar to the proof for (α, β)-metrics from [11]. □

By putting $\rho = 0$, one could easily show that Lemma 3.2 is satisfied for (F, β)-metrics.

Corollary 3.3. *Let M be an n-dimensional manifold. For any Finsler metric F and 1-form $\beta = b_i y^i$ with $\|\beta\|_F < b_0$, $\bar{F} = F\phi(s)$ is a Finsler metric if and only if the positive C^∞ function $\phi = \phi(s)$ satisfies $\phi(s) - s\phi'(s) > 0$, $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, when $n \geq 3$, or $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, when $n = 2$, where $s = \frac{\beta}{F}$ and b are arbitrary numbers with $|s| \leq b < b_0$.*

The formula for $(\bar{g}^{ij})$ can be obtained from [11, Lemma 1.1.1],

$$\bar{g}^{ij} = \frac{1}{\bar{\rho}}\Big[g^{ij} - \frac{\bar{\delta}}{1 + b^2\bar{\delta}}b^i b^j - \frac{\bar{\mu}}{1 + \bar{\mu}\bar{Y}^2}(\ell^i + \bar{\lambda}b^i)(\ell^j + \bar{\lambda}b^j)\Big], \tag{17}$$

where $(g^{ij}) = (g_{ij})^{-1}$, $b^2 = b_i b^i = g^{ij}b_i b_j$ and $\bar{\delta} := \frac{1}{\eta}\big(\eta_0 - \frac{\eta_1^2}{\eta_2}\big)$, $\bar{\mu} := \frac{\eta_2}{\eta}$, $\bar{\lambda} := \frac{\bar{\epsilon} - \bar{\delta}s}{1 + b^2\bar{\delta}}$, $\bar{\epsilon} := \frac{\eta_1}{\eta_2}$, $\bar{Y}^2 := 1 + (\bar{\epsilon} + \bar{\lambda})s + \bar{\epsilon}\bar{\lambda}b^2$. Differentiating (15) with respect to y^k, the Cartan tensor $\bar{C}_{ijk}$ is given by

$$\bar{C}_{ijk} = \eta C_{ijk} + \frac{\eta'}{2F}h_{ijk} + \frac{\eta_0'}{2F}m_i m_j m_k, \tag{18}$$

where $h_{ijk} := m_i h_{jk} + m_j h_{ik} + m_k h_{ij}$. By (17) and (18) we can obtain

$$\begin{aligned}\bar{C}^i_{jk} =& C^i_{jk} + \frac{\eta'}{2\eta F}h^i_{jk} + \frac{\eta_0'}{2\eta F}m^i m_j m_k - \frac{1}{2\eta F}\Big\{\big[2\rho\eta + \eta'(b^2 - s^2)\big]h_{jk}\\ &+ \big[2\eta' + \eta_0'(b^2 - s^2)\big]m_j m_k\Big\} \times \Big\{\Big[\frac{\bar{\delta}}{1 + \bar{\delta}b^2} + \frac{\bar{\mu}\bar{\lambda}^2}{1 + \bar{\mu}\bar{Y}^2}\Big]b^i + \frac{\bar{\mu}\bar{\lambda}^2}{1 + \bar{\mu}\bar{Y}^2}\ell^i\Big\}.\end{aligned} \tag{19}$$

For 1-form $\beta = b_i(x, y)y^i$ where b_i is an h-vector, we have

$$r_{ij} := \frac{1}{2}\big(b_{i|j} + b_{j|i}\big), \qquad s_{ij} := \frac{1}{2}\big(b_{i|j} - b_{j|i}\big). \tag{20}$$

where "$|$" denotes the horizontal covariant derivative with respect to the Cartan connection of F. Moreover, we define $r_{i0} := r_{ij}y^j$, $r_j := b^i r_{ij}$, $r_0 := r_j y^j$, $r_{00} = r_{ij}y^i y^j$, $s_{i0} := s_{ij}y^j$, $s_j := b^i s_{ij}$, $s_0 := s_j y^j$, $s^i_0 = g^{ij}s_{j0}$. Then $\dot{\partial}_k s_{ij} = \frac{1}{2}\big(\rho_j\ell_{ik} - \rho_i\ell_{jk}\big)$, $\dot{\partial}_k s_{i0} = \frac{1}{2}\rho_0\ell_{ik} + s_{ik}$, where $\rho_i = \partial_i\rho$ and $\rho_0 = \rho_k y^k$.

4. Spray coefficients of (F, β)-metrics

In this section we are going to calculate the spray coefficients of (F, β)-metrics. First assume that β is a 1-form with h-vector.

Differentiating (11) with respect to x^j and using (7) and the second equations in (9) and (10) yield

$$\partial_j \bar{\ell}_i = \phi\Big[\ell_{ir} N^r_j + \ell_r \Gamma^r_{ij}\Big] + \frac{\phi''}{F}\Big[b_{0|j} + m_r N^r_j\Big] m_i + \phi'\Big[b_{i|j} + (\rho - s)\ell_{ir} N^r_j + m_r \Gamma^r_{ij}\Big]. \quad (21)$$

Next, we deal with $\bar{\ell}_{i|j} = 0$, that is $\partial_j \bar{\ell}_i = \bar{\ell}_{ir} \bar{N}^r_j + \bar{\ell}_r \bar{\Gamma}^r_{ij}$. Let us define

$$D^i_{jk} := \bar{\Gamma}^i_{jk} - \Gamma^i_{jk}, \quad D^i_j := D^i_{jk} y^k = \bar{N}^i_j - N^i_j, \quad D^i := D^i_j y^j = 2\bar{G}^i - 2G^i. \quad (22)$$

Then $\partial_j \bar{\ell}_i = \bar{\ell}_{ir}(D^r_j + N^r_j) + \bar{\ell}_r(D^r_{ij} + \Gamma^r_{ij})$.

Putting (11) and (12) in the above equation yields

$$\partial_j \bar{\ell}_i = \bar{\ell}_{ir} D^r_j + \bar{\ell}_r D^r_{ij} + \Big[(\phi - s\phi' + \rho\phi')\ell_{ir} + \frac{\phi''}{F} m_i m_r\Big] N^r_j + \Big[\phi \ell_r + \phi' m_r\Big]\Gamma^r_{ij}. \quad (23)$$

By comparing (21) and (23), we get the following $\phi' b_{i|j} = \bar{\ell}_{ir} D^r_j + \bar{\ell}_r D^r_{ij} - \frac{\phi''}{F} m_i b_{0|j}$. Thus by (20) we have

$$2\phi' r_{ij} = \bar{\ell}_{ir} D^r_j + \bar{\ell}_{jr} D^r_i + 2\bar{\ell}_r D^r_{ij} - \frac{\phi''}{F}\big[m_i b_{0|j} + m_j b_{0|i}\big], \quad (24)$$

$$2\phi' s_{ij} = \bar{\ell}_{ir} D^r_j - \bar{\ell}_{jr} D^r_i - \frac{\phi''}{F}\big[m_i b_{0|j} - m_j b_{0|i}\big]. \quad (25)$$

Contracting (24) and (25) by y^j implies that

$$2\phi' r_{i0} = \bar{\ell}_{ir} D^r + 2\bar{\ell}_r D^r_i - \frac{\phi''}{F} r_{00} m_i, \quad (26)$$

$$2\phi' s_{i0} = \bar{\ell}_{ir} D^r - \frac{\phi''}{F} r_{00} m_i. \quad (27)$$

Subtracting (27) from (26) yields

$$\phi'(r_{i0} - s_{i0}) = \bar{\ell}_r D^r_i. \quad (28)$$

Contracting (28) by y^i leads to

$$\phi' r_{00} = \bar{\ell}_r D^r. \quad (29)$$

To obtain the spray coefficients of $\bar{F}$, first we must prove the following lemma.

Lemma 4.1. *The system of algebraic equations*
(i) $\bar{\ell}_{ir} A^r = B_i$, *(ii)* $\bar{\ell}_r A^r = B$,
has a unique solution A^r *for given* B *and* B_i *such that* $B_i y^i = 0$. *The solution is given by*

$$A^i = \frac{F}{\phi - s\phi' + \rho\phi'} B^i + \frac{1}{\phi}\big(B - \frac{F}{\lambda}\phi' B_r b^r\big)\ell^i - \frac{F\phi''(B_r b^r)}{\lambda(\phi - s\phi' + \rho\phi')} m^i,$$

where $B^i = g^{il} B_l$ *and* $m^i = g^{il} m_l$.

Proof. Contracting (12) by b^i implies that

$$\bar{\ell}_{ir} b^i = \frac{\lambda}{F} m_r, \quad (30)$$

where $\lambda := \phi - s\phi' + \rho\phi' + (b^2 - s^2)\phi''$.

Then contracting equation (i) by b^i and using (30), we get the following

$$\frac{\lambda}{F} m_r A^r = B_r b^r. \quad (31)$$

Substituting (11) in equation (ii) yields $\phi\ell_r A^r + \phi' m_r A^r = B$. Putting (31) in this equation we get

$$\ell_r A^r = \frac{1}{\phi}\big(B - \frac{F}{\lambda}\phi' B_r b^r\big). \tag{32}$$

Substituting (12) in equation (i) and using the fact that $\ell_{ir} = \frac{1}{F}\big(g_{ir} - \ell_i\ell_r\big)$, we get

$$g_{ir}A^r = \frac{F}{\phi - s\phi' + \rho\phi'}B_i + (\ell_r A^r)\ell_i - \frac{\phi''}{\phi - s\phi' + \rho\phi'}(m_r A^r)m_i.$$

Contracting this equation by g^{ij} and using (31) and (32) complete the proof. □

Now, we are able to obtain the spray coefficients of $\bar{F}$.

By contracting (27) by b^i and using the above relations, we get $\frac{\lambda}{F}m_r D^r = 2\phi' s_0 + \frac{\phi''}{F}r_{00}(b^2 - s^2)$. The equations (27) and (29) constitute a system of algebraic equations in $\ell_r D^r$ and $m_r D^r$ whose solution from Lemma 4.1 is given by

$$D^i = \frac{F}{\phi - s\phi' + \rho\phi'}B^i + \frac{1}{\phi}\big(B - \frac{F}{\lambda}\phi' B_r b^r\big)\ell^i - \frac{F\phi''}{\lambda(\phi - s\phi' + \rho\phi')}B_r b^r m^i,$$

where $B^i = 2\phi' s^i_0 + \frac{\phi''}{F}r_{00}m^i$, $B = \phi' r_{00}$, $B_r b^r = 2\phi' s_0 + \frac{\phi''}{F}(b^2 - s^2)r_{00}$. Since $D^i = 2\bar{G}^i - 2G^i$, we get the following theorem.

Theorem 4.2. *Let $\bar{F}$ be an (F,β)-metric with h-vector b_i. Then the spray coefficients of $\bar{F}$ are given by*

$$\begin{aligned}2\bar{G}^i = 2G^i &+ \frac{2F\phi'}{\phi - s\phi' + \rho\phi'}s^i_0 + \frac{\big[\phi'(\phi - s\phi' + \rho\phi') - s\phi\phi''\big]\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{\phi(\phi - s\phi' + \rho\phi')\lambda}\ell^i \\ &+ \frac{\phi''\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{(\phi - s\phi' + \rho\phi')\lambda}b^i.\end{aligned} \tag{33}$$

Corollary 4.3. *Let $\bar{F}$ be an (F,β)-metric. Then the spray coefficients of $\bar{F}$ are given by*

$$\begin{aligned}2\bar{G}^i = 2G^i &+ \frac{2F\phi'}{\phi - s\phi'}s^i_0 + \frac{\big[\phi'(\phi - s\phi') - s\phi\phi''\big]\big[(\phi - s\phi')r_{00} - 2F\phi' s_0\big]}{\phi(\phi - s\phi')(\phi - s\phi' + (b^2 - s^2)\phi'')}\ell^i \\ &+ \frac{\phi''\big[(\phi - s\phi')r_{00} - 2F\phi' s_0\big]}{(\phi - s\phi')(\phi - s\phi' + (b^2 - s^2)\phi'')}b^i.\end{aligned} \tag{34}$$

5. Cartan connection of (F,β)-metrics

Here the Cartan connection coefficients of (F,β)-metrics are calculated. Differentiating (12) with respect to x^k and using (9) and (10), we get

$$\begin{aligned}\partial_k\bar{\ell}_{ij} = &\big[\phi - s\phi' + \rho\phi'\big]\partial_k\ell_{ij} + \frac{\phi''}{F}(\rho - s)\big[b_{0|k} + m_r N^r_k\big]\ell_{ij} + \phi'\rho_k\ell_{ij} + \frac{\phi'''}{F^2}\big[b_{0|k} + m_r N^r_k\big]m_i m_j \\ &- \frac{\phi''}{F^2}m_i m_j\partial_k F + \frac{\phi''}{F}m_j\big[b_{i|k} + (\rho - s)\ell_{ir}N^r_k - \frac{1}{F}m_r N^r_k\ell_i + m_r\Gamma^r_{ik} - \frac{1}{F}b_{0|k}\ell_i\big]\end{aligned}$$

$$+\frac{\phi''}{F}m_i\big[b_{j|k}+(\rho-s)\ell_{rj}N^r_j-\frac{1}{F}m_rN^r_k\ell_j+m_r\Gamma^r_{jk}-\frac{1}{F}b_{0|k}\ell_j\big]. \tag{35}$$

With the help of $\bar{\ell}_{ij|k}=0$, that is $\partial_k\bar{\ell}_{ij}=\bar{\ell}_{ijr}\bar{N}^r_k+\bar{\ell}_{rj}\bar{\Gamma}^r_{ik}+\bar{\ell}_{ir}\bar{\Gamma}^r_{jk}$, and by (22) we have $\partial_k\bar{\ell}_{ij}=\bar{\ell}_{ijr}(D^r_k+N^r_k)+\bar{\ell}_{rj}(D^r_{ik}+\Gamma^r_{ik})+\bar{\ell}_{ir}(D^r_{jk}+\Gamma^r_{jk})$. Putting the values of $\bar{\ell}_{ir}$, $\bar{\ell}_{rj}$ and $\bar{\ell}_{ijr}$ from (12) and (13) in the above equation yields

$$\begin{aligned}\partial_k\bar{\ell}_{ij}=&\bar{\ell}_{ijr}D^r_k+\bar{\ell}_{rj}D^r_{ik}+\bar{\ell}_{ir}D^r_{jk}+\Big\{\big[\phi-s\phi'+\rho\phi'\big]\ell_{ijr}+\frac{\phi''}{F}(\rho-s)\big[m_r\ell_{ij}+m_j\ell_{ir}+m_i\ell_{jr}\big]\\&+\frac{\phi'''}{F^2}m_im_jm_r-\frac{\phi''}{F^2}\big[m_im_j\ell_r+m_im_r\ell_j+m_jm_r\ell_i\big]\Big\}N^r_k\\&+\Gamma^r_{ik}\Big\{\big[\phi-s\phi'+\rho\phi'\big]\ell_{rj}+\frac{\phi''}{F}m_rm_j\Big\}+\Gamma^r_{jk}\Big\{\big[(\phi-s\phi'+\rho\phi'\big]\ell_{ir}+\frac{\phi''}{F}m_im_r\Big\}.\end{aligned} \tag{36}$$

By comparing (35) and (36) and using (8) and the fact that $\partial_kF=\ell_rN^r_k$ we get the following

$$\begin{aligned}\bar{\ell}_{ijr}D^r_k+\bar{\ell}_{rj}D^r_{ik}+\bar{\ell}_{ir}D^r_{jk}=&\phi'\rho_k\ell_{ij}+\frac{\phi''}{F}(\rho-s)b_{0|k}\ell_{ij}+\frac{\phi''}{F}\Big[m_jb_{i|k}+m_ib_{j|k}\Big]\\&-\frac{\phi''}{F^2}b_{0|k}\big[m_i\ell_j+m_j\ell_i\big]+\frac{\phi'''}{F^2}b_{0|k}m_im_j.\end{aligned} \tag{37}$$

Contracting (37) by y^k yields

$$\begin{aligned}\bar{\ell}_{ijr}D^r+\bar{\ell}_{rj}D^r_i+\bar{\ell}_{ir}D^r_j=&\phi'\rho_0\ell_{ij}+\frac{\phi''}{F}(\rho-s)r_{00}\ell_{ij}+\frac{\phi''}{F}\Big[m_jb_{i|0}+m_ib_{j|0}\Big]\\&-\frac{\phi''}{F^2}r_{00}\big[m_i\ell_j+m_j\ell_i\big]+\frac{\phi'''}{F^2}r_{00}m_im_j.\end{aligned} \tag{38}$$

Substituting (25) in equation (38) implies that

$$\bar{\ell}_{ir}D^r_j=Q_{ij}, \tag{39}$$

where

$$\begin{aligned}Q_{ij}:=&-\frac{1}{2}\bar{\ell}_{ijr}D^r+\phi's_{ij}+\frac{1}{2}\rho_0\phi'\ell_{ij}+\frac{\phi''}{F}\big(m_ir_{j0}+m_js_{i0}\big)+\frac{\phi''}{2F}(\rho-s)r_{00}\ell_{ij}\\&-\frac{\phi''}{2F^2}r_{00}(m_i\ell_j+m_j\ell_i)+\frac{\phi'''}{2F^2}r_{00}m_im_j.\end{aligned}$$

From (27), we see $Q_{ij}y^i=0$. On the other hand, the equation (28) may be written as

$$\bar{\ell}_rD^r_j=Q_j, \tag{40}$$

where $Q_j:=\phi'(r_{j0}-s_{j0})$. The equations (40) and (39) constitute the system of algebraic equations whose solution from Lemma 4.1 is given by

$$D^i_j=\frac{F}{\phi-s\phi'+\rho\phi'}Q^i_j+\frac{1}{\phi}\big(Q_j-\frac{F}{\lambda}\phi'Q_{rj}b^r\big)\ell^i-\frac{F\phi''}{\lambda(\phi-s\phi'+\rho\phi')}Q_{rj}b^rm^i,$$

here $Q^i_j=g^{ir}Q_{rj}$. Then by (22) we have

$$\bar{N}^i_j=N^i_j+\frac{F}{\phi-s\phi'+\rho\phi'}Q^i_j+\frac{1}{\phi}\big(Q_j-\frac{F}{\lambda}\phi'Q_{rj}b^r\big)\ell^i-\frac{F\phi''}{\lambda(\phi-s\phi'+\rho\phi')}Q_{rj}b^rm^i. \tag{41}$$

Finally, applying Christoffel process with respect to indices i, j, k in equation (37) we obtain

$$\bar{\ell}_{rj} D^r_{ik} = M_{jik}, \tag{42}$$

where

$$\begin{aligned} M_{jik} := &-\frac{1}{2}\big[\bar{\ell}_{ijr} D^r_k + \bar{\ell}_{jkr} D^r_i - \bar{\ell}_{kir} D^r_j\big] + \frac{1}{2}\phi'\big[\rho_k \ell_{ij} + \rho_i \ell_{jk} - \rho_j \ell_{ik}\big] \\ &+ \frac{\phi''}{F}\big[m_j r_{ik} + m_i s_{jk} + m_k s_{ji}\big] + \frac{\phi''}{2F}(\rho - s)\big[b_{0|k}\ell_{ij} + b_{0|i}\ell_{jk} - b_{0|j}\ell_{ik}\big] \\ &+ \frac{\phi'''}{2F^2}\big[b_{0|k} m_i m_j + b_{0|i} m_k m_j - b_{0|j} m_i m_k\big] \\ &- \frac{\phi''}{2F^2}\big[b_{0|k}(m_i \ell_j + m_j \ell_i) + b_{0|i}(m_j \ell_k + m_k \ell_j) - b_{0|j}(m_i \ell_k + m_k \ell_i)\big]. \end{aligned}$$

Moreover, by (38) we get $M_{jik} y^j = 0$. Besides, the equation (24) may be written as

$$\bar{\ell}_r D^r_{ik} = M_{ik}, \tag{43}$$

where $M_{ik} := \phi' r_{ik} - \frac{1}{2}\bar{\ell}_{ir} D^r_k - \frac{1}{2}\bar{\ell}_{rk} D^r_i + \frac{\phi''}{2F}\big[m_i b_{0|k} + m_k b_{0|i}\big]$. Applying Lemma 4.1 to equations (42) and (43) implies that

$$D^i_{jk} = \frac{F}{\phi - s\phi' + \rho\phi'} M^i_{jk} + \frac{1}{\phi}\big(M_{jk} - \frac{F}{\lambda}\phi' M_{rjk} b^r\big)\ell^i - \frac{F\phi''}{\lambda(\phi - s\phi' + \rho\phi')} M_{rjk} b^r m^i,$$

where $M^i_{jk} = g^{ir} Q_{rjk}$. Then by (22) we get

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{F}{\phi - s\phi' + \rho\phi'} M^i_{jk} + \frac{1}{\phi}\big(M_{jk} - \frac{F}{\lambda}\phi' M_{rjk} b^r\big)\ell^i - \frac{F\phi''}{\lambda(\phi - s\phi' + \rho\phi')} M_{rjk} b^r m^i. \tag{44}$$

Theorem 5.1. *Let $C\bar{\Gamma} = (\bar{\Gamma}^i_{jk}, \bar{N}^i_j, \bar{C}^i_{jk})$ be the Cartan connection for the Finsler space $(M, \bar{F})$ where $\bar{F}$ is an (F, β)-metric with h-vector b_i. Then the Cartan connection is completely determined by the equations* (19)*,* (41) *and* (44)*.*

6. Proof of Theorem 1.1

Proof. Suppose that F and $\bar{F}$ be projectively related i.e. $\bar{G}^i - G^i = P y^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\bar{F}$ and F, respectively and $P = P(x, y)$ is a scalar function on the slit tangent bundle TM_0. By (22) we have $D^i = 2P y^i$. Putting it in (33) we get

$$\begin{aligned} 2P y^i = &\frac{2F\phi'}{\phi - s\phi' + \rho\phi'} s^i_0 + \frac{\big[\phi'(\phi - s\phi' + \rho\phi') - s\phi\phi''\big]\big[(\phi - s\phi' + \rho\phi') r_{00} - 2F\phi' s_0\big]}{\phi(\phi - s\phi' + \rho\phi')\lambda}\ell^i \\ &+ \frac{\phi''\big[(\phi - s\phi' + \rho\phi') r_{00} - 2F\phi' s_0\big]}{(\phi - s\phi' + \rho\phi')\lambda} b^i. \end{aligned} \tag{45}$$

Contracting (45) by $y_i := g_{ij}y^j$ and using the facts that $s^i_0 y_i = 0$ and $\ell^i y_i = F$, we obtain $P = \frac{\phi'\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2F\lambda\phi}$. Now let $\bar{F}$ be projectively flat; then one has $2\bar{G}^i = 2G^i + D^i = 2\bar{P}y^i$. Using the same calculations as above, by (33) one gets

$$h_{ij}G^j + \frac{F\phi'}{\phi - s\phi' + \rho\phi'}s_{i0} + \frac{\phi''\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2(\phi - s\phi' + \rho\phi')\lambda}m_i = 0.$$

Conversely, putting (2) in (33) yields that

$$\begin{aligned}\bar{G}^i =& G^i + \Big(\frac{F\phi'}{\phi - s\phi' + \rho\phi'}s_{r0} + \frac{\phi''\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2(\phi - s\phi' + \rho\phi')\lambda}m_r\Big)g^{ri}\\ &+ \frac{\phi'\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2\phi\lambda}\ell^i\\ =& G^i - h_{rj}g^{ri}G^j + \frac{\phi'\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2\phi\lambda}\ell^i\\ =& \Big(\ell_j G^j + \frac{\phi'\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2\phi\lambda}\Big)\ell^i = \bar{P}y^i.\end{aligned}$$

This completes the proof. □

6.1 Proof of Corollary 1.2

Note that for m-root Finsler metrics we have [16]:

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i}y^i, \quad A_{0l} = A_{x^r y^l}y^r, \tag{46}$$

and $2G^i = A^{ir}(A_{0r} - A_{x^r})$. Also, it is not hard to get $A_i = mA^{1-\frac{2}{m}}y_i$, and $A^r_i = (mA^{1-\frac{2}{m}}y^r)_{.i} = mA^{1-\frac{2}{m}}\Big(\delta^r_i + (m-2)\ell_i\ell^r\Big)$. Then after some calculations we have

$$2h_{ij}G^j = mA^{1-\frac{2}{m}}\Big(A_{0i} - A_{x^i} - (m-1)A_0 A^{-\frac{1}{m}}\ell_i\Big). \tag{47}$$

Putting the above equations in (2) yields that

$$-m(m-1)A_0 y_i A^{1-\frac{4}{m}} + m(A_{0i} - A_{x^i})A^{1-\frac{2}{m}} + 2s_{i0}A^{\frac{1}{m}} = 0.$$

By the following lemma, the above equation yields $A_{x^i} = 0$ and $s_{ij} = 0$ for $m \neq 5$. For $m = 5$ we get the same conclusion just by separating rational and irrational parts of equation.

Lemma 6.1. *Let $F = \sqrt[m]{A}$ ($m > 2$, $m \neq 5$), be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the equation $\Psi A^{1-\frac{4}{m}} + \Omega A^{1-\frac{2}{m}} + \Theta A^{\frac{1}{m}} = 0$ holds, where Ψ, Ω and Θ are homogeneous polynomials in y. Then $\Psi = \Omega = \Theta = 0$.*

Corollaries 1.3 and 1.4 are proven in a similar manner.

7. (F,β)-metrics of Douglas type

In [4], Douglas introduced the local functions $D^i_{j\ kl}$ on TM_0 defined by

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\Big).$$

It is easy to verify that $D := D^i_{j\ kl} dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on TM_0. D is called the Douglas tensor. The Finsler space (M,F) is called a Douglas space if and only if $G^i y^j - G^j y^i$ is a homogeneous polynomial of degree three in y^i [1].

7.1 Proof of Theorem 1.5

By (33) we get $\bar{G}^i y^j - \bar{G}^j y^i = G^i y^j - G^j y^i + H^{ij}$, where

$$H^{ij} := \frac{F\phi'}{\phi - s\phi' + \rho\phi'}(s^i_0 y^j - s^j_0 y^i) + \frac{\phi''\big[(\phi - s\phi' + \rho\phi')r_{00} - 2F\phi' s_0\big]}{2(\phi - s\phi' + \rho\phi')\lambda}(b^i y^j - b^j y^i).$$

With the help of the above definition, if F and $\bar{F}$ are Douglas metrics then H^{ij} must be a homogeneous polynomial of degree three in y^i.

By this theorem one could obtain many new Douglas metrics from a given one.

7.2 Proof of Corollary 1.6

(i) Putting $F = \sqrt[m]{A}$ and $\phi(s) = 1+s$ in (3) yields $2H^{ij} - (s^i_0 y^j - s^j_0 y^i)\sqrt[m]{A} = 0$. Then by separating rational and irrational parts of the above equation one gets $s^i_0 y^j = s^j_0 y^i$ and thus $s_{ij} = 0$.

(ii) Here $F = \sqrt[m]{A}$ and $\phi(s) = \frac{1}{1-s}$; then one has $\phi'(s) = \frac{1}{(1-s)^2}$, $\phi''(s) = \frac{2}{(1-s)^3}$, $\phi(s) - s\phi'(s) = \frac{1-2s}{(1-s)^2}$, $\lambda = \frac{1+2b^2-3s}{(1-s)^3}$. Putting them in (3) yields

$$\begin{aligned}(1-2s)(1+2b^2-3s)H^{ij} &- (1+2b^2-3s)\sqrt[m]{A}(s^i_0 y^j - s^j_0 y^i)\\ &- \Big((1-2s)r_{00} - 2s_0\sqrt[m]{A})\Big)(b^i y^j - b^j y^i) = 0.\end{aligned}$$

Multiplying above equation by $A^{\frac{2}{m}}$ yields

$$\begin{aligned}6\beta^2 H^{ij} &+ \beta\big[2r_{00}(b^i y^j - b^j y^i) - (4b^2+5)H^{ij}\big]A^{\frac{1}{m}}\\ &+ \big[(1+2b^2)H^{ij} + 3\beta(s^i_0 y^j - s^j_0 y^i) - r_{00}(b^i y^j - b^j y^i)\big]A^{\frac{2}{m}}\\ &+ \big[2s_0(b^i y^j - b^j y^i) - (1+2b^2)(s^i_0 y^j - s^j_0 y^i)\big]A^{\frac{3}{m}} = 0.\end{aligned}$$

Similar to Lemma 6.1 ($m > 3$), one could easily get $H_{ij} = 0$, $r_{00} = 0$ and $s_{ij} = 0$, which yields $b_{i|j} = 0$.

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