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INVARIANTS, SOLUTIONS AND INVOLUTION OF HIGHER ORDER DIFFERENTIAL SYSTEMS

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Abstract. The paper is concerned with the interpretation of the fixed points of an involution as invariant solutions under certain Lie algebra of symmetries of a given equation. Our aim is to study the involutivity in terms of the symmetries of an equation. We prove that if $\pi : E \to M$ is a fiber bundle and $\nabla : T^*M \to J^1T^*M$ is a linear connection on the base space, then there exists a unique involutive linear automorphism, α_∇ in J^1J^1E, that commutes with the projections π_{11} and $J^1\pi_{1,0}$. Moreover, we prove that the space $J^k(\pi)$ is the quotient space of the iterated sesqui-holonomics jets $\hat{J}^1J^{k-1}(\pi)$ relative to the subgroup of symmetries determined by some involution α_g.

1. Introduction

We know [2] that an anti-symplectic involution on a symplectic manifold (M, ω) is a map $\alpha : M \to M$ such that $\alpha^*\omega = \omega$ and $\alpha^2 = \mathrm{Id}|_M$. The fixed point set of an anti-symplectic involution is always a Lagrangian manifold. A classical construction of such an involution is given by complex conjugation when M is a smooth complex subvariety of the complex projective space $\mathbb{P}^n$ cut out by polynomials with real coefficients. In this case the fixed point locus is just the intersection with the real projective space $\mathbb{RP}^n$. On another side, using the properties of the fixed point, Villaroel showed in [9] the following:
(i) The n-planes P in a submanifold $J^k(E, n)$, that are horizontal with respect to the projection π_k and contained in $J^{k+1}(E, n)$, are fixed by a canonical involution.

(ii) If a submanifold $L \subset J^{k+1}(E, n)$ has at each point an n-plane fixed by the canonical involution, then L generates an involutive n-distribution on E.
In this paper, we review the notion of involution and the constructions shown by Villaroel [9] to discuss the interpretation of fixed points of an involution in terms of the invariant solutions under certain Lie algebra of symmetries of a given equation.

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In the second section we investigate the basic notion of jet spaces and have a brief review of iterated jet space with some important identifications. We prove that if $\pi : E \to M$ is a fiber bundle and $\nabla : T^*M \to J^1T^*M$ is a linear connection on the base space, then there exists a unique involutive linear automorphism, α_∇ in J^1J^1E, that commutes with the projections π_{11} and $J^1\pi_{10}$.

In the next section we review the notion of the symbol of a manifold and finally the last section is about the invariant solutions of an equation and we prove that the space $J^k(\pi)$ is the quotient space of the iterated sesqui-holonomics jets $\hat{J}^1J^{k-1}(\pi)$ relative to the subgroup of symmetries determined by some involution $\alpha_{\mathfrak{g}}$. The meaning of all these symbols will be explained later in the next sections.

2. Jet spaces

Let E be an $(n+m)$-dimensional manifold. Two n-dimensional manifolds L_1 and L_2 are k-equivalent in $x \in L_1 \cap L_2$ if L_1 and L_2 have contact of order k in x. This equivalence defines $J^k(E,n) = \{[L]^k_x \mid \dim L = n, x \in L\}$, where $[L]^k_x$ denotes the k jets of L in x and is the class of equivalence under the relation of contact. The important application that relates the distinct jet spaces is $\pi_{k,k-1} : J^k(E,n) \to J^{k-1}(E,n)$, where $\pi_{k,k-1}([L]^k_x) = [L]^{k-1}_x$. So the jet spaces are related in the following way,

$$E \xleftarrow{\pi_{1,0}} J^1(E,n) \xleftarrow{\pi_{2,1}} \cdots \longleftarrow J^{k-1}(E,n) \xleftarrow{\pi_{k,k-1}} J^k(E,n) \ .$$

In the case of a fiber bundle $\pi : E \to M$ where $\dim M = n$ and $\dim E = n+m$, consider all submanifolds of dimension n that are images of local sections of π; we denote these submanifolds by $\Gamma_{loc}(\pi)$. If $s \in \Gamma_{loc}(\pi)$, with $s : U \to E$ and U an open set of M, put $s(U) = L$. We define $[s]^k_x = [L]^k_{s(x)}$ for $x \in U$ and the space of k-jets of the so-defined bundle π is denoted by $J^k\pi$. An important application is the following: $j_k(L) : L \to J^k(E,n)$, $y \mapsto [L]^k_y$. In similar way for the k-jets of a fiber bundle, $j_k(s) : U \to J^k(E,n)$, $x \mapsto [s]^k_x$.

We will denote $\operatorname{Im} j_k(L) = L^{(k)}$. If L is locally represented by section s, then we denote $\operatorname{Im} j_k(L) = \Gamma^k_s$. Now we use Greek letters $\mu, \lambda, \ldots$ to refer to basis coordinates and Latin letters $i, j, \ldots$ to refer to fiber coordinates. The k-jet space $J^k\pi$ is a differentiable manifold and a fiber vector bundle with respect to the projection, $\pi_k : J^k\pi \to M$, $[s]^k_x \mapsto x$. If (x^μ, u^j) are local coordinates in E, then (x^μ, u^j, u^j_σ) are local coordinates for $J^k\pi$, where

$$u^j_\sigma([L]^k_x) = \frac{\partial^{|\sigma|} s^j}{\partial x^\sigma}(x)\,, \tag{1}$$

and $u^j = s^j(x^\mu)$ is a local representation of L, $\sigma = (i_1, \ldots, i_n)$ is a multi-index such that $0 \le |\sigma| \le k$, $|\sigma| = i_1 + \cdots i_n$. A system of differentiable equations of order k is then a submanifold $\mathcal{E} \subseteq J^k(E,n)$ or $\mathcal{E} \subseteq J^k(\pi)$. Since the representation in coordinates of $j_k(L)$ is $\Big(x^\mu, s^j, \dfrac{\partial^{|\sigma|} s^j}{\partial x^\sigma}\Big)$, it is natural to define a solution of $\mathcal{E}$ as a submanifold L, such that $j_k(L) \subseteq \mathcal{E}$. The notations and others facts on jet spaces used in this article are taken from the Russian school [1].

2.1 Cartan distribution

An integral plane in $J^k\pi$ is an n-dimensional plane of the form $T_\theta L^{(k)}$ that projects horizontally with respect to the projection π_k. For $\theta = [L]^k_x \in J^k\pi$ we call the tangent space $T_\theta L^{(k)}$ the integral plane at the point θ. We denote by $\mathcal{C}^k_\theta$ the linear span of all integral planes at the point θ. It defines the distribution $\theta \mapsto \mathcal{C}^k_\theta$. This distribution is denoted by $\mathcal{C}^k$ and is called the Cartan distribution on $J^k\pi$. It is generated by the derived formal operators

$$D_\mu = \frac{\partial}{\partial x^\mu} + \sum_{j,\sigma} u^j_{\sigma\mu}\frac{\partial}{\partial u^j_\sigma}\ , \text{ where } u^j_{\sigma\mu}(s^j) = \frac{\partial^{|\sigma|+1}s^j}{\partial x^\sigma \partial x^\mu}\ .$$

In terms of forms it is described by the Cartan 1-forms, $\omega^j_\sigma = du^j_\sigma - \sum_{\mu,\sigma} u^j_{\sigma\mu}dx^\mu$. These forms annihilate the operators D_μ. In general any horizontal sub-space H of the Cartan plane $\mathcal{C}^k_\theta$ with respect to the projection $\pi_{k,k-1}$ has $\dim H \leq n$. The important fact about the Cartan distribution is explained by the following theorem.

THEOREM 2.1. *L is a solution of the equation $\mathcal{E}$ if and only if $L^{(k)}$ is a maximal integral submanifold of $\mathcal{C}^k(\mathcal{E})$, where $\mathcal{C}^k_\theta(\mathcal{E}) = \mathcal{C}^k_\theta \cap T_\theta\mathcal{E}, \theta \in \mathcal{E}$, is the induced Cartan distribution on $\mathcal{E}$.*

2.2 Iterated jet spaces J^1J^1E

Let $L \subset E$ be an n-dimensional submanifold and $x \in L$. The application $j_1(L) : U \to J^1E$ defines a local section of π_1 and determines the n-dimensional submanifold in J^1E denoted by $L^{(1)}$. The space consisting of all contact elements of the form $[L^{(1)}]^1_x$ is the space of iterated jets J^1J^1E. If (x^μ, u^j, u^j_μ) are the coordinates in J^1E, then we will denote the coordinates in J^1J^1E by $(x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu)$. By taking the derivatives of fiber coordinates with respect to the base coordinates, we obtain the new coordinates in J^1J^1E . We have then two projections in the iterated jet space J^1J^1E:

$$\pi_{1,1} : J^1J^1E \to J^1E, \quad (x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu) \mapsto (x^\mu, u^j, u^j_\mu),$$

and $$J^1\pi_{1,0} : J^1J^1E \to J^1E, \quad (x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu) \mapsto (x^\mu, u^j, (u^j)_\mu).$$

These two projections define the corresponding affine bundles.

We denote by $\hat{J}^1J^1E$ the subset of J^1J^1E that consists of the elements of contact that fulfill the condition $(u^j)_\mu = u^j_\mu$. This space will be called the space of sesqui-holonomics jets. The coordinates in this space are given by $(x^\mu, u^j, u^j_\mu, (u^j_\lambda)_\mu)$. We note that both projections coincide and $\hat{J}^1J^1E = \ker(\pi_{1,1} - J^1\pi_{1,0})$. We can define from the sesqui-holonomics jets the space of jets of order 2, J^2E, like those jets that fulfill $(u^j_\lambda)_\mu = (u^j_\mu)_\lambda = u^j_{\mu\lambda}$. Some other important facts about iterated jet spaces can be found in [6, 7] The affine bundle $\pi_{1,1}$ is modeled on the vector bundle $T^*M \otimes_{J^1E} VJ^1E$ and the affine bundle $J^1\pi_{1,0}$ is modeled on the vector bundle $J^1(T^*M \otimes_E VE)$, see for example [4, Proposition 12.11]. We can prove that there exists a linear morphism between the two affine bundles $\pi_{1,1}$ and $J^1\pi_{1,0}$, that we call the involution. This involution depends on a linear connection on the tangent bundle of the base space M. For this we consider the following diagram

$$\begin{tikzcd} J^1J^1E \arrow[r] & T^*M \otimes_{J^1E} VJ^1E \arrow[rr, "\alpha_\nabla"] \arrow[dr, "\pi_{1,1}"'] & & J^1(T^*M \otimes_E VE) \arrow[r] \arrow[dl, "J^1\pi_{1,0}"] & J^1J^1E \\ & & J^1E & & \end{tikzcd}$$

where α_∇ is the linear morphism that links the vector bundle models of the corresponding affine bundles $\pi_{1,1}$ and $J^1\pi_{1,0}$. The operator ∇ is a linear connection in the base space. The existence of this linear morphism is proved by direct construction. First we consider some important identifications and morphisms. There is a linear morphism that relates J^1TE and TJ^1E. Consider coordinates $(x^\mu, u^j, \dot{x}^\mu, \dot{u}^j)$ in TE and $(x^\mu, u^j, u^j_\lambda)$ in J^1E: then $(x^\mu, u^j, \dot{x}^\mu, \dot{u}^j, (u^j)_\mu, (\dot{x}^\mu)_\lambda, (\dot{u}^j)_\lambda)$ are coordinates in J^1TE and $(x^\mu, u^j, u^j_\mu, \dot{x}^\mu, \dot{u}^j, \dot{u}^j_\lambda)$ in TJ^1E. We define the linear morphism $\kappa : J^1TE \to TJ^1E$, $\dot{u}^j_\lambda = (\dot{u}^j)_\lambda - (\dot{x}^\mu)_\lambda (u^j)_\mu$, having noted that for the case $(\dot{x}^\mu)_\lambda = 0$ we have the isomorphism $\iota : VJ^1E \to J^1VE$, $(\dot{u}^j)_\lambda = \dot{u}^j_\lambda$. There is another important linear fiber morphism. If $\pi : E \to M$ is a bundle with coordinates (x^μ, u^l), $\eta : W \to M$ a vector bundle with coordinates (x^μ, w^i) and $\rho : F \to E$ a vector bundle that fibers over M with coordinates (u^l, f^j), then we can consider the tensor product $W \otimes_E F$ with coordinates (x^μ, u^j, t^{ij}) and $J^1(W \otimes_E F)$ with coordinates $(x^\mu, u^j, t^{ij}, (u^j)_\mu, (t^{ij})_\mu)$. We have also the vector bundle $J^1W \otimes_{J^1E} J^1F$ with coordinates $(x^\mu, u^j, u^j_\mu, v^{ij}, v^{ij}_\mu)$; then the universal property of the tensor product induces a linear fiber morphism defined by

$$\begin{aligned} \tau : J^1W \otimes_{J^1E} J^1F &\to J^1(W \otimes_E F) \\ t^{ij} &= v^{ij} = w^i f^j \,, \\ (t^{ij})_\mu &= w^i_\mu f^j + w^i f^j_\mu \,. \end{aligned} \tag{2}$$

There is another important embedding for algebraic purposes. In J^1J^1E we have the following embedding Λ: $J^1J^1E \hookrightarrow T^*M \otimes_{J^1E} TJ^1E$, where $\Lambda = dx^\mu \otimes d_\mu$ and $d_\mu = \partial_\mu + u^j_\mu \partial_j + (u^j)_\mu \hat{\partial}_j + (u^j_\lambda)_\mu \partial^\lambda_j$ is the total derivative operator. We define the following operator in $T^*M \otimes TJ^1E$ by $\hat{d}_\mu = \partial_\mu + (u^j)_\mu \partial_j + u^j_\mu \hat{\partial}_j + (u^j_\lambda)_\mu \partial^\lambda_j$. We note that the vertical part of the operators d_μ and $\hat{d}_\mu$ with respect to the projection $\pi_{1,0} : J^1J^1E \to J^1E$ are $(d_\mu)_V = (u^j)_\mu \hat{\partial}_j + (u^j_\lambda)_\mu \partial^\lambda_j$ and $(\hat{d}_\mu)_V = u^j_\mu \hat{\partial}_j + (u^j_\lambda)_\mu \partial^\lambda_j$, respectively. We consider the difference of these two vertical parts $(\hat{d}_\mu)_V - (d_\mu)_V = (u^j_\mu - (u^j)_\mu)\hat{\partial}_j \in VJ^1E$ and we define the linear application $\psi : J^1J^1E \to VJ^1E$, $(x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu) \mapsto (\hat{d}_\mu)_V - (d_\mu)_V$.

Proposition 2.2. *The application ψ is well defined.*

Proof. If $(x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu)$ are coordinates for J^1J^1E then the coordinates transformation formulas for u^j_μ and $(u^j)_\mu$ are (see [7, Chapter 4]):

$$u'^j_\mu = \frac{\partial x^\lambda}{\partial x'^\mu}\left(\partial_\lambda + u^i_\lambda \partial_i\right) u'^j \,,$$

$$(u'^j)_\mu = \frac{\partial x^\lambda}{\partial x'^\mu}\left(\partial_\lambda + (u^i)_\lambda \partial_i\right) u'^j \,.$$

Then

$$u'^j_\mu - (u'^j)_\mu = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial u'^j}{\partial u^i}\left(u^i_\lambda - (u^i)_\lambda\right) \,.$$

□

Theorem 2.3. *If $\pi : E \to M$ is a fiber bundle and $\nabla : T^*M \to J^1T^*M$ a linear connection in the cotangent bundle T^*M, then there exists a unique involutive linear automorphism α_∇ in J^1J^1E that commutes with the projections $\pi_{1,1}$ and $J^1\pi_{1,0}$. That is, the following diagram commutes:*

$$\begin{tikzcd} J^1J^1E \arrow[rr, "\alpha_\nabla"] \arrow[dr, "\pi_{1,1}"'] & & J^1J^1E \arrow[dl, "J^1\pi_{1,0}"] \\ & J^1E & \end{tikzcd}$$

Proof. We define an affine morphism by $\alpha_\nabla : J^1J^1E \longrightarrow J^1J^1E$, $\quad \alpha_\nabla = \tau \circ (\nabla \otimes \psi)$, where τ is the fiber morphism defined by (2) and ψ is the application of Proposition 2.2. We describe this application in coordinates. If (x^μ) are local coordinates in M with a local basis (dx^μ) for the sections on the bundle $T^*M \to M$, then the linear connection ∇ in local coordinates has the form $\nabla_\mu = \partial_\mu + \Gamma^\nu_{\mu\lambda}\partial_\nu \otimes dx^\lambda$. Then we have $(\nabla \otimes \psi)^j_\mu = u^j_\mu + \Gamma^\lambda_{\mu\nu}(u^j_\lambda - (u^j)_\lambda)\,dx^\nu$ and $(\alpha_\nabla)^j_{\mu\eta} = (\tau \circ (\nabla \otimes \psi))^j_{\mu\eta} = \partial_\eta\big(u^j_\mu + \Gamma^\lambda_{\mu\nu}(u^j_\lambda - (u^j)_\lambda)\,dx^\nu\big) = (u^j_\mu)_\eta + \Gamma^\lambda_{\mu\eta}(u^j_\lambda - (u^j)_\lambda)$, where we observe, that τ is simply a derivation and the operator ∂_η is C^∞-linear on 1-forms. Then the coordinate representation of the linear automorphism α_∇ is $(x^\mu, u^j, u^j_\mu, (u^j)_\mu, (u^j_\lambda)_\mu) \mapsto \big(x^\mu, u^j, (u^j)_\mu, u^j_\mu, (u^j_\mu)_\lambda + \Gamma^\nu_{\mu\lambda}(u^j_\nu - (u^j)_\nu)\big)$. With this coordinate form, we can see that $\alpha^2_\nabla = \mathrm{Id}$ and that the diagram commutes, that is, $J^1\pi_{1,0} \circ \alpha_\nabla = \pi_{1,1}$. □

Example 2.4. In the sesqui-holonomics case, that is, $(u^j)_\mu = u^j_\mu$ we have the canonical involution $\alpha : \hat{J}^1J^1E \to \hat{J}^1J^1E$, $(x^\mu, u^j, u^j_\mu, (u^j_\lambda)_\mu) \mapsto (x^\mu, u^j, u^j_\mu, (u^j_\mu)_\lambda)$, This is the canonical involution [9].

Example 2.5. If $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is the trivial bundle, then we denote by $J^k(n,m)$ the associated jet space $J^k(\mathbb{R}^{m+n}, n)$ to the bundle π. For simplicity in notation we denote the coordinates in $J^1(2,1)$ by (x, y, z, z_x, z_y) and the coordinates in $J^1(J^1(2,1), 2)$ by $(x, y, z, z_x, z_y, (z_x)_x, (z_x)_y, (z_y)_x, (z_y)_y)$. In $J^1(J^1(2,1), 2)$ the Cartan distribution, is determined by the fields $D_\lambda = \frac{\partial}{\partial x^\lambda} + \sum_{j,\mu}(u^j_\mu)_\lambda \frac{\partial}{\partial u^j_\mu}$ $\lambda = 1, 2$. Then these fields are

$$\begin{aligned} D_1 &= \frac{\partial}{\partial x} + z_x\frac{\partial}{\partial z} + (z_x)_x\frac{\partial}{\partial z_x} + (z_y)_x\frac{\partial}{\partial z_y}\,, \\ D_2 &= \frac{\partial}{\partial y} + z_y\frac{\partial}{\partial z} + (z_x)_y\frac{\partial}{\partial z_x} + (z_y)_y\frac{\partial}{\partial z_y}\,. \end{aligned} \tag{3}$$

Let $F \in C^\infty(J^1(2,1))$; then F determines in a canonical way a scalar differential operator of first order on the set of sections of the bundle $\pi : \mathbb{R}^2 \to \mathbb{R}$ (see [1, p. 126]). Then the formal derivatives (3) of F define scalars differential operators of order 1 on the bundle $\pi : J^1(2,1) \to M$, by means of

$$D_x F = F_x + F_z z_x + F_{z_x}(z_x)_x + F_{z_y}(z_y)_x ,$$
$$D_y F = F_y + F_z z_y + F_{z_x}(z_x)_y + F_{z_y}(z_y)_y .$$

We note that the Cartan distribution in $J^1(J^1(2,1),2)$ is described by the 1-forms

$$\begin{cases} \omega_0 &= dz - z_x dx - z_y dy\,, \\ \omega_1 &= dz_x - (z_x)_x dx - (z_x)_y dy\,, \\ \omega_2 &= dz_y - (z_y)_x dx - (z_y)_y dy\,. \end{cases}$$

Whereas the Cartan distribution in $J^2(2,1)$ is described by the 1-forms

$$\begin{cases} \omega^1_{(00)} &= dz - z_x dx - z_y dy\,, \\ \omega^1_{(10)} &= dz_x - z_{xx} dx - z_{xy} dy\,, \\ \omega^1_{(01)} &= dz_y - z_{yx} dx - z_{yy} dy\,. \end{cases}$$

EXAMPLE 2.6. We consider the system of differential equations in $J^1(3,1)$ with canonical coordinates $(x, y, z, u, u_x, u_y, u_z)$

$$\begin{cases} F^1 : u_z + y u_x = 0\,, \\ F^2 : u_y = 0\,. \end{cases}$$

This system is described by the vector valued function $F = (F^1, F^2)$ and determines the submanifold $\mathcal{E} = F^{-1}(0)$ in $J^1(3,1)$. We consider the system in the space of sesquiholonomics jets $\hat{J}^1 J^1(3,1)$ by prolonging the equation $\mathcal{E} \subset J^1(3,1)$ in $\hat{J}^1 J^1(3,1)$. This prolongation is done by the total derivatives

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + (u_x)_x \frac{\partial}{\partial u_x} + (u_y)_x \frac{\partial}{\partial u_y} + (u_z)_x \frac{\partial}{\partial u_z}\,,$$
$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + (u_x)_y \frac{\partial}{\partial u_x} + (u_y)_y \frac{\partial}{\partial u_y} + (u_z)_y \frac{\partial}{\partial u_z}\,,$$
$$D_z = \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial u} + (u_x)_z \frac{\partial}{\partial u_x} + (u_y)_z \frac{\partial}{\partial u_y} + (u_z)_z \frac{\partial}{\partial u_z}\,.$$

In this manner the prolonged system takes the form

$$\begin{cases} (u_z)_x + y(u_y) = 0 \\ (u_z)_y + u_x + y(u_x)_y = 0 \\ (u_z)_z + y(u_x)_z = 0 \\ (u_y)_x = 0 \\ (u_y)_y = 0 \\ (u_z)_z = 0\,. \end{cases}$$

We note that we cannot find the integrability condition $u_x = 0$. If we take in consideration the points fixed by the canonical involution α of Example 2.4, we have that

$(u_x)_y = (u_y)_x$, $(u_z)_x = (u_x)_z$ and we arrive to the integrability condition $u_x = 0$ for this system.

2.3 Symmetries of equations

Let P be a distribution in E and $F : E \to E$ a diffeomorphism. We will say that F is a *finite symmetry* of P if $F_* P_\theta \subseteq P_{F(\theta)}$ for any $\theta \in E$, that is, F preserves the distribution P. If the distribution P is described by the 1-forms ω^j_σ, then F is a finite symmetry if $F^*\omega^j_\sigma \subseteq \mathcal{I}(P)$, for all j and σ, where $\mathcal{I}(P)$ is the ideal generated by $\{\omega^j_\sigma\}$. A field X in E is an *infinitesimal symmetry* if the flow that determines X, say $A_t : E \to E$, consists of finite symmetries.

Example 2.7. If $P = \mathcal{C}$, then a diffeomorphism that preserves P is called a Lie finite symmetry. A field X in $J^k(\pi)$, whose flow consists of Lie finite symmetries, is called infinitesimal Lie symmetry.

Let us denote the set of all infinitesimal symmetries of the distribution P by $\mathrm{Sym}(P)$. It is an $\mathbb{R}$-Lie algebra with respect to the Lie bracket. An infinitesimal symmetry X is called *characteristic* if X is in P and the set of all characteristics will be denoted by $\mathrm{Char}(P)$. The characteristics from $\mathrm{Char}(P)$ with respect to the Lie bracket form an ideal of the Lie algebra $\mathrm{Sym}(P)$. If $X \in \mathrm{Char}(P)$ and L is a maximal integral submanifold of P, then X is tangent to L. We are interested in considering the quotient Lie algebra $\mathrm{sym}(P) = \frac{\mathrm{Sym}(P)}{\mathrm{Char}(P)}$. The elements of $\mathrm{sym}(P)$ are called simply symmetries.

2.4 Generating section of a Lie field

Let us consider a fiber bundle $\pi : E \to M$, with $\dim E = n+m$, $\dim M = n$ and a Lie field X in $J^k(\pi)$, see Example 2.7 above. Such a Lie field X is,in adapted coordinates, given by (see [1, Theorem 3.3 and 3.4])

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \sum_{0 \le |\sigma| \le k} b^j_\sigma \frac{\partial}{\partial u^j_\sigma} ,$$

where σ is a multi-index as in (1). Now we consider a section s of the fiber bundle π. The graph of the section $j_k(s)$ of the fiber bundle π_k is a maximal integral manifold in $J^k(\pi)$ of the Cartan distribution and we will denote it by Γ^k_s. Let us consider the 1-parameter group $\{\phi_t\}$ corresponding to the field X. We note that for t sufficiently small $\phi_t(\Gamma^k_s)$ is the graph of the k-jet of a section s_t, that is, $\phi_t(\Gamma^k_s) = \Gamma^k_{s_t}$. In other words $\Gamma^k_{s_t}$ is an infinitesimal deformation of Γ^k_s. The velocity of such deformation is $\frac{d}{dt}\big|_{t=0} s_t$. To measure this velocity, we need to know how Γ^k_s is deformed locally by ϕ_t. This deformation is a consequence of the vertical component of X since the component tangent to Γ^k_s displaces the submanifold along itself not deforming it. Now we calculate the vertical component of the Lie field X. The rate of change in the vertical direction of each component is given by $\frac{d}{dt}\big|_{t=0} s^j_t = X_V(s^j_t) = \left(b^j - \sum_{i=1}^n a_i u^j_i\right)\Big|_{\Gamma^k_s}$, where X_V is the vertical part of the Lie field X, that is, $(\pi_k)_*(X_V) = 0$. If we take another section, the rate of change is calculated by the same formula evaluated on the graph of the k-jet section, so we will call the vector valued function $\varphi = (\varphi^1, \dots, \varphi^m)$ given by $\varphi^j = b^j - \sum_{i=1}^n a_i u_i^j$ the *generating section* of the Lie field X. The generating section measures the infinitesimal deformation of the graph of k-jet sections, that are maximal integral submanifolds of the Cartan distribution. The following result establishes that the Lie field and the generating section are in biunivocal correspondence (see [1, Proposition 3.5]).

THEOREM 2.8. *Lie field in $J^k\pi$ is univocally determined by its generating section $\varphi = (\varphi^1, \dots, \varphi^m)$. The components of the generating section are given by $\varphi^j = X \lrcorner\, \omega^j_{(0,\dots,0)}$, where $\omega^j_{(0,\dots,0)} = du^j - \sum_{i=1}^n u_i^j\, dx_i$.*

Let P be a distribution on M. We define the normal fibration to P, given by $\mathcal{N}_P = \frac{TM}{P}$ whose fibers are the vectors spaces T_xM/P_x, $x \in M$. We consider the $C^\infty(M)$-module $D(P)$ of all derivations in M that are in P, that is, $X \in D(P)$ if and only if $X_x \in P_x$ for all $x \in M$. Then $[X] = [Y] \in \Gamma(\mathcal{N}_P)$ if $X - Y \in P$, where $\Gamma(\mathcal{N}_P)$ is the $C^\infty(M)$-module of sections of $\mathcal{N}_P$. From now on, we will write $[X] = X \bmod P$. Let us consider the application $D(P) \times D(P) \to \Gamma(\mathcal{N}_P)$ given by

$$(X, Y) \mapsto [X, Y] \bmod P\,. \tag{4}$$

From $[fX, gY] = fg[X, Y] + fX(g)Y + gY(f)X$, it follows that $[fX, gY] \bmod P = fg[X, Y] \bmod P$, so (4) defines a $C^\infty(M)$-bilinear application and determines a 2-form with values in $\Gamma(\mathcal{N}_P)$, that is called the curvature of the distribution P. This curvature is denoted by Ω_P. The value of Ω_P at the point $x \in M$, is given by $\Omega_P(X_x, Y_x) = [X, Y] \bmod P$, where X, Y are extensions of $X_x, Y_x \in T_xM$. Let $\mathrm{Ann}(P_x)$ be the set of 1-forms $\omega_x \in T_x^*M$ such that $\omega_x(X) = 0$ if $X \in P_x$. We will call the $C^\infty(M)$-module $\mathrm{Ann}(P_x)$ the annihilator of P_x. We note that $\mathcal{N}^*_{P_x} = \mathrm{Ann}(P_x)$ and $P_x^* = T^*M/\mathrm{Ann}(P_x)$. Let $\omega_1 \dots \omega_n$ be a local base for $\mathrm{Ann}(P_x)$ and $Z_1, \dots, Z_n$ the dual base for $\mathcal{N}_P$; then, for all X, Y, $\Omega_P(X, Y) = -\sum_{i=1}^n d\omega_i(X, Y)Z_i$, that is, $\Omega_P = -\sum_{i=1}^n d\omega_i \otimes Z_i$.

The construction of the curvature of a distribution is very similar to the curvature of a connection. This curvature Ω_P is referred to as the *Levi form* (see [5]). We will say that the distribution P is flat if $\Omega_P = 0$. We will say that a distribution P is completely integrable if for each point $x \in M$, there exists an integral submanifold N of P such that $\dim N$ equals the rank of the distribution P. This is established by

THEOREM 2.9 (Frobenius). *A distribution P is completely integrable if and only if P is flat. Moreover if P is completely integrable and if L_1 and L_2 are maximal integral submanifolds of P such that $x \in L_1 \cap L_2$, then $L_1 = L_2$ in some neighborhood of x.*

3. On the involutive systems and the geometric symbol

From now on let $\Omega \subset \mathbb{R}^n$ be some convenient open set and $E = \Omega \times \mathbb{R}^m$; in this case the jet bundle J^kE can be identified with cartesian product $J^kE \simeq \Omega \times \mathbb{R}^m \times$

$\mathbb{R}^{mn_1} \times \cdots \times \mathbb{R}^{mn_k}$, where n_j is the number of partial derivatives of order j. Putting $d_j = 1+n_1+\cdots+n_j$, it is straightforward to show that $n_j = \binom{n+j-1}{j}$ and $d_j = \binom{n+j}{j}$. Hence $\dim(J^kE) = n+m\,d_k$ and the number $m\,d_k$ is the fiber dimension of J^kE. The coordinates of J^kE are denoted by $(x^\mu, u^j, \ldots, u^j_\sigma)$ where σ is a multi-index such that $|\sigma| = k$ (see Section 2). In these coordinates a system of PDEs can be represented as the zero set of some map $f\colon J^kE \to \mathbb{R}^k$: $\mathcal{E}^k\colon f(x^\mu, u^j, \ldots, u^j_\sigma) = 0$. Now that the basic objects of study are defined, we return to the original problems:

(i) define the canonical form,

(ii) give a criterion to recognize if the given system is in the canonical form,

(iii) for a given system, construct the canonical form.

To accomplish these tasks there are only two fundamental operations available: prolongation (differentiation) and projection (elimination). By differentiating the system with respect to all independent variables we get a new system, the prolongation of the original system:

$$\mathcal{E}^{k+1}: \begin{cases} \dfrac{\partial f}{\partial x^1}(x^\mu, u^j, \ldots) = 0, \\ \quad\vdots \\ \dfrac{\partial f}{\partial x^n}(x^\mu, u^j, \ldots) = 0, \\ f(x^\mu, u^j, \ldots) = 0. \end{cases} \tag{5}$$

By further differentiating we can similarly define $\mathcal{E}^{k+r} \subset J^{k+r}E$ for all r. The projection is in some sense the inverse operation. Now we define the maps that relates the different jet spaces (see Section 2), $\pi_{k+r,k}\colon J^{k+r}E \to J^kE$; in local coordinates this simply means that we "forget" the highest derivatives. Restricting this map to the differential equation gives the map $\pi_{k+r,k}\colon \mathcal{E}^{k+r} \to \mathcal{E}^k$. The image of this map is denoted by $\mathcal{E}^k_{(r)}$.

Note that this map is well defined since we always have $\mathcal{E}^k_{(r)} \subseteq \mathcal{E}^k$. If the inclusion is strict this means that by differentiating and eliminating we have found *integrability conditions*; i.e., equations of order k which are algebraically independent of the original equations and which are also satisfied by the solutions of the system.

At this point it is convenient to recall a regularity assumption. While the initial system $\mathcal{E}^k$ is a subbundle by definition, this does not necessarily imply that all $\mathcal{E}^{k+s}_{(r)}$, in principle, are only subsets of the relevant jet bundle, were also subbundles.

Definition 3.1. A differential equation $\mathcal{E}^k$ is called *regular*, if $\mathcal{E}^{k+s}_{(r)}$ is a subbundle for all $r,\ s \geq 0$.

Definition 3.2. A system $\mathcal{E}^k$ is *formally integrable* if $\mathcal{E}^{k+r}_{(1)} = \mathcal{E}^{k+r}$ for all $r \geq 0$.

Now, it is well known that some properties of PDEs depend only on highest order derivatives terms in the system. The information of this highest part is encoded in the symbol of the system. There are two types of symbols: the geometric symbol and the principal symbol. We consider in this section the geometric symbol. To simplify, we restrict our attention to linear problems in a system of coordinates.

So consider a linear PDE of order k given by

$$\mathcal{E}^k : A(s) = \sum_{|\mu|\leq k} a^\mu(x)\partial_\mu s = f, \tag{6}$$

where $a^\mu(x)$ are matrices of order $k \times m$ and the components of a^μ are differentiable functions. Let $\Omega \subset \mathbb{R}^n$ be the domain where the system is given and let $E_0 = \Omega \times \mathbb{R}^m$ and $E_1 = \Omega \times \mathbb{R}^k$.

Therefore a solution of $\mathcal{E}^k$ is a section s of E_0 such that $j_k(s)$ takes its values in $\mathcal{E}^k$: f is a section of E_1 and the operator A is a function $A : C^\infty(E_0) \to C^\infty(E_1)$. The information of the highest order derivatives is in the matrices a^μ with $|\mu| = k$, these togheter define the symbol.

Definition 3.3. Consider the system in (6) and let M_k be the following matrix $M_k = (a^{\mu_1}, a^{\mu_2}, \ldots, a^{\mu_{n_k}})$, where $\mu_1 > \mu_2 > \cdots > \mu_{n_k}$, $|\mu_i| = k$, and the matrix M_k is obtained by joining the various matrices. The *geometric symbol* $\mathcal{M}_k$ is the family of vector spaces defined by the kernels of M_k.

So to each point $p \in \Omega$ a certain vector space is attached. If the dimension of this vector space does not depend on p, then the symbol is in fact a vector bundle. This will be assumed in the sequel, so it is possible to discuss the properties of the symbol without specifying the base point. One may also call the matrix M_k the symbol of $\mathcal{E}^k$; i.e., one identifies the object (the family of vector spaces or bundle) and its representation.

Definition 3.4. Let us suppose that the symbol M_k is in the row echelon form. A jet coordinate u^i_μ is a *leader*, if there is a row whose first nonzero element is in the column which corresponds to u^i_μ. Let $\beta^{(l)}_k$ be the number of leaders of class l. These are the *indices* of M_k.

Definition 3.5. The symbol $\mathcal{M}_k$ is *involutive* if and only if $\operatorname{rank}(M_{k+1}) = \sum_{l=1}^n l\beta^{(l)}_k$.

This criterion is quite reasonable: one must find the row echelon form of M_k to determine the indices. Then one must differentiate the equations to obtain M_{k+1}. This is also simple. Finally one must calculate the rank of M_{k+1} which is also a matter of linear algebra. For the following theorems we refer the reader to [8, Chapter 7].

Theorem 3.6. *If the symbol $\mathcal{M}_k$ is involutive and $\mathcal{E}^k_{(1)} = \mathcal{E}^k$, then $\mathcal{E}^k$ is formally integrable.*

When the symbol is involutive, it is sufficient to test if there are integrability conditions using only one prolongation. If the initial system is not involutive we have:

Theorem 3.7. *For any symbol $\mathcal{M}_k$, there exists $k' \geq k$ such that $\mathcal{M}_{k'}$ is involutive.*

There is an explicit bound for the number k'. Let us define $\hat{k}$ recursively by $\hat{k}(0, m, 1) = 0$, $\hat{k}(n, m, 1) = m\binom{a+n-1}{n} + a + 1$, where $a = \hat{k}(n-1, m, 1)$, and

$\hat{k}(n,m,q) = \hat{k}(n,b,1)$, where $b = m\binom{k+n-1}{n}$. Then $k' \leq \hat{k}$. Moreover, one can show that if $\mathcal{M}_k$ is involutive, then $\mathcal{M}_k$ is also involutive. Then one can find an involutive form of a system given in the following way:
(i) The system is prolonged until its symbol becomes involutive.

(ii) The system is prolonged and projected once to verify if there are integrability conditions.

(iii) If there are no new equations in the previous step, the system is now involutive. If it is not the case, return to step one.
This is the *Cartan-Kuranishi completion algorithm* [8]. The next theorem shows that, under the appropriate hypothesis, the general algorithm is finite.

THEOREM 3.8. *For a regular system $\mathcal{E}^k$ there are numbers r and s such that $\mathcal{E}^{k+r}_{(s)}$ is involutive.*

In what follows, we try to explain what involutivity means.

Let $x = (x^\mu)$ be the independent variables and $u = (u^j)$ the dependent variables. Let $\sigma \in \mathbb{N}^n$ be a multi-index and let $|\sigma| = \sigma_1 + \cdots + \sigma_n$ and $x^\sigma = (x^1)^{\sigma_1} \cdots (x^n)^{\sigma_n}$. The derivatives are denoted by $\frac{\partial^{|\sigma|} u^j}{\partial (x^1)^{\sigma_1} \cdots \partial (x^n)^{\sigma_n}} = \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma} = \partial_\sigma u^j = u^j_\sigma$ (see (1)). It is useful to order the multi-indices with the degree reverse lexicographic order: $\alpha < \sigma$ if $\begin{cases} |\alpha| < |\sigma|, \text{ or} \\ |\alpha| = |\sigma|, \ \alpha_i = \sigma_i \text{ for } 1 \leq i < j \text{ and } \alpha_j > \sigma_j \,. \end{cases}$

In the same way we can order monomials and derivatives using the order of the multi-indices: $x^\alpha < x^\sigma$, if $\alpha < \sigma$, $u^i_\alpha < u^j_\sigma$, if $\alpha \leq \sigma$ and $i > j$.

So for example if $n = m = |\sigma| = 2$ we have $u^1_{02} > u^2_{02} > u^1_{11} > u^2_{11} > u^1_{20} > u^2_{20}$. The class of the multi-index σ (resp. monomials x^μ, derivatives u^j_σ) is k, if $\sigma_1 = \cdots = \sigma_{k-1} = 0$ and $\sigma_k \neq 0$. So in the example there are two derivatives of class two and four derivatives of class one.

EXAMPLE 3.9. Consider the following example, due to Janet [3]

$$\mathcal{E}^2 : \begin{cases} u_{002} - x_2 u_{200} = 0, \\ u_{020} = 0. \end{cases}$$

Prolonging this system

$$\mathcal{E}^3 : \begin{cases} u_{003} - x_2 u_{201} = 0, \\ u_{012} - x_2 u_{210} - u_{200} = 0, \\ u_{102} - x_2 u_{300} = 0, \\ u_{021} = u_{030} = u_{120} = 0, \\ \mathcal{E}^2. \end{cases}$$

Let us now apply the algorithm of Cartan-Kuranishi. Consider the symbols $M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -x_2 & -x_2 & -x_2 \end{pmatrix}$. The columns of the matrices are ordered using the monomial order, i.e., a total order on the set of all (monic) monomials in a given polynomial ring, satisfying the property of respecting multiplication. In the case of M_2 this gives $002 > 011 > 020 > 101 > 110 > 200$.

Therefore, the first column of M_2 corresponds to a monomial derivative of class three, the second and the third to monomials derivative of class two and the rest to monomials of class one. The matrix M_2 is in row echelon and therefore, its leaders are u_{002} and u_{020}, and its indices are $\beta_2^{(3)} = \beta_2^{(1)} = 1$ and $\beta_2^{(1)} = 0$. Then the rank is $M_3 = 6$. The criterion is not satisfied: $\operatorname{rank} M_3 = 6 > 5 = \beta_2^{(1)} + 2\beta_2^{(2)} + 3\beta_2^{(3)}$. In consequence, the symbol $\mathcal{M}_2$ is not involutive. When doing the same analysis with $\mathcal{E}^3$, we find that $\mathcal{M}_3$ is involutive. But going to the step 2 in the algorithm, we get a new condition of integrability:

$$\mathcal{E}^3_{(1)} : \begin{cases} u_{210} = 0, \\ \mathcal{E}^3. \end{cases}$$

Then one starts again in the step 1, with the system $\mathcal{E}^3_{(1)}$. Now the symbol $\mathcal{M}_3^{(1)}$ is not involutive, but $\mathcal{M}_4^{(1)}$ is. However, again in the step 2 one obtains an integrability condition:

$$\mathcal{E}^4_{(2)} : \begin{cases} u_{400} = 0, \\ \mathcal{E}^4_{(1)}. \end{cases}$$

Then, for the third time, one returns to the step 1. Now, a prolongation shows that $\mathcal{M}^5_{(2)}$ is involutive and, in the step 2 no new integrability conditions will appear. Therefore, $\mathcal{E}^5_{(2)}$ is an involutive system and then the system $\mathcal{E}^4_{(2)}$ is formally integrable.

4. Invariant solutions of an equation

Let $\mathfrak{g}$ be a Lie sub-algebra of the the algebra of symmetries of the equation $\mathcal{E}$ and f_0 a solution of $\mathcal{E}$. We will say that f_0 is a $\mathfrak{g}$-*invariant* solution of $\mathcal{E}$ if, for each $X \in \mathfrak{g}$, the one-parameter group of X, say $\{A_t\}$, leaves the solution invariant, that is $A_t(\Gamma^k_{f_0}) = \Gamma^k_{f_t} \implies \Gamma^k_{f_t} = \Gamma^k_{f_0}$, whenever $\Gamma^k_{f_t}$ projects horizontally. We recall that $\Gamma^k_f = \operatorname{Im} j_k(L)$, where $L = \operatorname{Im}(f)$. If the Lie algebra consists of only one generator, X_φ, where φ is the corresponding generating section, then we will say that f_0 is X-*invariant* or φ-*invariant*. Let us suppose now that the Lie algebra $\mathfrak{g}$ is finitely generated by $X_1, \ldots, X_s$. To obtain a $\mathfrak{g}$-invariant solution, proceed as follows. We consider an equation $\mathcal{E}$ in $J^k(\pi)$ given locally by the system $\{F^1(\theta) = 0, \ldots, F^r(\theta) = 0\}$, with $F^j \in C^\infty(J^k(\pi))$ and for all $i = 1, \ldots, s$, φ^i is the generating section associated to the symmetry X_i. Then, by definition of $\mathfrak{g}$-invariant solution, this is determined by resolving the system of equations

$$F^j(\theta) = 0, \quad \varphi^i(\theta) = 0, \quad i = 1, \ldots, s,\ j = 1, \ldots, r\,. \tag{7}$$

We are interested in studying the involutivity in terms of the symmetries of an equation, and this can be conceptualized as follows:

$$\boxed{\mathfrak{g}\text{-invariant solution of } \mathcal{E}} \iff \boxed{\text{Set of fixed points of an involution } \alpha_{\mathfrak{g}}}$$

More precisely, if $\alpha_{\mathfrak{g}}\colon \hat{J}^1 J^{k-1}(\pi) \to \hat{J}^1 J^{k-1}(\pi)$ is an involution and $\tilde{\mathcal{E}} \subset \hat{J}^1 J^{k-1}(\pi)$ such that $\pi_{1,1}(\tilde{\mathcal{E}}) = \mathcal{E}$, then $\alpha_{\mathfrak{g}}(\hat{\theta}) = \hat{\theta}$ if and only if there exist a $\mathfrak{g}$-invariant solution f, $\theta \in \Gamma_f^k$ where $\alpha_{\mathfrak{g}}$ is a transformation of $\hat{J}^1 J^{k-1}(\pi)$ such that $\alpha_{\mathfrak{g}}^2 = \mathrm{Id}$ and $\pi_{1,1}(\hat{\theta}) = \theta$.

Example 4.1. If $\pi : E \to M$ is a fiber vector bundle, then a vector field on $J^k\pi$ which takes values in the tangent space of E is called a *generalized vector field*, also called Lie-Backlund vector field. A generalized vector field on $J^k\pi$ which projects under π_k to zero on M is called a vertical generalized vector field or an *evolutionary vector field*.

We consider the empty equation, i.e., $\mathcal{E} = J^2(\pi)$, with $\dim \pi = 1$, and $\dim M = 2$. Let $\tilde{\mathcal{E}}$ be the embedding of $\mathcal{E}$ in $\hat{J}^1 J^1(\pi)$. Let us suppose that the coordinates in $\hat{J}^1 J^1(\pi)$ are as in Example 2.5: $(x, y, z, z_x, z_y, (z_x)_x, (z_y)_x, (z_x)_y, (z_y)_y)$. Let us consider the evolutionary field, $X = [(z_x)_y - (z_y)_x]\partial_z$ in $\hat{J}^1 J^1(\pi)$. This is a Lie field, and the generating function is given by Theorem 2.8: $\varphi = \omega_0(X) = (z_x)_y - (z_y)_x$, where $\omega_0 = dz - z_x dx - z_y dy$ is the basic 1-contact form. The function φ determines the $\mathfrak{g}$-invariant solution given by the points $\theta \in \hat{J}^1 J^1(\pi)$ such that $\varphi(\theta) = 0$, that is $(z_x)_y - (z_y)_x = 0$, with $\mathfrak{g} = \{X\}$. The equation $\varphi = 0$ determines the transformation $\alpha_\varphi : \hat{J}^1 J^1(\pi) \to \hat{J}^1 J^1(\pi)$ given by $((z_x)_y, (z_y)_x) \circ \alpha_\varphi = ((z_y)_x, (z_x)_y)$. We note that $\alpha_\varphi^2 = \mathrm{Id}$, that is, α_φ is an involution. So the φ-invariant solution corresponds to the set of fixed points of α_φ, even more, we have $\frac{\hat{J}^1 J^1(\pi)}{\alpha_\varphi} = J^2(\pi)$, where we have identified $(z_x)_y$ and $(z_y)_x$ with z_{xy} in $J^2(\pi)$ (see Section 2.2)

Let us note that we can introduce coordinates in $\frac{\hat{J}^1 J^1(\pi)}{\alpha_\varphi}$, through the coordinates of $\hat{J}^1 J^1(\pi)$ using the identification of α_φ. In the general case it is complicated to express the coordinates of the quotient space. Generalizing the previous example we have the following theorem.

Theorem 4.2. *The space $J^k(\pi)$ is the quotient space of the iterated sesqui-holonomics jets $\hat{J}^1 J^{k-1}(\pi)$ relative to the subgroup of the algebra of symmetries determined by a suitable involution $\alpha_{\mathfrak{g}}$.*

Proof. Given the multi-index $\sigma = \{i_1, \ldots, i_{k-1}\}$ define the vertical fields $X^j_{\sigma\lambda_{(l)}} = \big[(u^j_{i_1,\ldots,\lambda_{(l)},\ldots,i_{k-1}})_{i_l} - (u^j_{i_1,\ldots,i_{k-1}})_\lambda\big]\partial_j$, with $0 < |\sigma| \le k-1$, $l = 1, \ldots, k-1$, $\lambda_{(l)} = 1, \ldots, n$ and $j = 1, \ldots, \dim \pi$. Let us note that the fields $X^j_{\sigma\lambda_{(l)}}$ are Lie fields in $\hat{J}^1 J^{k-1}(\pi)$. Then the algebra of Lie symmetries $\mathfrak{g}$, generated by these fields is a Lie sub-algebra of symmetries in $\hat{J}^1 J^{k-1}(\pi)$. The $\mathfrak{g}$-invariant solution of the empty equation $\hat{J}^1 J^{k-1}(\pi)$ is given by solving the system (7):

$$\begin{cases} \hat{\theta} \in \hat{J}^1 J^{k-1}(\pi) & \text{(empty equation)} \\ \varphi^j_{\sigma\lambda_{(l)}}(\hat{\theta}) = 0 & \text{(generating sections)} \end{cases}$$

where $\varphi^j_{\sigma\lambda_{(l)}}$ are the generating sections that correspond to the Lie fields $X^j_{\sigma\lambda_{(l)}}$ with $0 < |\sigma| \le k-1$, $l = 1, \ldots, k-1$, $\lambda_{(l)} = 1, \ldots, n$ and $j = 1, \ldots, \dim \pi$ (see Theorem 2.8).

Then $\varphi^j_{\sigma\lambda_{(l)}} = X_{\sigma\lambda_{(l)}} \lrcorner\, \omega^j_0$. In this way we get

$$\underbrace{(u^j_{i_1,\ldots,\lambda_{(l)},\ldots,i_{k-1}}}_{\lambda \text{ in the } l \text{ position}})_{i_l} = (u^j_{i_1,\ldots,i_{k-1}})_\lambda = \underbrace{(u^j_{i_1,\ldots,\lambda_{(s)},\ldots,i_{k-1}}}_{\lambda \text{ in the } s \text{ position}})_{i_s} \tag{8}$$

for every multi-index σ, $s, l = 1, \ldots, k-1$ and $\lambda = 1, \ldots, n$. The equations (8) determine the transformation $\alpha_{\mathfrak{g}} : \hat{J}^1 J^{k-1}(\pi) \to \hat{J}^1 J^{k-1}(\pi)$, given by $(u^j_{i_1,\ldots,\lambda_{(l)},\ldots,i_{k-1}})_{i_l} \circ \alpha_{\mathfrak{g}} = (u^j_{i_1,\ldots,\lambda_{(s)},\ldots,i_{k-1}})_{i_s}$. So we have that $\alpha^2_{\mathfrak{g}} = \mathrm{Id}$. In the sesqui-holonomics jet space $\hat{J}^1 J^{k-1}\pi$, we make the identifications by equations (8). We denote the quotient space by $\frac{\hat{J}^1 J^{k-1}\pi}{\alpha_{\mathfrak{g}}}$ and the identification with the k-jet space $J^k\pi$ is done by $[(u^j_\sigma)_\mu] = u^j_{\sigma\mu}$ where [ ] represent the equivalent class modulo the relations (8) and σ is a multi-indice such that $|\sigma| = k-1$. $\square$

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