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ON A GENERALIZED FIXED POINT THEOREM IN INCOMPLETE SOFT METRIC SPACES

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Abstract. In this paper, we introduce the notion of orthogonal relation on a soft set (F, A) and some related concepts. This notion allows us to consider fixed point theorem in SO-complete instead of complete soft metric spaces introduced by Yazar et.al. (Filomat 30:2 (2016), 269–279). Then, the existence and uniqueness of soft fixed points for a generalized soft contractive mapping are proved. Also, some examples are given to support that our main theorem is a real extension of Yazar et.al.

1. Introduction

Soft theory was initiated by the Russian researcher Molodtsov [9] in 1999. Molodtsov proposed the soft set which provides a completely new approach for modeling vagueness and uncertainty. Due to the fact that many mathematical objects such as fuzzy sets, topological spaces, rough sets (see [7, 9]) can be considered as particular types of soft sets, it is a very general tool to handle objects which are defined in terms of loose or general set of characteristics. The study of soft set operations is vital for mathematicians and computer scientists to develop the theory of soft topological spaces. Maji et al. [8] introduced some basic algebraic operations on soft sets. Das and Samanta [3, 4] introduced a different notion of soft metric space by using a different concept of soft point and investigated some important properties of these spaces. In 2016, Yazar et.al [11] by using the concept soft metric spaces defined in [4], investigated some fixed points for soft contractive mappings on soft metric spaces. Especially, Eshaghi et.al. [6] introduced the notion of orthogonal set. They gave an extension of Banach contraction principle in incomplete metric spaces. The main theorem presented in [6] is given by the following result.

THEOREM 1.1. *Let* $(X, \perp, d)$ *be an* O*-complete metric space* (*not necessarily complete*) *and* $0 < \lambda < 1$. *Let* $f : X \to X$ *be* $\perp$*-continuous,* $\perp$*-contraction with Lipschitz*

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constant λ and $\perp$-preserving. Then f has a unique fixed point $x^* \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n(x) = x^*$ *for all $x \in X$.*

For more details about orthogonal space, we refer the reader to [1, 2, 6].

The aim of this paper is to initiate the study of soft fixed point theory in the framework of incomplete soft metric spaces and to prove a soft generalized version of Banach's fixed point theorem. Also, we construct an example which shows that the main theorem of this paper is a real extension of [11, Theorem 4.8].

Let us recall some basic definitions and known results in the soft set theory.

Throughout this paper, X refers to an initial universe, and E is the set of parameters for X. Denote by $P(X)$ the family of all subsets of X.

Definition 1.2 ([9]). If F is a set valued mapping on $A \subseteq E$ taking values in $P(X)$, then a pair (F, A) is called a soft set over X.

Definition 1.3 ([8]). A soft set (F, A) over X is said to be a null soft set over X if $F(e) = \emptyset$ for all $e \in A$.

A soft set (F, A) over X is said to be an absolute soft set, denoted by $\tilde{X}$ if $F(e) = X$ for all $e \in A$.

Definition 1.4 ([9]). The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and for all $e \in C, H(e) = F(e) \cap G(e)$. This is denoted by $(F, A)\tilde{\cap}(G, B) = (H, C)$.

The union of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$

$$H(e) = \begin{cases} F(e), & e \in A/B, \\ G(e), & e \in B/A, \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 1.5 ([5]). Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$ for all $e \in E$, respectively. The set of all soft real numbers is denoted by $\mathbb{R}(E)$.

Definition 1.6 ([5]). Let $\tilde{r}, \tilde{s}$ be two soft real numbers. Then the following statements hold:

(i) $\tilde{r}\tilde{\leq}\tilde{s}$ if $\tilde{r}(e) \leq \tilde{s}(e)$ for all $e \in E$; (ii) $\tilde{r}\tilde{\geq}\tilde{s}$ if $\tilde{r}(e) \geq \tilde{s}(e)$ for all $e \in E$;

(iii) $\tilde{r}\tilde{<}\tilde{s}$ if $\tilde{r}(e) < \tilde{s}(e)$ for all $e \in E$; (iv) $\tilde{r}\tilde{>}\tilde{s}$ if $\tilde{r}(e) > \tilde{s}(e)$ for all $e \in E$.

We say that a real soft number $\tilde{r}$ is non-negative whenever $\tilde{r} \tilde{\geq} \tilde{0}$. The set of all positive soft real numbers is denoted by $\mathbb{R}(E)^*$.

Definition 1.7 ([4]). A soft set (F, E) over X is said to be a soft point, denoted by $\tilde{x}_e$, if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \emptyset$ for all $e' \in E/\{e\}$.

Two soft points $\tilde{x}_e, \tilde{y}_{e'}$ are said to be equal if $e = e'$ and $x = y$ (see [4]). Thus $\tilde{x}_e \neq \tilde{y}_{e'}$ if $x \neq y$ or $e \neq e'$. Also, a soft point $\tilde{x}_e$ is said to belong to a soft set (F, E) if $e \in E$ and $\{x\} \subset F(e)$. We write $\tilde{x}_e \in (F, E)$.

Every soft set can be expressed as a union of all soft points belonging to it as $(F, E) = \bigcup_{\tilde{x}_e \in (F,E)} \{\tilde{x}_e\}$. Conversely, any set of soft points can be considered as a soft set.

The set of all soft points of absolute soft set $\tilde{X}$ is denoted by $SP(\tilde{X})$.

Definition 1.8 ([4]). A mapping $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft metric on the soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:
(i) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \tilde{0} \Longleftrightarrow \tilde{x}_e = \tilde{y}_{e'}$; (ii) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = \tilde{d}(\tilde{y}_{e'}, \tilde{x}_e)$;
(iii) $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) \tilde{\leq} \tilde{d}(\tilde{x}_e, \tilde{z}_{e''}) + \tilde{d}(\tilde{z}_{e''}, \tilde{y}_{e'})$ for all $\tilde{z}_{e''} \in SP(\tilde{X})$.
The triple $(\tilde{X}, \tilde{d}, E)$ is called a soft metric space.

Definition 1.9 ([4]). The sequence $\{\tilde{x}^n_{e_n}\}$ of soft points in $(\tilde{X}, \tilde{d}, E)$ is called a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\epsilon} \tilde{>} \tilde{0}$ there is $n_0 \in \mathbb{N}$ such that $\tilde{d}(\tilde{x}^n_{e_n}, \tilde{x}^m_{e_m}) \tilde{<} \tilde{\epsilon}$ for all $m, n \geq n_0$.

Also, the sequence $\{\tilde{x}^n_{e_n}\}$ is said to be convergent to $\tilde{x}^0_{e_0}$ in $\tilde{X}$, denoted by $\tilde{x}^n_{e_n} \xrightarrow{\tilde{d}} \tilde{x}^0_{e_0}$, if there is a soft point $\tilde{x}^0_{e_0}$ such that for every $\tilde{\epsilon} \tilde{>} \tilde{0}$ there is $n_0 \in \mathbb{N}$ for which $\tilde{d}(\tilde{x}^n_{e_n}, \tilde{x}^0_{e_0}) \tilde{<} \tilde{\epsilon}$ for all $n \geq n_0$.

Definition 1.10 ([11]). Let $(\tilde{X}, \tilde{d}, E)$ and $(\tilde{Y}, \tilde{d}', E')$ be two soft metric spaces. The mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{Y}, \tilde{d}', E')$ is a soft mapping, where $f : X \to Y$ and $\varphi : E \to E'$ are two mappings.

The soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is said to be soft contractive mapping whenever $\exists\ \tilde{0} < \tilde{\lambda} \tilde{<} \tilde{1},\ \forall \tilde{x}_e, \tilde{y}_{e'} \in SP(\tilde{X}),\ \tilde{d}\big((f,\varphi)(\tilde{x}_e), (f,\varphi)(\tilde{y}_{e'})\big) \tilde{\leq} \tilde{\lambda}\ \tilde{d}(\tilde{x}_e, \tilde{y}_{e'})$.

2. Orthogonal sets and their relations to soft sets

Let $\tilde{X}$ be an absolute soft set and E be a parameter set and $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$.

We start our work with the following definition.

Definition 2.1. Let $\tilde{\perp} \subseteq SP(\tilde{X}) \times SP(\tilde{X})$ be a relation in $SP(\tilde{X})$. If $\tilde{\perp}$ satisfies the following condition

$$\exists\ x_{e_0} \in SP(\tilde{X}) : (\forall y_e \in SP(\tilde{X}),\ y_e \tilde{\perp}\ x_{e_0}) \ \text{ or } \ (\forall y_e \in SP(\tilde{X}),\ x_{e_0} \tilde{\perp}\ y_e),$$

then $\tilde{X}$ is called an *orthogonal soft set* (briefly *O-soft set*). We denote this O-soft set by $(\tilde{X}, \tilde{\perp})$.

In the above definition, we say that x_{e_0} is an orthogonal soft element. Also two soft elements x_e and $y_{e'}$ are said to be $\tilde{\perp}$-comparable, if $x_e \tilde{\perp} y_{e'}$ or $y_{e'} \tilde{\perp} x_e$.

As an illustration, let us consider the following examples:

EXAMPLE 2.2. Let X be the set of all blood types O, A, B, AB and E be the set of types "positive" and "negative", i.e. $X = \{A, B, AB, O\}$ and $E = \{+, -\}$. Consider the relation $\tilde{\perp}$ on $SP(\tilde{X})$ as $x_e \tilde{\perp} y_{e'}$ iff "x" of the type "e" can give blood to "y" of the type "e'". Thus, if x_{e_0} is a person such that his (her) blood type is "O" of the type "$-$" , then we have $x_{e_0} \tilde{\perp} y_e$ for all $y_e \in SP(\tilde{X})$.

Notice that, in the above example x_{e_0} may be a person with blood type "AB" of the type "$+$". In this case, we have $y_e \tilde{\perp}\, x_{e_0}$ for all $y_e \in SP(\tilde{X})$.

The above example shows that the orthogonal soft element is not necessarily unique.

EXAMPLE 2.3. Suppose $X = \{1, 2, \ldots, 10\}$ and $E = \{e_1, e_2, e_3\}$. Define $x_{e_i} \tilde{\perp} y_{e_j}$ iff $x \mid y$ and $i \leq j$. If x_e is a soft point that $x = 1$ and $e = e_1$, then $x_e \tilde{\perp} y_{e'}$ for all $y_{e'} \in SP(\tilde{X})$.

EXAMPLE 2.4. Let X be the set of all σ-algebras on $\mathbb{R}$ and E be the set of all nonempty open subsets of $\mathbb{R}$. For x_e and $y_{e'}$ in $SP(\tilde{X})$ define $x_e \tilde{\perp} y_{e'}$ iff x is the generated σ-algebra by e and y is the generated σ-algebra by e' and the generated σ-algebra by e is a subset of the generated σ-algebra by e'. It is easy to show that the generated σ-algebra by $\mathbb{R}$ is an orthogonal element of $(\tilde{X}, \tilde{\perp})$.

DEFINITION 2.5. Let $(\tilde{X}, \tilde{\perp})$ be an O-soft set. A sequence $\{x^n_{e_n}\}_{n \in \mathbb{N}}$ is called a *strongly orthogonal soft sequence* (briefly, *SO-soft sequence*) if $(\forall n, k; x^n_{e_n} \,\tilde{\perp}\, x^{n+k}_{e_{n+k}})$ or $(\forall n, k; x^{n+k}_{e_{n+k}} \,\tilde{\perp}\, x^n_{e_n})$.

DEFINITION 2.6. The quarter $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$ is said to be an orthogonal soft metric space whenever $(\tilde{X}, \tilde{\perp})$ is an O-soft set and $(\tilde{X}, \tilde{d}, E)$ is a soft metric space.

DEFINITION 2.7. Let $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$ be an orthogonal soft metric space. $\tilde{X}$ is said to be strongly orthogonal complete (briefly, *SO-complete*) if every Cauchy SO-soft sequence in $SP(\tilde{X})$ is convergent to some soft point in $SP(\tilde{X})$.

Notice that every complete soft metric space is SO-complete. By means of the next example we show that the converse is not true.

EXAMPLE 2.8. Let X be a Banach space with norm $\|\cdot\|$ and $f : X \to X$ be a continuous mapping and Picard operator, that is, there exists $x^* \in X$ for which $\lim_{n\to\infty} f^n(y) = x^*$ for all $y \in X$. Suppose B is a non closed subset of X such that $x^* \in B$ and $E = Fix(f) = \{x \in X \mid f(x) = x\}$. Since B is not closed, then there exist $x' \in X$ and $\{x_n\} \subseteq B$ with $x' \notin B$ and $x_n \to x'$. Since f is a Picard operator and continuous mapping, then $f(x^*) = x^*$ and so $E \neq \emptyset$.

Define a soft metric $\tilde{d}$ on $\tilde{X}$ as $\tilde{d}(x_e, y_{e'}) = \|x - y\| + \|e - e'\|$. Note that $(\tilde{B}, \tilde{d}, E)$ is not a complete soft metric space. To see this, consider the soft sequence $\{x^n_{e_n}\}$ such that for all $n \in \mathbb{N}$, $x^n := x_n$ and $e_n := x^*$. It is easy to see that $\{x^n_{e_n}\}$ converges to soft point $x'_{x^*} \notin SP(\tilde{B})$. We define a relation $\tilde{\perp}$ on $\tilde{B}$ as follows

$$x_e \tilde{\perp}\, y_{e'} \iff x = \lim_{n\to\infty} f^n(y) \quad \text{and} \quad e = f(e').$$

Clearly, if $x^0_{e_0}$ is a soft point such that $x^0 = e_0 = x^*$, then this soft point is an orthogonal element of $(\tilde{X}, \tilde{\perp})$.

Now, we show that $(\tilde{B}, \tilde{d}, E, \tilde{\perp})$ is SO-complete. Take a Cauchy SO-soft sequence $\{x^n_{e_n}\}$ in $\tilde{B}$. Definition of orthogonality implies that for all $n, k \in \mathbb{N}$, $x^n = x^*$ and $e_n = e_{n+k}$. This show that $\{x^n_{e_n}\}$ converges to a soft point $x^*_{x^*} \in \tilde{B}$.

Definition 2.9. Consider the orthogonal soft metric space $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$. A soft mapping $(f, \varphi) : (\tilde{X}, \tilde{d}, E) \to (\tilde{X}, \tilde{d}, E)$ is called strongly orthogonal soft continuous ((briefly *SO-soft continuous*) at soft point $\tilde{x}_e$ in $SP(\tilde{X})$, if for each SO-soft sequence $\{\tilde{x}^n_{e_n}\}$ in $SP(\tilde{X})$, if $\tilde{x}^n_{e_n} \xrightarrow{\tilde{d}} \tilde{x}_e$, then $(f, \varphi)(\tilde{x}^n_{e_n}) \xrightarrow{\tilde{d}} (f, \varphi)(\tilde{x}_e)$. Also, (f, φ) is SO-soft continuous on $\tilde{X}$ if (f, φ) is SO-soft continuous at each $\tilde{x}_e \in SP(\tilde{X})$.

It is easy to see that every soft continuous mapping is SO-soft continuous. The following example shows that the converse is not true.

Example 2.10. Let $X = \mathbb{R}$ and $E = \mathbb{N}$ with the Euclidean metric. Take a soft metric $\tilde{d}(\tilde{x}_e, \tilde{y}_{e'}) = |x - y| + \frac{1}{2}|e - e'|$ on $\tilde{X}$ and define two mappings $f : X \to X$ and $\varphi : E \to E$ as $f(x) = \begin{cases} \frac{1}{x}, & x \in \mathbb{Q}^c \\ 0, & x \in \mathbb{Q} \end{cases}$ and $\varphi(e) = e^2$ for all $x \in X$ and $e \in E$.

Notice that (f, φ) is not continuous. Because if $x^n = \frac{\sqrt{2}}{n}$ and $e_n = 1$ for each $n \in \mathbb{N}$, then $\tilde{x}^n_{e_n} \xrightarrow{\tilde{d}} \tilde{0}_1$ while $(f, \varphi)(\tilde{x}^n_{e_n}) \overset{\tilde{d}}{\not\to} (f, \varphi)(\tilde{0}_1)$.

Now we consider the following relation on $SP(\tilde{X})$

$$\tilde{x}_e \;\tilde{\perp}\; \tilde{y}_{e'} \iff xy = 0 \;\text{ and }\; ee' \leq \max\{e, e'\}.$$

We can see that (f, φ) is SO-soft continuous. If $\{\tilde{x}^n_{e_n}\}$ is an SO-soft sequence in $SP(\tilde{X})$ which converges to $\tilde{x}_e \in SP(\tilde{X})$. Applying definition $\tilde{\perp}$ we obtain that for n large enough, $x^n = 0$ and $e_n = 1$. This implies that $(f, \varphi)(\tilde{x}^n_{e_n}) = f(\tilde{x})^n_{\varphi(e_n)} \xrightarrow{\tilde{d}} \tilde{0}_1 = (f, \varphi)(\tilde{x}_e)$

Definition 2.11. Consider orthogonal soft metric space $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$. A soft mapping $(f, \varphi) : \tilde{X} \to \tilde{X}$ is called $\tilde{\perp}$-preserving, if $(f, \varphi)(\tilde{x}_e) \;\tilde{\perp}\; (f, \varphi)(\tilde{y}_{e'})$ whenever $\tilde{x}_e \;\tilde{\perp}\; \tilde{y}_{e'}$ and $\tilde{x}_e, \tilde{y}_{e'} \in SP(\tilde{X})$.

Example 2.12. Let $X = \{x : [0, 1] \to \mathbb{R} : x \text{ is continuous}\}$ with supremum norm $\|x\| = \sup\{x(t) : t \in [0, 1]\}$ and $E = [0, 1]$. Consider the relation $\tilde{\perp}$ as follows:

$$\tilde{x}_e, \tilde{y}_{e'} \in SP(\tilde{X}), \quad \tilde{x}_e \tilde{\perp} \tilde{y}_{e'} \iff x(e)y(e') = \min\{x(t)y(t) : \; t \in [0, 1]\}.$$

Obviously $(\tilde{X}, \tilde{\perp})$ is an O-soft set. Define $f : X \to X$ and $\varphi : E \to E$ as $f(x) = 2x$ and $\varphi(e) = e$ for all $x \in X$ and $e \in E$. We see that $(f, \varphi)(\tilde{x}_e) \tilde{\perp} (f, \varphi)(\tilde{y}_{e'})$ whenever $\tilde{x}_e \tilde{\perp} \tilde{y}_{e'}$. This show that (f, φ) is a $\tilde{\perp}$-preserving mapping.

3. The main theorem

In this section, we prove our main theorem. First, we need the following definition that is the soft version of definition by Wardowski [10].

DEFINITION 3.1. Let Ω be the set of all functions $F : \mathbb{R}(E)^* \to \mathbb{R}(E)$ satisfying the following conditions:
(F1) F is strictly increasing, i.e. for all $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}(E)^*$, with $\tilde{\alpha} \tilde{<} \tilde{\beta}$, $F(\tilde{\alpha}) \tilde{<} F(\tilde{\beta})$;

(F2) For each soft real sequence $\{\tilde{a}_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}(E)^*$, $\lim_{n\to\infty} \tilde{a}_n = \tilde{0}$ if and only if $\lim_{n\to\infty} F(\tilde{a}_n) = -\tilde{\infty}$ (that is, for every $e \in E$, $\lim_{n\to\infty} F(\tilde{a}_n)(e) = -\infty$);

(F3) There exists a soft real point $\tilde{0} \tilde{<} \tilde{k} \tilde{<} \tilde{1}$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\tilde{\alpha}) = \tilde{0}$.

DEFINITION 3.2. Let $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$ be an orthogonal soft metric space and $F \in \Omega$. A mapping $(f, \varphi) : \tilde{X} \to \tilde{X}$ is said to be $\tilde{\perp}_F$-contraction if there exists $\tilde{\tau} \tilde{>} \tilde{0}$ such that for all $\tilde{x}_e, \tilde{y}_{e'} \in SP(\tilde{X})$ with $\tilde{x}_e \tilde{\perp}\, \tilde{y}_{e'}$ we have

$$\tilde{d}((f,\varphi)(\tilde{x}_e), (f,\varphi)(\tilde{y}_{e'})) \tilde{>} \tilde{0} \Longrightarrow \tilde{\tau} + F(\tilde{d}((f,\varphi)(\tilde{x}_e), (f,\varphi)(\tilde{y}_{e'})) \tilde{\leq} F(\tilde{d}(\tilde{x}_e, \tilde{y}_{e'})).$$

THEOREM 3.3. *Let $(\tilde{X}, \tilde{d}, E, \tilde{\perp})$ be an SO-complete orthogonal soft metric space (not necessarily complete) and $F \in \Omega$. Assume that $(f, \varphi) : \tilde{X} \to \tilde{X}$ is an SO-continuous, $\tilde{\perp}_F$-contraction and $\tilde{\perp}$-preserving soft mapping. Then there exists a unique soft point $\tilde{x}^*_e \in SP(\tilde{X})$ for which $(f, \varphi)(\tilde{x}^*_e) = \tilde{x}^*_e$. Also, the soft mapping (f, φ) is a Picard operator, that is, $\lim_{n\to\infty} \tilde{d}((f, \varphi)^n(\tilde{y}_{e'}), \tilde{x}^*_e) = \tilde{0}$ for all $\tilde{y}_{e'} \in SP(\tilde{X})$.*

Proof. It follows from definition of orthogonality that there exists an orthogonal soft element $\tilde{x}_{e_0}$ such that $\Big[\forall \tilde{y}_{e'} \in SP(\tilde{X}),\ \ \tilde{x}_{e_0} \tilde{\perp} \tilde{y}_{e'}\Big]$ or $\Big[\forall \tilde{y}_{e'} \in SP(\tilde{X}),\ \ \tilde{y}_{e'} \tilde{\perp} \tilde{x}_{e_0}\Big]$.

Construct the soft sequence $\{\tilde{x}^n_{e_n}\}$ as follows:

$$\tilde{x}^1_{e_1} := (f,\varphi)(\tilde{x}_{e_0}), \quad \tilde{x}^2_{e_2} := (f,\varphi)(\tilde{x}^1_{e_1}) = (f,\varphi)^2(\tilde{x}_{e_0}), \ldots,$$
$$\tilde{x}^{n+1}_{e_{n+1}} := (f,\varphi)(\tilde{x}^n_{e_n}) = (f,\varphi)^{n+1}(\tilde{x}_{e_0})$$

for all $n \in \mathbb{N}$. Since f is $\tilde{\perp}$-preserving, we see that

$$[\ \forall n, k \in \mathbb{N},\ \ \tilde{x}^k_{e_k} = (f,\varphi)^k(\tilde{x}_{e_0}) \tilde{\perp} (f,\varphi)^k(\tilde{x}^n_{e_n}) = \tilde{x}^{n+k}_{e_{n+k}}\ ]$$

or

$$[\ \forall n, k \in \mathbb{N},\ \ \tilde{x}^{n+k}_{e_{n+k}} = (f,\varphi)^k(\tilde{x}^n_{e_n}) \tilde{\perp} (f,\varphi)^k(\tilde{x}_{e_0}) = \tilde{x}^k_{e_k}\ ].$$

This shows that the sequence $\{\tilde{x}^n_{e_n}\}$ is an SO-soft sequence. Put $\tilde{\zeta}_n := \tilde{d}(\tilde{x}^n_{e_n}, \tilde{x}^{n+1}_{e_{n+1}})$, $n = 0, 1, 2, \ldots$. If for some $n_0 \in \mathbb{N}$, $\tilde{x}^{n_0+1}_{e_{n_0+1}} = \tilde{x}^{n_0}_{e_{n_0}}$, then by the construction of $\{\tilde{x}^n_{e_n}\}$, the proof is finished. Now, let for each $n \in \mathbb{N}$, $\tilde{x}^{n+1}_{e_{n+1}} \neq \tilde{x}^n_{e_n}$. Since (f, φ) is an $\tilde{\perp}_F$-contraction, then we have for all $n \in \mathbb{N}$,

$$F(\tilde{\zeta_n}) \tilde{\leq} F(\tilde{\zeta_{n-1}}) - \tilde{\tau} \tilde{\leq} F(\tilde{\zeta}_{n-2}) - \tilde{2}\tilde{\tau} \tilde{\leq} \cdots \tilde{\leq} F(\tilde{\zeta}_0) - n\tilde{\tau}. \tag{1}$$

From the above relation, we obtain $\lim_{n\to\infty} F(\tilde{\zeta_n}) = -\tilde{\infty}$ which, together with (F2), gives

$$\lim_{n\to\infty} \tilde{\zeta}_n = 0. \tag{2}$$

It follows from $(F3)$ that there exists $\tilde{0}\tilde{<}\ \tilde{k}\ \tilde{<}\tilde{1}$ such that

$$\lim_{n\to\infty}\tilde{\zeta}_n^k F(\tilde{\zeta}_n)=\tilde{0}. \tag{3}$$

Applying (1), for each $n\in\mathbb{N}$ we have

$$\tilde{\zeta}_n^k F(\tilde{\zeta}_n)-\tilde{\zeta}_n^k F(\tilde{\zeta}_0)\tilde{\leq}\tilde{\zeta}_n^k(F(\tilde{\zeta}_0)-n\tilde{\tau})-\tilde{\zeta}_n^k F(\tilde{\zeta}_0)=-\tilde{\zeta}_n^k n\tilde{\tau}.$$

In the above, letting $n\to\infty$ by using (2) and (3) we have $\lim_{n\to\infty} n\tilde{\zeta}_n^k=0$. This shows that there exists $n_1\in\mathbb{N}$ such that $n\tilde{\zeta}_n^k\tilde{<}\tilde{1}$ for all $n>n_1$.

Thus, we have for all $n>n_1$

$$\tilde{\zeta}_n\tilde{\leq}\frac{\tilde{1}}{n^{\frac{1}{k}}}. \tag{4}$$

Now in order to show that the SO-soft sequence $\{\tilde{x}^n_{e_n}\}_{n\in\mathbb{N}}$ is Cauchy, consider $m,n\in\mathbb{N}$ such that $m>n\geq n_1$. From the definition of the soft metric and from (4) we get

$$\begin{aligned}\tilde{d}(\tilde{x}^n_{e_n},\tilde{x}^m_{e_m})&\tilde{\leq}\tilde{d}(\tilde{x}^n_{e_n},\tilde{x}^{n+1}_{e_{n+1}})+\tilde{d}(\tilde{x}^{n+1}_{e_{n+1}},\tilde{x}^{n+2}_{e_{n+2}})+\cdots+\tilde{d}(\tilde{x}^{m-1}_{e_{m-1}},\tilde{x}^m_{e_m})\\&=\tilde{\zeta}_n+\tilde{\zeta}_{n+1}+\cdots+\tilde{\zeta}_{m-1}\tilde{<}\sum_{i=n}^{\infty}\tilde{\zeta}_i\tilde{\leq}\sum_{i=n}^{\infty}\frac{\tilde{1}}{i^{\frac{1}{k}}}.\end{aligned}$$

the convergence of the series $\sum_{i=1}^{\infty}\frac{\tilde{1}}{i^{1/k}}$ implies that $\{\tilde{x}^n_{e_n}\}_{n\in\mathbb{N}}$ is a Cauchy SO-soft sequence.

Since $\tilde{X}$ is SO-complete, then there exists a soft point $\tilde{x}^*_e\in SP(\tilde{X})$ for which $\tilde{x}^n_{e_n}\xrightarrow{\tilde{d}}\tilde{x}^*_e$. The SO-continuity of (f,φ) implies that $(f,\varphi)(\tilde{x}^*_e)=\tilde{x}^*_e$. Therefore, $\tilde{x}^*_e$ is a soft fixed point of (f,φ).

To prove the uniqueness of the soft fixed point, let $\tilde{y}_{e'}$ be a soft fixed point of (f,φ). Then we have $(f,\varphi)^n(\tilde{y}_{e'})=\tilde{y}_{e'}$ for all $n\in\mathbb{N}$. By our choice of $\tilde{x}_{e_0}$ in the first part of the proof, we obtain that $[\tilde{x}_{e_0}\tilde{\perp}\tilde{y}_{e'}]$ or $[\tilde{y}_{e'}\tilde{\perp}\tilde{x}_{e_0}]$. Applying $\tilde{\perp}$-preserving of (f,φ), we see that $[(f,\varphi)^n(\tilde{x}_{e_0})\tilde{\perp}(f,\varphi)^n(\tilde{y}_{e'})]$ or $[(f,\varphi)^n(\tilde{y}_{e'})\tilde{\perp}(f,\varphi)^n(\tilde{x}_{e_0})]$, for all $n\in\mathbb{N}$. On the other hand, since (f,φ) is a $\perp_F$-contraction, then for all $n\in\mathbb{N}$,

$$\begin{aligned}F(\tilde{d}((f,\varphi)^n(\tilde{x}_{e_0}),(f,\varphi)^n(\tilde{y}_{e'})))&\leq F(\tilde{d}((f,\varphi)^{n-1}(\tilde{x}_{e_0}),(f,\varphi)^{n-1}(\tilde{y}_{e'})))-\tilde{\tau}\\&\tilde{\leq}\cdots\tilde{\leq}F(\tilde{d}(\tilde{x}_{e_0},\tilde{y}_{e'}))-n\tilde{\tau}.\end{aligned}$$

Letting $n\to\infty$, condition (F2) implies that $\tilde{x}^n_{e_n}\xrightarrow{\tilde{d}}\tilde{y}_{e'}$. This shows that $\tilde{x}^*_e=\tilde{y}_{e'}$.

Finally, in order to show that (f,φ) is a Picard operator, take an arbitrary soft point $\tilde{y}_{e'}$ in $SP(\tilde{X})$. Similarly, $[\tilde{x}_{e_0}\tilde{\perp}\tilde{y}_{e'}]$ or $[\tilde{y}_{e'}\tilde{\perp}\ \tilde{x}_{e_0}]$. Now, since (f,φ) is $\tilde{\perp}$-preserving, then $[(f,\varphi)^n(\tilde{x}_{e_0})\tilde{\perp}(f,\varphi)^n(\tilde{y}_{e'})]$ or $[(f,\varphi)^n(\tilde{y}_{e'})\tilde{\perp}(f,\varphi)^n(\tilde{x}_{e_0})]$, for all $n\in\mathbb{N}$. Hence, for all $n\in\mathbb{N}$, we get

$$F(\tilde{d}((f,\varphi)^n(\tilde{x}_{e_0}),(f,\varphi)^n(\tilde{y}_{e'})))\tilde{\leq}F(\tilde{d}(\tilde{x}_{e_0},\tilde{y}_{e'}))-n\tilde{\tau}.$$

Letting $n\to\infty$, by triangle inequality we obtain $(f,\varphi)^n(\tilde{y}_{e'})\xrightarrow{\tilde{d}}\tilde{x}^*_e$. □

Theorem 3.3 is an extension of [11, Theorem 4.8].

Corollary 3.4 ([11]). *Let $(\tilde{X},\tilde{d},E)$ be a soft complete metric space. Let $(f,\varphi):(\tilde{X},\tilde{d},E)\to(\tilde{X},\tilde{d},E)$ be a soft contractive mapping, that is, let there be a soft real number $\tilde{0}\tilde{<}\ \tilde{\lambda}\ \tilde{<}\tilde{1}$ such that for any two soft points $\tilde{x}_e,\tilde{y}_{e'}\in SP(\tilde{X})$ we have*

$\tilde{d}((f,\varphi)(\tilde{x}_{e_0}),(f,\varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{\lambda}\ \tilde{d}(\tilde{x}_{e_0},\tilde{y}_{e'})$. *Then* (f,φ) *has a unique soft fixed point in* $SP(\tilde{X})$.

Proof. Define the relation $\tilde{\perp}$ on $\tilde{X}$ as follows:

$$\tilde{x}_e \tilde{\perp}\ \tilde{y}_{e'} \iff \tilde{d}((f,\varphi)(\tilde{x}_{e_0}),(f,\varphi)(\tilde{y}_{e'})) \tilde{\leq} \tilde{d}(\tilde{x}_{e_0},\tilde{y}_{e'}).$$

Since (f,φ) is a soft contractive mapping, thus every two soft points of $SP(\tilde{X})$ are $\tilde{\perp}$-comparable. This shows that $(\tilde{X},\tilde{\perp})$ is an O- soft set. The completeness of $\tilde{X}$ implies the SO-completeness of $\tilde{X}$, also the continuity of (f,φ) implies the SO-continuity of (f,φ). It follows from definition of $\tilde{\perp}$ that (f,φ) is $\tilde{\perp}$-preserving.

Now, define $F : \mathbb{R}(E)^* \to \mathbb{R}(E)$ by $F(\tilde{\alpha}) = \tilde{\ln}(\alpha)$ for all $\tilde{\alpha} \in \mathbb{R}(E)^*$ and put $\tilde{\tau} = -\tilde{\ln}(\lambda)$, then the soft mapping (f,φ) is an $\tilde{\perp}_F$-contraction. The existence and uniqueness of soft fixed point is implied by Theorem 3.3. □

Now, we show that our theorem is a real extension of the main theorem of [11].

EXAMPLE 3.5. Let $X = [0, 10)$ and $E = [1,\infty)$ with Euclidean metric. Let $\tilde{d}(\tilde{x}_e,\tilde{y}_{e'}) = |x-y| + |e-e'|$ be a soft metric on $\tilde{X}$. Consider the orthogonal $\tilde{\perp}$ as follows:

$$\tilde{x}_e \tilde{\perp} \tilde{y}_{e'} \iff xy \leq \min\{x,y\} \text{ and } ee' \in \{e,e'\}.$$

It is easy to see that $\tilde{x}_{e_0}$ where $x = 0$ and $e_0 = 1$ is an orthogonal soft element for $\tilde{X}$. We define two mappings $f : X \to X$ and $\varphi : E \to E$ as

$$f(x) = \begin{cases} \frac{x}{2}, & x \leq 1, \\ \frac{10+x}{2}, & x > 1 \end{cases} \quad \text{and} \quad \varphi(e) = \frac{1+e}{2}$$

for all $x \in X$ and $e \in E$.

We have the following cases.

Case 1) $(\tilde{X},\tilde{d},E,\tilde{\perp})$ is SO-complete.

In fact, if $\{\tilde{x}^n_{e_n}\}$ is a Cauchy SO-soft sequence in $SP(\tilde{X})$, then for every large enough $n \in \mathbb{N}$, $x^n \leq 1$ and $e_n = 1$. Since $[0,1]$ with Euclidean metric is a complete metric space, then there exists a subsequence of $\{\tilde{x}^n_{e_n}\}$ that converges to soft point $\tilde{x}_e$ for which $x \leq 1$ and $e = 1$. Since every Cauchy sequence which has a convergent subsequence is convergent, so the SO-sequence $\{\tilde{x}^n_{e_n}\}$ is convergent to soft point $\tilde{x}_e$ in $\tilde{X}$. Notice that $\tilde{X}$ is not a complete soft metric space.

Case 2) (f,φ) is SO-soft continuous.

Take an SO-sequence $\{\tilde{x}^n_{e_n}\}$ that converges to soft point $\tilde{x}_e$. By Case 1, for every enough large $n \in \mathbb{N}$, $x^n \leq 1$ and $e_n = 1$. Also, $x \leq 1$ and $e = 1$. Thus, by definition of f and φ we have $(f,\varphi)(\tilde{x}^n_{e_n}) \xrightarrow{\tilde{d}} (f,\varphi)(\tilde{x}_e)$. This implies that (f,φ) is SO-soft continuous while it is not soft continuous.

Case 3) (f,φ) is $\tilde{\perp}$-preserving.

Let $\tilde{x}_e \tilde{\perp}\ \tilde{y}_{e'}$, The definition of $\tilde{\perp}$ shows that $x \leq 1$ and $y \leq 1$ and $e = 1$ or $e' = 1$. Hence, $f(x) \leq 1$ and $f(y) \leq 1$ and $\varphi(e) = 1$ or $\varphi(e') = 1$. This implies that $(f,\varphi)(\tilde{x}_e)\tilde{\perp}(f,\varphi)(\tilde{y}_{e'})$.

Case 4) (f, φ) is a $\tilde{\perp}_F$-contraction.

For each two $\tilde{\perp}$-comparable soft points $\tilde{x}_e, \tilde{y}_{e'}$, we can see that

$$\tilde{d}((f,\varphi)(\tilde{x}_e),(f,\varphi)(\tilde{y}_{e'}))\tilde{\leq}\,\frac{\tilde{1}}{2}\,\tilde{d}(\tilde{x}_e,\tilde{y}_{e'}).$$

By the above cases, we conclude the existence and uniqueness of soft fixed point by Theorem 3.3.

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