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ON GENERALIZED DISTANCE SPECTRAL RADIUS OF A BIPARTITE GRAPH

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Abstract. For a simple connected graph G, let $D(G)$, $Tr(G)$, $D^L(G)$ and $D^Q(G)$ respectively be the distance matrix, the diagonal matrix of the vertex transmissions, the distance Laplacian matrix and the distance signless Laplacian matrix of a graph G. The convex linear combination $D_\alpha(G)$ of $Tr(G)$ and $D(G)$ is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, $0 \leq \alpha \leq 1$. As $D_0(G) = D(G)$, $2D_{\frac{1}{2}}(G) = D^Q(G)$, $D_1(G) = Tr(G)$, this matrix reduces to merging the distance spectral, signless distance Laplacian spectral theories. In this paper, we study the spectral radius of the generalized distance matrix $D_\alpha(G)$ of a graph G. We obtain bounds for the generalized distance spectral radius of a bipartite graph in terms of various parameters associated with the structure of the graph and characterize the extremal graphs. For $\alpha = 0$, our results improve some previously known bounds.

1. Introduction

We consider only connected, undirected, simple and finite graphs. A graph is denoted by $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is *regular* if all its vertices are of the same degree. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of the shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $D(G)$, is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *transmission* $Tr_G(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph G is said to be k-*transmission regular* if $Tr_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph

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G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2}\sum_{v\in V(G)} Tr_G(v)$. For any vertex $v_i \in V(G)$, the transmission $Tr_G(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is called the *transmission degree sequence* of the graph G. The second transmission degree of v_i, denoted by T_i is given by $T_i = \sum_{j=1}^{n} d_{ij}Tr_j$. For other undefined notations and terminology, the readers are referred to [14].

Let $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [3, 4] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian matrix of* G. There is a growing interest among the researchers in the study of the spectral properties of $D(G), D^L(G)$ and $D^Q(G)$ and as such in the literature several papers can be seen regarding their spectral properties, like spectral radius, second largest eigenvalue, smallest eigenvalue etc. For some recent works, we refer to [1,2,6–11,13,16,17] and the references therein.

Recently, Cui et al. [5] introduced the *generalized distance matrix* $D_\alpha(G)$ as a convex combination of $Tr(G)$ and $D(G)$, defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \le \alpha \le 1$. Since $D_0(G) = D(G)$, $2D_{\frac{1}{2}}(G) = D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^L(G)$, any result regarding the spectral properties of generalized distance matrix, has its counterpart for each of these particular graph matrices, and these counterparts follow immediately from a single proof. In fact, this matrix reduces to merging the distance spectral and distance signless Laplacian spectral theories. Since the matrix $D_\alpha(G)$ is real symmetric, all its eigenvalues are real. Therefore, we can arrange them as $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$. The largest eigenvalue ∂_1 of the matrix $D_\alpha(G)$ is called the *generalized distance spectral radius* of G (from now onwards, we will denote $\partial_1(G)$ by $\partial(G)$). As $D_\alpha(G)$ is nonnegative and irreducible, by the Perron-Frobenius theorem, $\partial(G)$ is the unique eigenvalue and there is a unique positive unit eigenvector X corresponding to $\partial(G)$, which is called the *generalized distance Perron vector* of G.

The spectral radius of a general matrix M is an important area of research and as such the investigation of the spectral radius of matrices associated to a graph becomes interesting. When M is restricted to a particular graph matrix, the spectral radius has attracted much attention of the researchers as is clear from the fact that various papers can be found in the literature in this direction. For a particular graph matrix (like adjacency, Laplacian, signless Laplacian, etc), the much studied problem about the parameter spectral radius is to obtain bounds in terms of various graph parameters. Another problem worth to mention is to characterize the extremal graphs for the spectral radius of a graph matrix, in some classes of graphs. For some recent works we refer to [6, 9, 10, 13, 16, 17] and the references therein.

A column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_i, that is, $X(v_i) = x_i$ for $i = 1, 2, \ldots, n$.

Then,

$$X^T D_\alpha(G)X = \alpha \sum_{i=1}^{n} Tr(v_i)x_i^2 + 2(1-\alpha) \sum_{1\le i<j\le n} d(v_i, v_j)x_i x_j,$$

and λ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvector X if and only if $X \neq \mathbf{0}$ and

$$\lambda x_i = \alpha Tr(v_i)x_i + (1-\alpha)\sum_{j=1}^{n} d(v_i, v_j)x_j.$$

These equations are called the (λ, x)-eigenequations of G. For a normalized column vector $X \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh's principle, we have $\partial(G) \geq X^T D_\alpha(G)X$, with equality if and only if X is the generalized distance Perron vector of G.

In the rest of the paper, we obtain some bounds for the generalized distance spectral radius of a bipartite graph in terms of various graph parameters associated to the structure of the graph and characterize the extremal graphs. For $\alpha = 0$, our results improve some previously known bounds.

2. Bounds for the generalized distance spectral radius of bipartite graphs

We are going to obtain upper and lower bounds for the generalized spectral radius $\partial(G)$ in terms of transmission degree sequence and second transmission degree sequence, the diameter and the order of a bipartite graph G.

A connected graph G is said to be biregular if there exist positive integers a and b such that every vertex $v_i \in V(G)$ has either degree a or b, $a \neq b$. Similarly, a connected graph G is said to be transmission biregular if there exist positive integers a and b such that every vertex $v_i \in V(G)$ has either transmission a or b, $a \neq b$.

Now, we obtain a lower bound for the generalized distance spectral radius ∂ in terms of order n, the cardinality of the partite sets, the maximum vertex degree of partite sets, the minimum transmission degree of the partite sets and the parameter α of a bipartite graph G.

THEOREM 2.1. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$, such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let Δ_{V_1}, Δ_{V_2} and $Tr_{\min}(V_1), Tr_{\min}(V_2)$ be respectively the maximum degree and the minimum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then*

$$\partial(G) \geq \max_{i,j} \left\{ \frac{\alpha Tr_{\min}(V_1)+\alpha Tr_{\min}(V_2)+3(1-\alpha)(n-2)-(1-\alpha)(t_{1i}+t_{2j})+\sqrt{\theta}}{2} \right\}, \quad (1)$$

where $\theta = \big[\alpha Tr_{\min}(V_1)+\alpha Tr_{\min}(V_2)+3(1-\alpha)(n-2)-(1-\alpha)(t_{1i}+t_{2j})\big]^2-4\big(\alpha Tr_{\min}(V_1)$ $+3(1-\alpha)(n_1-1)-(1-\alpha)t_{1i}\big)\big(\alpha Tr_{\min}(V_2)+3(1-\alpha)(n_2-1)-(1-\alpha)t_{2j}\big)+4(1-\alpha)^2(3n_2$ $-2\Delta_{V_1})(3n_1-2\Delta_{V_2})$. *Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a bipartite* *degree biregular, transmission biregular graph having eccentricity of each vertex equal to* 3.

Proof. Consider a connected bipartite graph G of order n with bipartition $V(G) = V_1 \cup V_2$. Suppose that $V_1 = \{1, 2, \ldots, n_1\}$ and $V_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the generalized distance Perron vector of G. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Without loss of generality, assume that $x_i = \min\{x_k : k \in V_1\}$ and $x_j = \min\{x_k : k \in V_2\}$. We suppose that $1, 2, \ldots, t_{1i}$ are the vertices in V_1 at distance 2 from $v_i \in V_1$ and $n_1 + 1, n_1 + 2, \ldots, n_1 + t_{2j}$ are the vertices in V_2 at distance 2 from $v_j \in V_2$. From the i-th equation of $D_\alpha(G)X = \partial(G)X$, we obtain

$$\begin{aligned}
\partial x_i &= \alpha Tr_i x_i + (1-\alpha) \sum_{k=1, k\neq i}^{n} d_{ik} x_k \\
&= \alpha Tr_i x_i + (1-\alpha) \sum_{k=1, k\neq i}^{n_1} d_{ik} x_k + (1-\alpha) \sum_{k=n_1+1, k\neq i}^{n_1+n_2} d_{ik} x_k \\
&\geq \alpha Tr_{\min}(V_1) x_i + (1-\alpha)\big[2t_{1i} x_i + 3(n_1 - 1 - t_{1i}) x_i + d_i x_j + 3(n_2 - d_i) x_j\big] \\
&\geq \alpha Tr_{\min}(V_1) x_i + (1-\alpha)\big[2t_{1i} x_i + 3(n_1 - 1 - t_{1i}) x_i + (3n_2 - 2\Delta_{V_1}) x_j\big].
\end{aligned}$$

This shows that

$$\big[\partial - \alpha Tr_{\min}(V_1) - 2t_{1i}(1-\alpha) - 3(1-\alpha)(n_1 - 1 - t_{1i})\big] x_i \geq (1-\alpha)(3n_2 - 2\Delta_{V_1}) x_j. \tag{2}$$

Similarly, from the j-th equation of $D_\alpha(G)X = \partial(G)X$, we obtain

$$\begin{aligned}
\partial x_j &= \alpha Tr_j x_j + (1-\alpha) \sum_{k=1, k\neq j}^{n} d_{jk} x_k \\
&= \alpha Tr_j x_j + (1-\alpha) \sum_{k=1, k\neq j}^{n_1} d_{jk} x_k + (1-\alpha) \sum_{k=n_1+1, k\neq j}^{n_1+n_2} d_{jk} x_k \\
&\geq \alpha Tr_{\min}(V_2) x_j + (1-\alpha)\big[2t_{2j} x_j + 3(n_2 - 1 - t_{2j}) x_j + d_j x_i + 3(n_1 - d_j) x_i\big] \\
&\geq \alpha Tr_{\min}(V_2) x_j + (1-\alpha)\big[2t_{2j} x_j + 3(n_2 - 1 - t_{2j}) x_j + (3n_1 - 2\Delta_{V_2}) x_i\big].
\end{aligned}$$

This shows that

$$\big[\partial - \alpha Tr_{\min}(V_2) - 2t_{2j}(1-\alpha) - 3(1-\alpha)(n_2 - 1 - t_{2j})\big] x_j \geq (1-\alpha)(3n_1 - 2\Delta_{V_2}) x_i. \tag{3}$$

Multiplying the corresponding sides of the inequalities (2) and (3) and using the fact that $x_k > 0$ for all k, we obtain

$$\begin{aligned}
&\partial^2 - \big(\alpha(Tr_{\min}(V_1) + Tr_{\min}(V_2)) + 3(1-\alpha)(n-2) - (1-\alpha)(t_{1i} + t_{2j})\big)\partial \\
&\quad + (\alpha Tr_{\min}(V_1) + 3(1-\alpha)(n_1 - 1) - (1-\alpha)t_{1i})(\alpha Tr_{\min}(V_2) + 3(1-\alpha)(n_2 - 1) - (1-\alpha)t_{2j}) \\
&\quad - (1-\alpha)^2(3n_2 - 2\Delta_{V_1})(3n_1 - 2\Delta_{V_2}) \geq 0.
\end{aligned}$$

The result follows from the previous conclusions. Suppose that equalities occur in (1). Then equality occurs in each of the inequalities (2) and (3). Equality in (2) gives $Tr_k = Tr_{\min}(V_1)$, $d_k = \Delta_{V_1}$, for all $k \in V_1$ and

$$x_k = x_i, \quad d_{ik} = 2, \quad \text{for} \quad k \leq t_{1i}, \quad d_{ik} = 3, \quad \text{for} \quad k > t_{1i} \quad \text{for all} \quad k \in V_1$$

and $\quad x_k = x_j, \quad \text{if} \quad ki \in E(G), \ d_{ik} = 3, \quad \text{if} \quad ki \notin E(G) \quad \text{for all} \quad v_k \in V_2.$

Similarly, equality in (3) gives that $Tr_k = Tr_{\min}(V_2)$, $d_k = \Delta_{V_2}$, for all $k \in V_2$ and

$$x_k = x_j, \quad d_{jk} = 2, \text{ for } k \leq t_{2j}, \ d_{ik} = 3, \text{ for } k > t_{2j} \quad \text{for all} \quad k \in V_2$$

and $\quad x_k = x_i, \quad d_{jk} = 1, \text{ if } kj \in E(G), \ d_{jk} = 3, \text{ if } kj \notin E(G) \quad \text{for all} \quad v_k \in V_1.$

This shows that every vertex in V_1 has degree Δ_{V_1}, transmission degree $Tr_{\min}(V_1)$ and eccentricity at most 3 and every vertex in V_2 has degree Δ_{V_2}, transmission degree $Tr_{\min}(V_2)$ and eccentricity at most 3. This further shows that G is a bipartite degree biregular and transmission biregular graph having eccentricity at most 3. If eccentricity is at most 2, the connected bipartite graph, which is degree biregular and transmission biregular is K_{n_1,n_2}. On the other hand, if eccentricity is 3, the connected bipartite graph $G \neq K_{n_1,n_2}$, which is both degree biregular and transmission biregular should have all of its vertices of eccentricity equal to 3.

Conversely, if $G \cong K_{n_1,n_2}$, then $Tr_{\min}(V_1) = 2n - 2 - n_2$, $\Delta_{V_1} = n_2$, $t_{1i} = n_1 - 1$, $Tr_{\min}(V_2) = 2n - 2 - n_1$, $\Delta_{V_2} = n_1$, $t_{2j} = n_2 - 1$ and $\partial = \frac{1}{2}\Big((\alpha + 2)n - 4 + \sqrt{(n_1^2 + n_2^2)(\alpha - 2)^2 + 2n_1n_2(\alpha^2 - 2)}\Big)$ (see [15]). It can be seen that equality holds in (1). On the other hand, if G is a connected bipartite graph, which is degree biregular, transmission biregular and has all of its vertices of eccentricity equal to 3, then clearly from inequalities (2) and (3), it follows that equality holds. This completes the proof. □

Taking $\alpha = 0$, we obtain the following lower bound for the distance spectral radius ∂_1^D of a bipartite graph.

Corollary 2.2. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$, such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let Δ_{V_1}, Δ_{V_2} and $Tr_{\min}(V_1), Tr_{\min}(V_2)$ be respectively the maximum degree and the minimum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then*

$$\partial_1^D(G) \geq \max_{i,j} \left\{ \frac{3(n-2) - (t_{1i} + t_{2j}) + \sqrt{\theta}}{2} \right\},$$

where $\theta = \big[3(n-2) - (t_{1i} + t_{2j})\big]^2 - 4\big(3n_1 - 3 - t_{1i}\big)\big(3n_2 - 3 - t_{2j}\big) + 4(3n_2 - 2\Delta_{V_1})(3n_1 - 2\Delta_{V_2})$. Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a bipartite degree biregular, transmission biregular graph having eccentricity of each vertex equal to 3.

The following lower bound for the distance spectral radius ∂_1^D of a bipartite graph was obtained in [18]:

$$\partial_1^D(G) \geq n - 2 + \sqrt{n^2 - 4n_1n_2 + (3n_2 - 2\Delta_{V_1})(3n_1 - 2\Delta_{V_2})}, \tag{4}$$

where $n_1 = |V_1|$, $n_2 = |V_2|$ and $\Delta_{V_1}, \Delta_{V_2}$ are the maximum vertex degree in V_1, V_2 respectively.

It is easy to see that our lower bound given by Corollary 2.2 is always better than the lower bound given by (4).

Taking $\alpha = \frac{1}{2}$, we obtain the following lower bound for the distance signless Laplacian spectral radius ∂_1^Q of a bipartite graph.

Corollary 2.3. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$, such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let Δ_{V_1}, Δ_{V_2} and $Tr_{\min}(V_1), Tr_{\min}(V_2)$ be respectively the maximum degree and the minimum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then*

$$\partial_1^Q(G) \geq \max_{i,j}\Big\{\frac{Tr_{\min}(V_1) + Tr_{\min}(V_2) + 3(n-2) - (t_{1i} + t_{2j}) + \sqrt{\theta}}{2}\Big\},$$

where $\theta = \big[Tr_{\min}(V_1) + Tr_{\min}(V_2) + 3(n-2) - (t_{1i} + t_{2j})\big]^2 - 4\big(Tr_{\min}(V_1) + 3n_1 - 3 - t_{1i}\big)\big(Tr_{\min}(V_2) + 3n_2 - 3 - t_{2j}\big) + 4(3n_2 - 2\Delta_{V_1})(3n_1 - 2\Delta_{V_2})$. Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a bipartite degree biregular, transmission biregular graph having eccentricity of each vertex equal to 3.

Now, we obtain an upper bound for the generalized distance spectral radius ∂ in terms of order n, the cardinality of the partite sets, the minimum vertex degree of partite sets, the maximum transmission degree of the partite sets and the parameter α for a bipartite graph G.

Theorem 2.4. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$, such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let δ_{V_1}, δ_{V_2} and $Tr_{\max}(V_1), Tr_{\max}(V_2)$ be respectively the minimum degree and the maximum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then for $\gamma = t_{1i} + t_{2j}$*

$$\partial(G) \leq \frac{1}{2}\min_{i,j}\big\{\alpha Tr_{\max}(V_1) + \alpha Tr_{\max}(V_2) + (1-\alpha)(d(n-2) - (d-2)\gamma) + \sqrt{\phi}\big\}, \tag{5}$$

where $\phi = \big[\alpha Tr_{\max}(V_1) + \alpha Tr_{\max}(V_2) + d(1-\alpha)(n-2) - (1-\alpha)(d-2)(t_{1i} + t_{2j})\big]^2 - 4\big(\alpha Tr_{\max}(V_1) + (1-\alpha)(d(n_1-1) - (d-2)t_{1i})\big)\big(\alpha Tr_{\max}(V_2) + (1-\alpha)(d(n_2-1) - (d-2)t_{2j})\big) + 4(1-\alpha)^2(d(n_2-1) - (d-1)\delta_{V_1})(d(n_1-1) - (d-1)\delta_{V_2})$, where d is the diameter of G. Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a connected bipartite graph, which is degree biregular, transmission biregular and has the property that the eccentricity of every vertex $i \in V(G)$ is equal to diameter d.

Proof. Consider a connected bipartite graph G of order n with bipartition $V(G) = V_1 \cup V_2$ having diameter d. Suppose that $V_1 = \{1, 2, \ldots, n_1\}$ and $V_2 = \{n_1+1, \ldots, n_1+n_2\}$, $n_1 + n_2 = n$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the generalized distance Perron vector of G. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Without loss of generality, let us assume that $x_i = \max\{x_k : k \in V_1\}$ and $x_j = \max\{x_k : k \in V_2\}$. Suppose $1, 2, \ldots, t_{1i}$ are the vertices in V_1 at distance 2 from $v_i \in V_1$ and $n_1 + 1, n_1 + 2, \ldots, n_1 + t_{2j}$ are the vertices in V_2 at distance 2 from $v_j \in V_2$. From the i-th equation of $D_\alpha(G)X = \partial(G)X$, we obtain

$$\begin{aligned}\partial x_i &= \alpha Tr_i x_i + (1-\alpha)\sum_{k=1,k\neq i}^{n_1} d_{ik}x_k + (1-\alpha)\sum_{k=n_1+1,k\neq i}^{n_1+n_2} d_{ik}x_k \\ &\leq \alpha Tr_{\max}(V_1)x_i + (1-\alpha)\big[2t_{1i}x_i + d(n_1-1-t_{1i})x_i + d_i x_j + d(n_2-d_i)x_j\big]\end{aligned}$$

$$\leq \alpha Tr_{\max}(V_1)x_i+(1-\alpha)\big[2t_{1i}x_i+d(n_1-1-t_{1i})x_i+(dn_2-(d-1)\delta_{V_1})x_j\big].$$

This shows that

$$\big[\partial-\alpha Tr_{\max}(V_1)-(1-\alpha)(d(n_1-1)-(d-2)t_{1i})\big]x_i \leq (1-\alpha)(dn_2-(d-1)\delta_{V_1})x_j. \quad (6)$$

Similarly, from the j-th equation of $D_\alpha(G)X = \partial(G)X$, we obtain

$$\big[\partial-\alpha Tr_{\max}(V_2)-(1-\alpha)(d(n_2-1)-(d-2)t_{2j})\big]x_j \leq (1-\alpha)(dn_1-(d-1)\delta_{V_2})x_i. \quad (7)$$

Multiplying the corresponding sides of the inequalities (6) and (7) and using the fact that $x_k > 0$ for all k, we obtain

$$\partial^2-\big(\alpha(Tr_{\max}(V_1)+Tr_{\max}(V_2))+d(1-\alpha)(n-2)-(1-\alpha)(d-2)(t_{1i}+t_{2j})\big)\partial$$
$$+(\alpha Tr_{\max}(V_1)+(1-\alpha)(d(n_1-1)-(d-2)t_{1i}))(\alpha Tr_{\max}(V_2)+(1-\alpha)(d(n_2-1)-(d-2)t_{2j}))$$
$$-(1-\alpha)^2(dn_2-(d-1)\delta_{V_1})(dn_1-(d-1)\delta_{V_2}) \geq 0.$$

From this the result follows. Suppose equalities occur in (5). Then equality occurs in each of the inequalities (6) and (7). Equality in (6) gives that $Tr_k = Tr_{\max}(V_1)$, $d_k = \delta_{V_1}$, for all $k \in V_1$ and

$$x_k = x_i, \quad d_{ik} = 2, \quad \text{for} \quad k \leq t_{1i} \quad d_{ik} = d, \quad \text{for} \quad k > t_{1i} \quad \text{for all} \quad k \in V_1$$

and $$x_k = x_j, \quad \text{if} \quad ki \in E(G), \quad d_{ik} = d, \quad \text{if} \quad ki \notin E(G) \quad \text{for all} \quad v_k \in V_2.$$

Similarly, equality in (3) gives that $Tr_j = Tr_{\max}(V_2)$, $d_j = \delta_{V_2}$, for all $k \in V_2$ and

$$x_k = x_j, \quad d_{jk} = 2, \quad \text{for} \quad k \leq t_{2j} \quad d_{jk} = d, \quad \text{for} \quad k > t_{2j}, \quad \text{for all} \quad k \in V_2$$

and $$x_k = x_i, \quad \text{if} \quad kj \in E(G), \quad d_{jk} = d, \quad \text{if} \quad kj \notin E(G) \quad \text{for all} \quad v_k \in V_1.$$

We see that every vertex i in V_1 has vertex degree δ_{V_1}, transmission degree $Tr_{\max}(V_1)$ and distance of every vertex non-adjacent to vertex i is 2 or d from i and every vertex j in V_2 has vertex degree δ_{V_2}, transmission degree $Tr_{\max}(V_2)$ and distance of every vertex non-adjacent to vertex j is 2 or d from j. Also, we observe that G is a bipartite degree biregular and transmission biregular graph having the property that the distance of every non-adjacent vertex from vertex $i \in V(G)$ is equal to 2 or diameter d. Since G is a bipartite graph, therefore the only vertices adjacent to $i \in V_1$ are in V_2 and vice-versa. If $d \leq 2$, the connected bipartite graph, which is degree biregular, transmission biregular having the property that the distance of every non-adjacent from a vertex $i \in V(G)$ is equal to diameter d is K_{n_1,n_2}. On the other hand, if diameter $d \geq 3$, then equality occurs if the connected bipartite graph G is degree biregular, transmission biregular and has the property that the eccentricity of every vertex $i \in V(G)$ is equal to diameter d.

Conversely, it is easy to see that equality occurs in (5) if $G \cong K_{n_1,n_2}$ or G is a connected bipartite graph, which is degree biregular, transmission biregular and has the property that the eccentricity of every vertex $i \in V(G)$ is equal to diameter d. □

Taking $\alpha = 0$, we have the following upper bound for the distance spectral radius ∂_1^D of a bipartite graph.

Corollary 2.5. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let δ_{V_1}, δ_{V_2} and $Tr_{\max}(V_1), Tr_{\max}(V_2)$ be respectively the minimum degree and the maximum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in*

V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then, for $\gamma = t_{1i} + t_{2j}$,

$$\partial_1^D(G) \leq \frac{1}{2} \min_{i,j} \{d(n-2) - (d-2)\gamma + \sqrt{\phi}\},$$

where $\phi = [d(n-2)-(d-2)(t_{1i}+t_{2j})]^2 - 4(d(n_1-1)-(d-2)t_{1i})(d(n_2-1)-(d-2)t_{2j}) + 4(d(n_2-1)-(d-1)\delta_{V_1})(d(n_1-1)-(d-1)\delta_{V_2})$, where d is the diameter of G. Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a connected bipartite graph, which is degree biregular, transmission biregular and has the property that the eccentricity of every vertex $i \in V(G)$ is equal to diameter d.

Taking $\alpha = \frac{1}{2}$, we obtain the following upper bound for the distance Laplacian spectral radius ∂_1^Q of a bipartite graph.

Corollary 2.6. *Let G be a connected bipartite graph of order n with bipartition $V(G) = V_1 \cup V_2$, such that $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 + n_2 = n$. Let δ_{V_1}, δ_{V_2} and $Tr_{\max}(V_1), Tr_{\max}(V_2)$ be respectively the minimum degree and the maximum transmission degree among the vertices in V_1 and V_2. Let t_{ik} be the number of vertices in V_i, which are at distance 2 from $v_k \in V_i$, $i = 1, 2$. Then for $\gamma = t_{1i} + t_{2j}$*

$$\partial_1^Q(G) \leq \frac{1}{2} \min_{i,j} \{Tr_{\max}(V_1) + Tr_{\max}(V_2) + d(n-2) - (d-2)\gamma + \sqrt{\phi}\},$$

where $\phi = [Tr_{\max}(V_1) + Tr_{\max}(V_2) + d(n-2) - (d-2)(t_{1i} + t_{2j})]^2 - 4(Tr_{\max}(V_1) + (1 - d(n_1-1) - (d-2)t_{1i})(Tr_{\max}(V_2) + d(n_2-1) - (d-2)t_{2j}) + 4(d(n_2-1) - (d-1)\delta_{V_1})(d(n_1-1) - (d-1)\delta_{V_2})$, where d is the diameter of G. Equality occurs if and only if $G \cong K_{n_1,n_2}$ or G is a connected bipartite graph, which is degree biregular, transmission biregular and has the property that the eccentricity of every vertex $i \in V(G)$ is equal to diameter d.

Lemma 2.7 ([12]). *Let B and C be square nonnegative matrices having spectral radius $\rho(B)$ and $\rho(C)$. If B is irreducible, $B \geq C$ and $B \neq C$, then $\rho(B) \geq \rho(C)$.*

Using Lemma 2.7, we have the following observation.

Lemma 2.8. *Let G be a connected graph of order n and let u and v be two non-adjacent vertices of G. Let $G' = G + uv$ be the graph obtained from G by adding edge between u and v. Then $\partial(G) \geq \partial(G')$.*

For $0 \leq \alpha \leq 1$, from Lemma 2.8, it is clear that among all the bipartite graphs, the complete bipartite graph $K_{a,n-a}$, $n - a \leq a$, has the minimum generalized distance spectral radius and tree T has the maximum generalized distance spectral radius. It is shown in [15] that the generalized spectral radius of $K_{a,n-a}$ is

$$\partial(K_{a,n-a}) = \frac{(\alpha+2)n - 4 + \sqrt{n^2\alpha^2 - (n^2 + 2a^2 - 2an)4\alpha + 4(n^2 - 3an + 3a^2)}}{2}.$$

Consider the function $f(a) = n^2\alpha^2 - (n^2+2a^2-2an)4\alpha + 4(n^2-3an+3a^2)$, $a \in [1, \frac{n}{2}]$ and $\frac{1}{2} \leq \alpha \leq 1$. We have $f'(a) = (2n-4a)4\alpha + 4(6a-3n)$ giving $f'(a) < 0$, for all $a \in [1, \frac{n}{2}]$. This implies that $f(a)$ is a decreasing function of a, for all $a \in [1, \frac{n}{2}]$, which in turn implies that $\partial(K_{a,n-a})$ is a decreasing function of a for all $a \in [1, \frac{n}{2}]$. Therefore, it follows that $\partial(K_{2,n-2}) \geq \partial(K_{3,n-3}) \geq \cdots \geq \partial(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$. Thus, we have the following observation.

Theorem 2.9. *Among all the connected bipartite graphs the complete bipartite graph* $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ *has the minimum generalized distance spectral radius.*

We note that a tree T has the maximum generalized distance spectral radius. For $\alpha = 0$, it is known that among the trees the maximum generalized distance spectral radius is attained by a path P_n. Therefore, it will be of interest to determine the tree for $0 < \alpha \leq 1$, which has the maximum generalized distance spectral radius. So, we have the following problem.

Problem. For $0 < \alpha \leq 1$, among all trees, characterize the tree which has the maximum generalized distance spectral radius.

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