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МАТЕМАТИЧКИ ВЕСНИК
72, 4 (2020), 314–326
December 2020

research paper
оригинални научни рад

THE ZARIOUH'S PROPERTY (gaz) THROUGH LOCALIZED SVEP

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Abstract. In this paper we study the property (gaz) for a bounded linear operator $T \in L(X)$ on a Banach space X, introduced by Zariouh in [*Property (gz) for bounded linear operators*, Mat. Vesnik, **65(1)**(2013), 94–103], through the methods of local spectral theory. This property is a stronger variant of generalized a-Browder's theorem. In particular, we shall give several characterizations of property (gaz), by using the localized SVEP.

1. Introduction

The classical Browder's theorem and a-Browder's theorem for operators $T \in L(X)$, defined in Banach spaces X, admit some variants, as property (b), property (ab), and property (gb), that have been introduced in [8, 9]. All these properties, that are stronger versions than Browder's theorem and a-Browder's theorem, have been also studied by using methods of local spectral theory in [2] or [1, Chapter 5]. In this paper we consider a property, called property (gaz), introduced recently by Zariouh in [13] and, among other characterizations, we show that property (gaz) holds for T precisely when the dual T^* has the SVEP at the points λ that do not belong to the upper semi B-Weyl spectrum of T, while, dually, T^* has property (gaz) if and only if T has the SVEP at the points λ that do not belong to the upper semi B-Weyl spectrum of T^*. In the last part of the paper we show that property (gaz) may be also characterized by means of the quasi-nilpotent part $H_0(\lambda I - T)$, or by means of the analytic core $K(\lambda I - T)$, as λ ranges in a certain subset $\Delta_1^g(T)$ of the spectrum.

2. Definitions and preliminary results

Let $T \in L(X)$ be a bounded linear operator defined on an infinite-dimensional complex Banach space X, and denote by $\alpha(T)$ and $\beta(T)$, the dimension of the kernel $\ker T$

2010 Mathematics Subject Classification: 47A10, 47A11, 47A53, 47A55
Keywords and phrases: Property (gaz); localized SVEP; Browder type theorems.

and the codimension of the range $R(T) := T(X)$, respectively. Recall that $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$ if $\beta(T) < \infty$. The class of *Fredholm* operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, while the class of semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$. If $T \in \Phi_\pm(X)$ then its index is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The set of *Weyl operators* is defined by $W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\}$, the class of *upper semi-Weyl operators* is defined by $W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\}$, and class of *lower semi-Weyl operators* is defined by $W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}$. Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by $\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$ the *upper semi-Weyl spectrum*, defined by $\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}$, and the *lower semi-Weyl spectrum*, defined by $\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}$. Let $p := p(T)$ and $q := q(T)$ denote the *ascent* and the *descent* of the operator T, respectively. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Chapter 1]. Moreover, if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [11, Proposition 50.2].

The class of all *Browder operators* is defined as the set $B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}$; the class of all *upper semi-Browder operators* is defined $B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}$, and the class of all *lower semi-Browder operators* is defined $B_+(X) := \{T \in \Phi_-(X) : q(T) < \infty\}$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$ and $B_-(X) \subseteq W_-(X)$.

In the sequel we denote by $\sigma_{\mathrm{ap}}(T)$ the *approximate point spectrum*, defined as $\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$, where an operator is said to be *bounded below* if it is injective and has closed range. The classical *surjective spectrum* of T is denoted by $\sigma_{\mathrm{s}}(T)$.

An operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, or equivalently $\Delta(T) = p_{00}(T)$, where $\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$ and $p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$. The operator $T \in L(X)$ is said to satisfy *a-Brower's theorem* if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, or equivalently $\Delta_a(T) = p_{00}^a(T)$, where $\Delta_a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T)$ and $p_{00}^a(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. It is known that a-Browder's theorem entails Browder's theorem, see [1, Chapter 5] for details.

Semi-Fredholm operators have been generalized by Berkani [6, 7] in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm*, (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ (see [7]) with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl, lower semi-Weyl). The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not } B\text{-Weyl}\},$$

and the *upper semi B-Weyl spectrum* of T is defined by

$$\sigma_{\mathrm{ubw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi } B\text{-Weyl}\}.$$

Analogously, the *lower semi B-Weyl spectrum* of T is defined by

$$\sigma_{\mathrm{lbw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi } B\text{-Weyl}\}.$$

In the sequel we shall need the following punctured theorem which follows as a particular case of a result proved in [7, Corollary 3.2] for operators having topological uniform descent for $n \geq d$.

Theorem 2.1. *Suppose that $T \in L(X)$ is upper semi B-Fredholm. Then there exists an open disc $\mathbb{D}(0,\varepsilon)$ centered at 0 such that $\lambda I - T \in \Phi_+(X)$ for all $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$ and* $\operatorname{ind}(\lambda I - T) = ind(T)$ *for all $\lambda \in \mathbb{D}(0,\varepsilon)$. Moreover, if $\lambda \in \mathbb{D}(0,\varepsilon) \setminus \{0\}$ then* $\alpha(\lambda I - T) = \dim(\ker T \cap T^d(X))$ *for some $d \in \mathbb{N}$, so that $\alpha(\lambda I - T)$ is constant as λ ranges in $\mathbb{D}(0,\varepsilon) \setminus \{0\}$ and $\alpha(\lambda I - T) \leq \alpha(T)$ for all $\lambda \in \mathbb{D}(0,\varepsilon)$.*

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if $p(T) = q(T) < \infty$. Clearly, $T \in L(X)$ is Drazin invertible if and only if $\lambda I - T$ is invertible or λ is a pole of the resolvent. Drazin invertibility for bounded operators suggests the following definition.

Definition 2.2. An operator $T \in L(X)$ is said to be left Drazin invertible if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$ is said to be right Drazin invertible if $q := q(T) < \infty$ and $T^q(X)$ is closed. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a *left pole*. A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a *right pole*. A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.

It should be noted that there is a perfect duality, i.e., T (respectively, T^*) is left Drazin invertible if and only if T^* (respectively T) is right Drazin invertible. Furthermore, $T \in L(X)$ is Drazin invertible if and only if T is both left Drazin invertible and right Drazin invertible.

Denote by $\Pi(T)$, $\Pi_a(T)$ and $\Pi_s(T)$ the set of all poles, the set of left poles of T, and the set of right poles respectively. Clearly, $\Pi(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$, $\Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$ and $\Pi_s(T) = \sigma_{\mathrm{s}}(T) \setminus \sigma_{\mathrm{rd}}(T)$. Obviously, $\Pi(T) \subseteq \operatorname{iso}\sigma(T)$, and analogously we have $\Pi_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$ for all $T \in L(X)$. In fact, if $\lambda_0 \in \Pi_a(T)$ then $\lambda I - T$ is left Drazin invertible and hence $p(\lambda_0 I - T) < \infty$. Since $\lambda I - T$ has topological uniform descent (see [10] for definition and details), it then follows, from [10, Corollary 4.8], that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0. An analogous reasoning shows that $\Pi_s(T) \subseteq \operatorname{iso}\sigma_{\mathrm{s}}(T)$ for all $T \in L(X)$.

Obviously, $p_{00}^a(T) \subseteq \Pi_a(T)$ and $p_{00}(T) \subseteq \Pi(T)$ for every $T \in L(X)$. The *Drazin spectrum* is defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

the *left Drazin spectrum* is defined as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

while the *right Drazin spectrum* is defined as

$$\sigma_{\rm rd}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\}.$$

Evidently, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$, $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm ld}(T)$ and $\sigma_{\rm bw}(T) \subseteq \sigma_{\rm d}(T)$.

The proof of the following theorem may be found in [1, Theorem 1.143].

THEOREM 2.3. *For an operator $T \in L(X)$ the following statements hold:*
(i) If T is upper semi B-Weyl and $q(T) < \infty$, then T is Drazin invertible.

(ii) If T is lower semi B-Weyl and $p(T) < \infty$, then T is Drazin invertible.

(iii) If T is B-Weyl and either $p(T)$ and $q(T)$ are finite, then T is Drazin invertible.

LEMMA 2.4. *Let $T \in L(X)$. Then we have $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm a}(T) = \sigma(T)$. Analogously, $\sigma_{\rm rd}(T) = \sigma_{\rm d}(T) \Leftrightarrow \sigma_{\rm s}(T) = \sigma(T)$.*

Proof. Suppose that $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$. If $\lambda \notin \sigma_{\rm a}(T)$ then $\lambda I - T$ is upper semi-Browder, and hence left Drazin invertible. Since $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$ then $\lambda I - T$ is Drazin invertible, hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. This implies, by [1, Chapter 1] that $\alpha(\lambda I - T) = \beta(\lambda I - T)$, and since $\alpha(\lambda I - T) = 0$ we then have $\lambda \notin \sigma(T)$. Therefore, $\sigma_{\rm a}(T) = \sigma(T)$.

Conversely, assume that $\sigma_{\rm a}(T) = \sigma(T)$, and let $\lambda \notin \sigma_{\rm ld}(T)$. Then $\lambda I - T$ is left Drazin invertible, hence $p(\lambda I - T) < \infty$. There are two possibilities: $\lambda \notin \sigma_{\rm a}(T)$ or $\lambda \in \sigma_{\rm a}(T)$. If $\lambda \notin \sigma_{\rm a}(T) = \sigma(T)$ then $\lambda I - T$ is invertible and hence Drazin invertible, i.e., $\lambda \notin \sigma_{\rm d}(T)$. In the other case, where $\lambda \in \sigma_{\rm a}(T)$, we have $\lambda \in \sigma_{\rm a}(T) \setminus \sigma_{\rm ld}(T)$, so λ is a left pole and hence $\lambda \in \operatorname{iso} \sigma_{\rm a}(T) = \operatorname{iso} \sigma(T)$, i.e., $\lambda I - T$ is Drazin invertible, and consequently $\lambda \notin \sigma_{\rm d}(T)$ also in this case. Therefore, $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$.

Suppose that $\sigma_{\rm rd}(T) = \sigma_{\rm d}(T)$. If $\lambda \notin \sigma_{\rm s}(T)$ then $\lambda I - T$ is lower semi-Browder, and hence right Drazin invertible. Since $\sigma_{\rm rd}(T) = \sigma_{\rm d}(T)$ then $\lambda I - T$ is Drazin invertible, hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. This implies, by [1, Chapter 1] that $\alpha(\lambda I - T) = \beta(\lambda I - T)$, and since $\beta(\lambda I - T) = 0$ we then have $\lambda \notin \sigma(T)$. Therefore, $\sigma_{\rm s}(T) = \sigma(T)$.

Conversely, assume that $\sigma_{\rm s}(T) = \sigma(T)$, and let $\lambda \notin \sigma_{\rm rd}(T)$. Then $\lambda I - T$ is right Drazin invertible, hence $q(\lambda I - T) < \infty$. There are two possibilities: $\lambda \notin \sigma_{\rm s}(T)$ or $\lambda \in \sigma_{\rm s}(T)$. If $\lambda \notin \sigma_{\rm s}(T) = \sigma(T)$ then $\lambda I - T$ is invertible and hence Drazin invertible, i.e., $\lambda \notin \sigma_{\rm d}(T)$. In the other case, where $\lambda \in \sigma_{\rm s}(T)$, we have $\lambda \in \sigma_{\rm s}(T) \setminus \sigma_{\rm rd}(T)$, so λ is a right pole and hence $\lambda \in \operatorname{iso} \sigma_{\rm s}(T) = \operatorname{iso} \sigma(T)$, hence $\lambda I - T$ is Drazin invertible, i.e., $\lambda \notin \sigma_{\rm d}(T)$ also in this case. Therefore, $\sigma_{\rm rd}(T) = \sigma_{\rm d}(T)$. □

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of the spectrum.

Remark 2.5. Let $\lambda_0 \in \mathbb{C}$ and suppose that T has SVEP at the points λ of a punctured open disc $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Then T has SVEP at λ_0. Indeed, let $f : \mathbb{D}(\lambda_0, \varepsilon) \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$ holds for every $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. Choose $\mu \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ and let $\mathbb{D}(\mu, \delta)$ be an open disc contained in $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. The SVEP for T at μ entails $f(\lambda) = 0$ on $\mathbb{D}(\mu, \delta)$. Since f is continuous at λ_0 we then conclude that $f(\lambda_0) = 0$. Hence $f \equiv 0$ on $\mathbb{D}(\lambda_0, \varepsilon)$, thus T has the SVEP at λ_0.

Note that $p(\lambda I - T) < \infty \Longrightarrow T$ has SVEP at λ, and dually, $q(\lambda I - T) < \infty \Longrightarrow T^*$ has SVEP at λ, see [1, Chapter 2]. Moreover, from the definition of localized SVEP we easily obtain that if $\sigma_{\mathrm{a}}(T)$ does not cluster at λ then T has SVEP at λ, and, by duality, if $\sigma_{\mathrm{s}}(T)$ does not cluster at λ then T^* has SVEP at λ.

The *quasi-nilpotent part* of T is defined as $H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n(x)\|^{1/n} = 0\}$. For a bounded operator $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence $(x_n)_{n=0,1,\dots} \subset X$, such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \le c^n \|x\|$ for all $n \in \mathbb{N}$. Note that $T(K(T)) = K(T)$, see [1, Chapter 1].

The two subspaces $H_0(T)$ and $K(T)$ are in general not closed and $H_0(\lambda I - T)$ closed $\Longrightarrow T$ has SVEP at λ, see [1, Chapter 2]. Furthermore, if $\lambda \in \operatorname{iso}\sigma(T)$ then the decomposition $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$ holds. If λ is a pole of the resolvent of T of order p then $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ and $K(\lambda I - T) = (\lambda I - T)^p(X)$, see [1, Chapter 2].

3. Zariouh property (gaz)

Define $\Delta_a^g(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T)$ and $\Delta_1^g(T) := \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$. Since $\sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$, we then have $\Pi_a(T) \subseteq \Delta_a^g(T) \subseteq \Delta_1^g(T)$.

Definition 3.1. Let $T \in L(X)$.
1) T is said to verify *property (gaz)* if $\Delta_1^g(T) = \Pi_a(T)$.

2) T is said to verify *generalized a-Browder's theorem*, (gaB), if $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$, or equivalently $\Delta_a^g(T) = \Pi_a(T)$.
Generalized a-Browder's theorem and a-Browder's theorem are equivalent (see [5] or [1, Chapter 5]).

Property (gaz) may be characterized in several ways. The next theorem shows that the operators which satisfy this property have a very nice spectral structure.

Theorem 3.2. *Let $T \in L(X)$. Then the following statements are equivalent:*
(i) T has property (gaz);

(ii) generalized a-Browder's theorem holds and $\sigma_{\mathrm{a}}(T) = \sigma(T)$;

(iii) $\Delta_1^g(T) \subseteq iso\, \sigma_{\mathrm{a}}(T)$;

(iv) $\Delta_1^g(T) \subseteq \partial\sigma_{\mathrm{a}}(T)$, where $\partial\sigma_{\mathrm{a}}(T)$ is the boundary of $\sigma_{\mathrm{a}}(T)$;

(v) $int\, \Delta_1^g(T) = \emptyset$;

(vi) $\sigma(T) = \sigma_{\rm ubw}(T) \cup \partial\sigma_{\rm a}(T)$*;*

(vii) $\sigma(T) = \sigma_{\rm ubw}(T) \cup iso\,\sigma_{\rm a}(T)$*.*

Proof. The proof of the equivalence (i) $\Leftrightarrow$ (ii) may be found in [13].

(ii) $\Rightarrow$ (iii) $\Delta_1^g(T) = \Pi_a(T) \subseteq \mathrm{iso}\,\sigma_{\rm a}(T)$.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) Clear, since $\mathrm{iso}\,\sigma_{\rm a}(T) \subseteq \partial\sigma_{\rm a}(T)$.

(v) $\Rightarrow$ (ii) The condition $\mathrm{int}\,\Delta_1^g(T) = \emptyset$ entails that $\sigma_{\rm a}(T) = \sigma(T)$. Indeed, let $\lambda_0 \notin \sigma_{\rm a}(T)$ and suppose that $\lambda_0 \in \sigma(T)$. Then $\lambda_0 I - T$ is bounded below and hence there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$, centered at λ_0, such $\lambda I - T$ is bounded below for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. Observe that no point of $\mathbb{D}(\lambda_0, \varepsilon)$ belongs to $\partial\sigma_{\rm a}(T)$, since the boundary of the spectrum is always contained in the approximate point spectrum, see [1, Theorem 1.12]. Therefore, $\mathbb{D}(\lambda_0, \varepsilon) \subseteq \mathrm{int}\,\sigma(T)$, and since every bounded below operator is upper semi B-Weyl, we have $\mathbb{D}(\lambda_0, \varepsilon) \subseteq \mathrm{int}\,\Delta_1^g(T)$, a contradiction. So $\lambda_0 \notin \sigma(T)$ and hence $\sigma_{\rm a}(T) = \sigma(T)$.

The condition $\mathrm{int}\,\Delta_1^g(T) = \emptyset$ also entails that T satisfies generalized a-Browder's theorem. Indeed, $\Delta_1^g(T) = \sigma(T) \setminus \sigma_{\rm ubw}(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ubw}(T) = \Delta_a^g(T)$, and the condition $\mathrm{int}\,\Delta_a^g(T) = \emptyset$ is equivalent to saying that T satisfies generalized a-Browder's theorem, see [1, Theorem 5.40].

(ii) $\Leftrightarrow$ (vi) Suppose (ii). Generalized a-Browder's theorem is equivalent, see [1, Theorem 5.40], to the equality $\sigma_{\rm a}(T) = \sigma_{\rm ubw}(T) \cup \partial\sigma_{\rm a}(T)$, and since by assumption $\sigma_{\rm a}(T) = \sigma(T)$, we then obtain that the equality (vi) holds.

Conversely, if $\sigma(T) = \sigma_{\rm ubw}(T) \cup \partial\sigma_{\rm a}(T)$ then, since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm a}(T)$, we have $\sigma(T) \subseteq \sigma_{\rm a}(T)$, hence $\sigma(T) = \sigma_{\rm a}(T)$. Therefore, $\sigma_{\rm a}(T) = \sigma_{\rm ubw}(T) \cup \partial\sigma_{\rm a}(T)$, and this is equivalent to generalized a-Browder's theorem, again by [1, Theorem 5.40].

(ii) $\Leftrightarrow$ (vii) The argument is similar to that of the proof of (ii) $\Leftrightarrow$ (vi). Indeed, generalized a-Browder's theorem is equivalent, see [1, Chapter], to the equality $\sigma_{\rm a}(T) = \sigma_{\rm ubw}(T) \cup \mathrm{iso}\ \sigma_{\rm a}(T)$, and since by assumption $\sigma_{\rm a}(T) = \sigma(T)$, we then have (vii). Conversely, if $\sigma(T) = \sigma_{\rm ubw}(T) \cup \mathrm{iso}\ \sigma_{\rm a}(T)$ then, since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm a}(T)$, we have $\sigma(T) \subseteq \sigma_{\rm a}(T)$, hence $\sigma(T) = \sigma_{\rm a}(T)$. Therefore, $\sigma_{\rm a}(T) = \sigma_{\rm ubw}(T) \cup \mathrm{iso}\ \sigma_{\rm a}(T)$, and this is equivalent, by [1, Chapter 5], to generalized a-Browder's theorem. □

The equivalence (i) $\Leftrightarrow$ (ii) in the previous theorem was first proved in [13]. Property (gaz) is a rather strong property. The next corollary shows that this property entails that several spectra coincide.

Theorem 3.3. *Let* $\in L(X)$*. Then we have:*
(i) If T *has property* (gaz) *then*

$$\sigma_{\rm ubw}(T) = \sigma_{\rm bw}(T) = \sigma_{\rm ld}(T) = \sigma_{\rm d}(T). \tag{1}$$

Consequently, $\Pi(T) = \Pi_a(T)$*.*

(ii) If T^* *has property* (gaz) *then*

$$\sigma_{\rm ubw}(T^*) = \sigma_{\rm bw}(T^*) = \sigma_{\rm rd}(T) = \sigma_{\rm d}(T). \tag{2}$$

Consequently, $\Pi(T) = \Pi_s(T)$*.*

Proof. (i) By Theorem 3.2 we have $\sigma_{\mathrm{a}}(T) = \sigma(T)$ and hence, by Lemma 2.4, $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$. By Theorem 3.2 T also satisfies generalized a-Browder's theorem, so $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$. Moreover, since generalized a-Browder's theorem entails generalized Browder's theorem, we have $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$. Therefore, the equalities (1) hold.

Since $\sigma_{\mathrm{a}}(T) = \sigma(T)$ we also have $\Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$.

(ii) By Theorem 3.2 we have $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{a}}(T^*) = \sigma(T^*) = \sigma(T)$ and hence, by Lemma 2.4, $\sigma_{\mathrm{rd}}(T) = \sigma_{\mathrm{d}}(T)$. By Theorem 3.2 T^* also satisfies generalized a-Browder's theorem, so $\sigma_{\mathrm{ubw}}(T^*) = \sigma_{\mathrm{ld}}(T^*)$, or equivalently, $\sigma_{\mathrm{ubw}}(T^*) = \sigma_{\mathrm{rd}}(T)$. Moreover, since generalized a-Browder's theorem entails generalized Browder's theorem, we have $\sigma_{\mathrm{bw}}(T^*) = \sigma_{\mathrm{d}}(T^*) = \sigma_{\mathrm{d}}(T)$. Therefore, the equalities (2) hold.

Since $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{a}}(T^*) = \sigma(T^*) = \sigma(T)$, we have $\Pi_s(T) = \sigma_{\mathrm{s}}(T) \setminus \sigma_{\mathrm{rd}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$. □

Also the following properties, introduced in [9], may be though as stronger variants than Browder type theorems.

DEFINITION 3.4. Let $T \in L(X)$.
(i) T is said to satisfy *property* (b) if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T)$.

(ii) T is said to satisfy *property* (gb) if $\Delta_a^g(T) = \Pi(T)$.

By Theorem 3.2, if $T \in L(X)$ satisfies property (gaz), the equality $\sigma_{\mathrm{a}}(T) = \sigma(T)$ implies $\Delta_1^g(T) = \Delta_a^g(T) \subseteq \mathrm{iso}\, \sigma_{\mathrm{a}}(T)$, and this last inclusion is equivalent to property (gb), see [2]. Hence

$$\text{property } (gaz) \Rightarrow \text{property } (gb) \Rightarrow \text{generalized a-Browder's theorem.}$$

The following theorem establishes the exact relationship between property (gaz) and property (gb).

THEOREM 3.5. *Let $T \in L(X)$. Then the following statements are equivalent:*
(i) $T \in L(X)$ has property (gaz);

(ii) T has property (gb) and $\sigma_{\mathrm{a}}(T) = \sigma(T)$;

(iii) T has property (b) and $\sigma_{\mathrm{a}}(T) = \sigma(T)$;

(iv) T satisfies generalized a-Browder's theorem and $\sigma_{\mathrm{a}}(T) = \sigma(T)$;

(v) $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}^a(T)$.

Proof. (i) ⇒ (ii) As observed before, property (gaz) entails property (gb). We show the equality $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$. It is sufficient to prove $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{ld}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ld}}(T)$. There are two possibilities: $\lambda \notin \sigma_{\mathrm{a}}(T)$ or $\lambda \in \sigma_{\mathrm{a}}(T)$. Trivially, if $\lambda \notin \sigma_{\mathrm{a}}(T) = \sigma(T)$ then $\lambda I - T$ is invertible, so $\lambda \notin \sigma_{\mathrm{d}}(T)$. If $\lambda \in \sigma_{\mathrm{a}}(T)$ then $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Since T satisfies generalized Browder's theorem we have $\sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{ld}}(T)$, hence $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = \Pi(T)$, since T satisfies property (gb). Hence λ is a pole of T, and consequently $\lambda \notin \sigma_{\mathrm{d}}(T)$.

(ii) ⇒ (iii) ⇒ (iv) Clear.

(iv) ⇒ (v) Property (b) entails a-Browder's theorem, i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$. Since by assumption $\sigma_{\mathrm{a}}(T) = \sigma(T)$, we then have $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^a(T)$.

(v) $\Rightarrow$ (i) Let $\lambda_0 \in \Delta_1^g(T)$. Then, $\lambda_0 \in \sigma(T)$ and $\lambda_0 I - T$ is upper semi B-Weyl, so, by Theorem 2.1, there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that $\lambda I - T \in W_+(X)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, with $\operatorname{ind}(\lambda_0 I - T) = \operatorname{ind}(\lambda I - T) \leq 0$. Hence,

$$\lambda \in \sigma(T) \setminus \sigma_{\rm uw}(T) = p_{00}^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T),$$

so $p(\lambda I - T) < \infty$, and hence T has SVEP at every $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. By Remark 2.5 it then follows that T has SVEP also at λ_0, so $\lambda_0 I - T$ is left Drazin invertible, by [1, Theorem 2.97]. We also have $\lambda_0 \in \sigma_{\rm a}(T)$. Indeed, for every $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $\lambda I - T$ has closed range, being $\lambda I - T \in W_+(X)$, hence $\alpha(\lambda I - T) > 0$, since $\lambda \in \sigma_{\rm a}(T)$. From Theorem 2.1 it then follows that $0 < \alpha(\lambda I - T) < \alpha(\lambda_0 I - T)$, thus $\lambda_0 \in \sigma_{\rm a}(T)$. Therefore, $\lambda_0 \in \Pi_a(T)$, so $\Delta_1^g(T) \subseteq \Pi_a(T)$, and since the reverse inclusion is always true we then conclude that $\Delta_1^g(T) = \Pi_a(T)$. □

In [13], an operator T for which the equality $\sigma(T) \setminus \sigma_{\rm uw}(T) = p_{00}^a(T)$ holds is said to have *property* (az). Evidently, properties (gaz) and (az) are equivalent. These two properties are also equivalent to the properties (gah) and (ah) studied in [14].

The next theorem gives a local spectral characterization of property (gaz).

Theorem 3.6. *Let $T \in L(X)$. Then we have:*
(i) T^* has SVEP at the points $\lambda \notin \sigma_{\rm ubw}(T)$ if and only if property (gaz) holds for T.
(ii) T has SVEP at the points $\lambda \notin \sigma_{\rm ubw}(T^*)$ if and only if property (gaz) holds for T^*.

Proof. (i) Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\rm ubw}(T)$. The SVEP for T^* at the points $\lambda \notin \sigma_{\rm ubw}(T)$ entails generalized a-Browder's theorem. Indeed, if $\lambda \notin \sigma_{\rm ubw}(T)$ then the SVEP of T^* entails, by part (i) of Theorem 2.3, that $\lambda I - T$ is Drazin invertible, in particular left Drazin invertible, so $\lambda \notin \sigma_{\rm ld}(T)$. Therefore, $\sigma_{\rm ld}(T) \subseteq \sigma_{\rm ubw}(T)$. The reverse inclusion is true for every operator, so $\sigma_{\rm ld}(T) = \sigma_{\rm ubw}(T)$. Hence T satisfies generalized a-Browder's theorem.

On the other hand, if $\lambda \notin \sigma_{\rm ld}(T) = \sigma_{\rm ubw}(T)$, then $\lambda I - T$ is left Drazin invertible, and, by [1, Theorem 2.98], the SVEP for T^* at λ entails that $\lambda I - T$ is right Drazin invertible, thus $\lambda \notin \sigma_{\rm d}(T)$. Hence $\sigma_{\rm d}(T) \subseteq \sigma_{\rm ld}(T)$ and since the reverse inclusion is always true, we then have $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$, or equivalently, by Lemma 2.4, $\sigma_{\rm a}(T) = \sigma(T)$. By Theorem 3.2 it then follows that T has property (gaz).

Conversely, assume that T has property (gaz) and let $\lambda \notin \sigma_{\rm ubw}(T)$. From Corollary 3.3 then $\lambda \notin \sigma_{\rm d}(T)$, so $q(\lambda\lambda I - T) < \infty$ and hence T^* has SVEP at λ.

(ii) We show first that the SVEP for T at the points $\lambda \notin \sigma_{\rm ubw}(T^*)$ entails generalized a-Browder's theorem for T^*. Let $\lambda \notin \sigma_{\rm ubw}(T^*)$. Then $\lambda I - T^*$ is upper semi B-Weyl, hence is quasi-Fredholm, see [1, Chapter 1] for definition and details, or equivalently $\lambda I - T$ is quasi-Fredholm, by [1, Theorem 1.104]. The SVEP of T^* at λ implies that $\lambda I - T$ is right Drazin invertible, hence, by duality, $\lambda I - T^*$ is left Drazin invertible, so $\lambda \notin \sigma_{\rm ld}(T^*)$. Therefore, $\sigma_{\rm ld}(T^*) \subseteq \sigma_{\rm ubw}(T^*)$, and since the opposite inclusion is true, it then follows that $\sigma_{\rm ld}(T^*) = \sigma_{\rm ubw}(T^*)$, i.e., T^* satisfies generalized a-Browder's theorem.

We show now that $\sigma_{\rm ld}(T^*) = \sigma_{\rm d}(T^*)$. Let $\lambda \notin \sigma_{\rm ld}(T^*)$. Then $\lambda I - T^*$ is left Drazin invertible, hence both $\lambda I - T^*$ and $\lambda I - T$ are quasi-Fredholm. Since $\sigma_{\rm ubw}(T^*) \subseteq \sigma_{\rm ld}(T^*)$ we have $\lambda \notin \sigma_{\rm ubw}(T^*)$, so T has SVEP at λ. By [1, Theorem 2.97] then

$\lambda I - T$ is left right invertible, hence $\lambda I - T^*$ is right Drazin invertible. This shows that $\sigma_{\mathrm{d}}(T^*) \subseteq \sigma_{\mathrm{ld}}(T^*)$, and hence $\sigma_{\mathrm{d}}(T^*) = \sigma_{\mathrm{ld}}(T^*)$, By Lemma 2.4 it then follows that T^* satisfies property (gaz). Conversely, assume that T^* has property (gaz). From part (ii) of Corollary 3.3 we have $\sigma_{\mathrm{ubw}}(T^*) = \sigma_{\mathrm{d}}(T^*)$. Hence, if $\lambda \notin \sigma_{\mathrm{ubw}}(T^*)$ then λ is a pole of T^*, or equivalently, by [1, Theorem 4.2], λ is a pole of T. From $p(\lambda I - T) < \infty$ we then conclude that T has SVEP at λ. □

Corollary 3.7. *If T^* has SVEP then property (gaz) holds for T, while if T has SVEP then property (gaz) holds for T^*.*

Corollary 3.7 applies to several classes of operators; the SVEP for T is for instance satisfied by the class $H(p)$, where T is said to be $H(p)$ if there exists a natural $p := p(\lambda)$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ for all $\lambda \in \mathbb{C}$. Indeed, property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [1, Chapter 4] for details.

The following example shows that the result observed in Theorem 3.6 is not true whenever we replace the assumption that T^* (respectively, T) has SVEP with the assumption that T has SVEP (respectively, T^*). Recall that $T \in L(X)$ is said to be *a-polaroid* if every isolated element of $\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent.

Example 3.8. Let R denote the classical right shift in the Hilbert space $\ell_2(\mathbb{N})$, defined as $R(x_1, x_2, \dots,) := (0, x_1, x_2, \dots)$ for all $x = (x_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$, and denote by L the left shift in the Hilbert space $\ell_2(\mathbb{N})$, defined as $L(x_1, x_2, \dots,) := (x_2, x_3, \dots)$ for all $x = (x_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$. It is known that the adjoint L' is R, and that $R' = L$. Moreover, R has SVEP, while L does not have SVEP at 0. By Corollary 3.7 every left shift operator satisfies property (gaz), since $L' = R$ has SVEP. By Theorem 3.2 then property (gaz) fails for R, since $\sigma(R) = \mathbf{D}(0,1)$, $\mathbf{D}(0,1)$ the closed disc in $\mathbb{C}$, while $\sigma_{\mathrm{a}}(R) = \partial\mathbf{D}(0,1)$. This example also shows that property (gaz) for a bounded operator T is not transmitted, in general, to its adjoint.

The right shift also provides an example of operator which satisfies property (gb) but not property (gaz). Indeed, since $\operatorname{iso}\sigma_{\mathrm{a}}(R) = \emptyset$, R is a-polaroid. Since R has SVEP then R satisfies property (gb), by [2, Corollary 3.11].

Property (gaz) may be characterized in a very simple way.

Corollary 3.9. *Let $T \in L(X)$. Then T has property (gaz) $\Leftrightarrow$ $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$.*

Proof. If T has property (gaz) then, by Theorem 3.3, $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$. Conversely, suppose that $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$. If $\lambda \notin \sigma_{\mathrm{ubw}}(T)$ then $\lambda I - T$ is Drazin invertible, hence $q(\lambda I - T) < \infty$ and this implies that T^* has SVEP at λ. By Theorem 3.6 then T has property (gaz). □

Corollary 3.10. *If $T \in L(X)$ then the following statements are equivalent:*
(i) T has property (gaz);

(ii) T satisfies generalized a-Browder's theorem and $\sigma_{\mathrm{bw}}(T) \cap \Delta_1^g(T) = \emptyset$;

(iii) T satisfies generalized Browder's theorem and $\sigma_{\mathrm{bw}}(T) \cap \Delta_1^g(T) = \emptyset$.

Proof. (i) ⇒ (ii) If T satisfies (gaz) then, by Theorem 3.3, $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T)$, so $\sigma_{\mathrm{bw}}(T) \cap \Delta_1^g(T) = \emptyset$.

(ii) ⇒ (iii) Clear, since generalized a-Browder's theorem entails generalized Browder's theorem.

(iii) ⇒ (i) By Corollary 3.9 it suffices to prove the equality $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$, and for that we have only to show the inclusion $\sigma_{\mathrm{d}}(T) \subseteq \sigma_{\mathrm{ubw}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ubw}}(T)$. Then either $\lambda \notin \sigma(T)$ or $\lambda \in \sigma(T)$. Trivially, $\lambda \notin \sigma_{\mathrm{d}}(T)$ in the first case. If $\lambda \in \sigma(T)$ then $\lambda \in \Delta_1^g(T)$, hence $\lambda I - T$ is B-Weyl. Generalized Browder's theorem for T yields that $\lambda I - T$ is Drazin invertible, thus $\lambda \notin \sigma_{\mathrm{d}}(T)$ in the second case, too. □

Since the dual of the left shift L in $\ell_2(\mathbb{N})$ has SVEP we have that $\sigma_{\mathrm{a}}(L) = \sigma(L)$, see [1, Theorem 2.68]. We can say much more.

Corollary 3.11. *For the left shift L in $\ell_2(\mathbb{N})$ we have $\sigma_{\mathrm{ubw}}(L) = \sigma_{\mathrm{bw}}(L) = \sigma_{\mathrm{ld}}(L) = \sigma_{\mathrm{d}}(L) = \sigma_{\mathrm{a}}(L) = \sigma(L) = \mathbf{D}(0,1)$.*

Proof. The equalities $\sigma_{\mathrm{ubw}}(L) = \sigma_{\mathrm{bw}}(L) = \sigma_{\mathrm{ld}}(L) = \sigma_{\mathrm{d}}(L)$, $\sigma_{\mathrm{a}}(L) = \sigma(L) = \mathbf{D}(0,1)$ are consequences of property (gaz) for L. Clearly, if $\lambda \notin \sigma_{\mathrm{d}}(L)$, then $\lambda I - L$ is Drazin invertible, and hence $\lambda \in \mathrm{iso}\,\sigma(L)$, or $\lambda I - T$ is invertible. Since $\mathrm{iso}\,\sigma(L) = \emptyset$ then $\lambda \notin \sigma(L)$, so $\sigma_{\mathrm{d}}(L) = \sigma(L)$. □

Let $\rho_{\mathrm{a}}(T) = \mathbb{C} \setminus \sigma_{\mathrm{a}}(T)$ and $\rho(T) = \mathbb{C} \setminus \sigma(T)$. In [3] it has been proved that $\rho_{\mathrm{uw}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{uw}}(T)$ is connected if and only if $\rho_{\mathrm{a}}(T)$ is connected and T satisfies a-Browder's theorem, or equivalently generalized a-Browder's theorem. In [3] it has been shown that $\rho_{\mathrm{w}}(T)$ is connected if and only if $\rho(T)$ is connected and T satisfies Browder's theorem. We can improve this result.

Theorem 3.12. *Let $T \in L(X)$ be such that $\rho_{\mathrm{uw}}(T)$ is connected. Then T satisfies property (gaz).*

Proof. Since, as noted above, T satisfies a-Browder's theorem, it suffices, by Theorem 3.2, to prove that $\sigma_{\mathrm{a}}(T) = \sigma(T)$. Since $\rho_{\mathrm{a}}(T)$ is connected then $\rho(T)$ is connected, i.e., there is no bounded open connected component of $\rho(T)$. Let Ω be unique unbounded open connected component of $\rho(T)$. Evidently, $\Omega \subseteq \rho_{\mathrm{a}}(T) \subseteq \rho_{\mathrm{sf}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{sf}}(T)$, where $\sigma_{\mathrm{sf}}(T)$ denotes the semi-Fredholm spectrum of T, and Ω is also unique unbounded open connected component of $\rho_{\mathrm{a}}(T)$. Now, let $\lambda \notin \sigma_{\mathrm{a}}(T)$. Then $\lambda \in \rho_{\mathrm{a}}(T)$, and since $\rho_{\mathrm{a}}(T)$ is connected, then λ belongs to Ω. By [4, Theorem 2.5] we then have that T^* has SVEP at λ, so $q(\lambda I - T) < \infty$, by [1, Theorem 2.98], and hence $\beta(\lambda I - T) \leq \alpha(\lambda I - T) = 0$, see [1, Theorem 1.22]. Thus $\lambda \notin \sigma(T)$, and hence $\sigma_{\mathrm{ap}}(T) = \sigma(T)$. □

Property (gaz) may be also characterized by means of the quasi-nilpotent part as follows.

Theorem 3.13. *Let $T \in L(X)$. Then the following statements are equivalent:*
(i) T has property (gaz);

(ii) For every $\lambda \in \Delta_1^g(T)$ there exists a natural number $\nu := \nu(\lambda)$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$ and $\sigma(T) = \sigma_{\mathrm{a}}(T)$:

(iii) $H_0(\lambda I - T)$ *is closed for all* $\lambda \in \Delta_1^g(T)$ *and* $\sigma(T) = \sigma_{\mathrm{a}}(T)$.

(iv) For every $\lambda \in \Delta_1^g(T)$ *there exists a natural number* $\nu := \nu(\lambda)$ *such that* $K(\lambda I - T) = (\lambda I - T)^\nu(X)$ *and* $\sigma(T) = \sigma_{\mathrm{a}}(T)$.

Proof. (i) $\Rightarrow$ (ii) Assume property (gaz) for T. Then every $\lambda \in \Delta_1^g(T)$ is a left pole of T, hence, see [1, Theorem 4.3], there exists a natural number $\nu := \nu(\lambda)$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$. Furthermore, $\sigma(T) = \sigma_{\mathrm{a}}(T)$ by Theorem 3.2.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let $\lambda \notin \sigma_{\mathrm{ubw}}(T)$. Since $H_0(\lambda I - T)$ is closed then T has SVEP at λ, and since $\lambda I - T$ has topological uniform descent, by [1, Theorem 2.97], then $\lambda I - T$ is left Drazin invertible, so $\lambda \notin \sigma_{\mathrm{ld}}(T)$, and consequently $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ubw}}(T)$. From this we obtain that T has property (gaz).

(ii) $\Rightarrow$ (iv) Since $\sigma_{\mathrm{a}}(T) = \sigma(T)$, every point $\lambda \in \Delta_1^g(T)$ is an isolated point of $\sigma(T)$, hence $X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^\nu \oplus K(\lambda I - T)$, from which we obtain $(\lambda I - T)^\nu(X) = K(\lambda I - T)$.

(iv) $\Rightarrow$ (i) If $\lambda \in \Delta_a^g(T)$ from the inclusion $\Delta_a^g(T) \subseteq \Delta_1^g(T)$ we know that $K(\lambda I - T) = (\lambda I - T)^\nu(X)$ for some $\nu \in \mathbb{N}$. By [2, Theorem 3.8], then T satisfies property (gb). Since by assumption $\sigma(T) = \sigma_{\mathrm{a}}(T)$, it the follows that T satisfies (gaz), by Theorem 3.5. □

Let M, N be two closed linear subspaces of X and define $\delta(M, N) := \sup\{\operatorname{dist}(u, N) : u \in M, \|u\| = 1\}$, in the case $M \neq \{0\}$, otherwise set $\delta(\{0\}, N) = 0$ for any subspace N. According to [12, §2, Chapter IV], the *gap* between M and N is defined by $\widehat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}$. The function $\widehat{\delta}$ is a metric on the set of all linear closed subspaces of X and the convergence $M_n \to M$ is obviously defined by $\widehat{\delta}(M_n, M) \to 0$ as $n \to \infty$.

In the following we need the following elementary lemma.

Lemma 3.14. *If* $T \in L(X)$ *is injective and upper semi* B*-Fredholm then* T *is bounded below.*

Proof. If T is upper semi B-Fredholm then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed. By assumption $\alpha(T) < \infty$, and this implies that $\alpha(T^n) < \infty$, so T^n is upper semi-Fredholm and by the classical Fredholm theory we deduce that T is upper semi-Fredholm. Consequently, $T(X)$ is closed and hence T is bounded below. □

Theorem 3.15. *For a bounded operator* $T \in L(X)$ *the following statements are equivalent:*

(i) T *satisfies property* (gaz)*;*

(ii) The mapping $\lambda \mapsto \ker(\lambda I - T)$ *is discontinuous at every* $\lambda \in \Delta_1^g(T)$ *in the gap metric.*

Proof. (i) $\Rightarrow$ (ii) Suppose that T satisfies (gaz). If $\lambda_0 \in \Delta_1^g(T) = \Pi_a(T)$ then $\lambda_0 I - T$ is upper semi B-Weyl and $\lambda_0 \in \sigma_{\mathrm{a}}(T)$. Note that $\alpha(\lambda_0 I - T) > 0$. Indeed, if $\alpha(\lambda_0 I - T) = 0$ then, by Lemma 3.14 we would have that $\lambda_0 I - T$ is bounded below, i.e., $\lambda_0 \notin \sigma_{\mathrm{a}}(T)$. On the other hand, by Theorem 3.2, there exists a disc $\mathbb{D}(\lambda_0)$

centered at λ_0 such that $\alpha(\lambda I - T) = 0$ for all $\mathbb{D}(\lambda_0) \setminus \{\lambda_0\}$, hence the mapping $\lambda \mapsto \ker(\lambda I - T)$ is discontinuous at λ_0 in the gap metric.

(ii) $\Rightarrow$ (i) We show that $\Delta_1^g(T) \subseteq \operatorname{iso}\sigma_{\rm a}(T)$, so Theorem 3.2 applies. Let $\lambda_0 \in \Delta_1^g(T)$ be arbitrary. Then $\lambda_0 I - T$ is upper semi B-Weyl. By Theorem 2.1 we know that there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that, $\lambda I - T$ is upper Weyl for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $\alpha(\lambda I - T)$ is constant as λ ranges on $\mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, $\operatorname{ind}(\lambda I - T) = \operatorname{ind}(\lambda_0 I - T)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$, and $0 \le \alpha(\lambda I - T) \le \alpha(\lambda_0 I - T)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. Since the mapping $\lambda \mapsto \ker(\lambda I - T)$ is discontinuous at every $\lambda \in \Delta_1^g(T)$ then $0 \le \alpha(\lambda I - T) < \alpha(\lambda_0 I - T)$ for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. We show that

$$\alpha(\lambda I - T) = 0 \quad \text{for all } \lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}. \tag{3}$$

To see this, suppose that there exists $\lambda_1 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ such that $\alpha(\lambda_1 I - T) > 0$. Clearly, $\lambda_1 \in \Delta_1^g(T)$, so, arguing as for λ_0, we obtain a $\lambda_2 \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0, \lambda_1\}$ such that $0 < \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T)$, and this is impossible since $\alpha(\lambda I - T)$ is constant for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. Therefore (3) is satisfied and since $\lambda I - T$ is upper Weyl for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, the range $(\lambda I - T)(X)$ is closed for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, thus $\lambda_0 \in \operatorname{iso}\sigma_{\rm a}(T)$, as desired. □

Set $E(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}$, and $E_a(T) := \{\lambda \in \operatorname{iso}\sigma_{\rm a}(T) : 0 < \alpha(\lambda I - T)\}$. Evidently, if T has property (gaz) then $E(T) = E_a(T)$ and $\Pi(T) = \Pi_a(T)$, since $\sigma(T) = \sigma_{\rm a}(T)$ and $\sigma_{\rm ld}(T) = \sigma_{\rm d}(T)$.

Definition 3.16. We say that $T \in L(X)$ satisfies *generalized Weyl's theorem*, in symbol (gW), if $\sigma(T) \setminus \sigma_{\rm bw}(T) = E(T)$. $T \in L(X)$ is said to satisfy the *generalized a-Weyl's theorem*, abbreviated (gaW), if $\Delta_a^g(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ubw}(T) = E_a(T)$. $T \in L(X)$ is said to satisfy the *generalized property* (gw), abbreviated (gw), if the equality $\sigma_{\rm a}(T) \setminus \sigma_{\rm ubw}(T) = E(T)$ holds.

Note that either of properties (gaW) and (gw) entails (gW), see [1, Chapter 6]. If T has property (gaz), the equality $\sigma_{\rm a}(T) = \sigma(T)$ entails that $\Delta_a^g(T) = E_a(T)$ if and only if $\Delta_a^g(T) = E(T)$, so (gaW) and (gw) are equivalent for T. The following example shows that, in general, (gaW), (gw) and (gaz) are independent.

Example 3.17. Property (gaz), (gaW) or (gw) are independent. To see this, consider the weighted right shift T on the Hilbert space $\ell^2(\mathbb{N})$, defined as $T(x_1, x_2, \dots) := (0, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$ for all $(x_n) \in \ell^2(\mathbb{N})$. T is quasi-nilpotent and hence has SVEP, so its adjoint T' satisfies property (gaz). On the other hand we have $E(T') = E_a(T') = \{0\} \neq \sigma_{\rm a}(T') \setminus \sigma_{\rm ubw}(T') = \emptyset$, so T' does not satisfy (gaW) and (gw).

To show an example of operator for which (gaW) and (gw) hold, but not (gaz), consider a right shift R in $\ell^2(\mathbb{N})$. As observed before, property (gaz) fails for R. We have $\sigma_{\rm ubw}(R) = \sigma_{\rm a}(R) = \partial\sigma(R)$. The inclusion $\sigma_{\rm ubw}(R) \subseteq \sigma_{\rm a}(R) = \partial\sigma(R) = \partial\mathbf{D}(0,1)$ is obvious. Suppose that there exists $\lambda \notin \sigma_{\rm ubw}(R)$ such that $\lambda \in \sigma_{\rm a}(R)$. Since R has SVEP at λ then, by [1, Theorem 2.97], $\lambda \in \operatorname{iso}\sigma_{\rm a}(T)$, and this is impossible, since $\operatorname{iso}\sigma_{\rm a}(T) = \emptyset$. Hence, $\sigma_{\rm ubw}(R) = \sigma_{\rm a}(R)$, so $\Delta_a^g(R) = \emptyset$. On the other hand, $E_a(R) = E(R) = \emptyset$, so R satisfies both (gaW) or (gw).

Theorem 3.18. *Let $T \in L(X)$. Then we have:*

(i) If T satisfies either (gaW), or (gw), and $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T)$ then T has property (gaz).

(ii) If T satisfies property (gaz) and $E(T) = \Pi(T)$ then T satisfies (gaW), or equivalently, T satisfies (gw).

Proof. (i) From assumption we have $\Delta_1^g(T) = \sigma(T) \setminus \sigma_{\mathrm{bw}}(T)$. But property (gaW) entails generalized a-Browder's theorem, hence $\Delta_1^g(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T) \subseteq \Pi_a(T)$. The converse inclusion $\Pi_a(T) \subseteq \Delta_1^g(T)$ is always true, so $\Pi_a(T) = \Delta_1^g(T)$.

(ii) If T satisfies property (gaz) and $E(T) = \Pi(T)$ then $E_a(T) = E(T)$, and, by Theorem 3.3, we have $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T)$. Hence $\Pi(T) = \Pi_a(T) = \Delta_1^g(T) = \Delta_a^g(T)$. Evidently, property (gaW) and (gw) are equivalent, since $E_a(T) = E(T)$. □

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(received 14.01.2019; in revised form 15.05.2019; available online 07.01.2020)

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