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# $I$-SECOND SUBMODULES OF A MODULE

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**Abstract**. Let $R$ be a commutative ring with identity, $I$ an ideal of $R$, and $M$ be an $R$-module. In this paper, we will introduce the concept of $I$-second submodules of $M$ as a generalization of second submodules of $M$ and obtain some related results.

## 1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [9]. A non-zero submodule $N$ of $M$ is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [14].

A proper ideal $P$ of $R$ is *weakly prime* if for $a, b \in R$ with $0 \neq ab \in P$, either $a \in P$ or $b \in P$. Weakly prime ideals were studied in some detail in [3]. A proper submodule $N$ of $M$ is called *weakly prime* if for $r \in R$ and $m \in M$ with $0 \neq rm \in N$, either $m \in N$ or $r \in (N :_R M)$ [10].

Let $I$ be an ideal of $R$. In [1], the author gave a generalization of weakly prime ideals and said that such ideals I-prime ideals. A proper ideal $P$ of $R$ is called *$I$-prime ideal* if for $a, b \in R$, $ab \in P \setminus IP$, implies $a \in P$ or $b \in P$ [1]. Akray and Hussein in [2] extended $I$-prime ideals to $I$-prime submodules. A proper submodule $P$ of $M$ is called an *$I$-prime submodule* of $M$ if for $r \in R$, $m \in M$, $rm \in P \setminus IP$ implies that $m \in P$ or $r \in (P :_R M)$ [2].

The main purpose of this paper is to introduce and study the notion of $I$-second submodules of an $R$-module $M$ as a dual notion of $I$-prime submodules, where $I$ is an ideal of $R$ and investigate some properties of this class of modules.

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## 2. Main results

A proper submodule $N$ of an $R$-module $M$ is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [11].

We use the following basic fact without further comment.

REMARK 2.1. Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

LEMMA 2.2. *[4, 2.10] For a submodule $S$ of an $R$-module $M$ the following statements are equivalent:*
*(a) $S$ is a second submodule of $M$;*

*(b) $S \neq 0$ and $rS \subseteq K$, where $r \in R$ and $K$ is a submodule of $M$, implies that either $rS = 0$ or $S \subseteq K$;*

*(c) $S \neq 0$ and $rS \subseteq L$, where $r \in R$ and $L$ is a completely irreducible submodule of $M$, implies that either $rS = 0$ or $S \subseteq K$.*

THEOREM 2.3. *Let $I$ be an ideal of $R$. For a non-zero submodule $S$ of an $R$-module $M$ the following statements are equivalent:*
*(a) For each $r \in R$, a submodule $K$ of $M$, $r \in (K :_R S) \setminus (K :_R (S :_M I))$ implies that $S \subseteq K$ or $r \in Ann_R(S)$;*

*(b) For each $r \notin (rS :_R (S :_M I))$, we have $rS = S$ or $rS = 0$;*

*(c) $(K :_R S) = Ann_R(S) \cup (K :_R (S :_M I))$, for any submodule $K$ of $M$ with $S \not\subseteq K$;*

*(d) $(K :_R S) = Ann_R(S)$ or $(K :_R S) = (K :_R (S :_M I))$, for any submodule $K$ of $M$ with $S \not\subseteq K$.*

*Proof.* (a) $\Rightarrow$ (b) Let $r \notin (rS :_R (S :_M I))$. Then as $rS \subseteq rS$, we have $S \subseteq rS$ or $rS = 0$ by part (a). Thus $rS = S$ or $rS = 0$.

(b) $\Rightarrow$ (a) Let $r \in R$ and $K$ be a submodule of $M$ such that $r \in (K :_R S) \setminus (K :_R (S :_M I))$. Then if $r \in (rS :_R (S :_M I))$, then $r \in (K :_R (S :_M I))$ which is a contradiction. Thus $r \notin (rS :_R (S :_M I))$. Now by part (b), $rS = S$ or $rS = 0$. So $S \subseteq K$ or $rS = 0$, as needed.

(a) $\Rightarrow$ (c) Let $r \in (K :_R S)$ and $S \not\subseteq K$. If $r \notin (K :_R (S :_M I))$, then $r \in Ann_R(S)$ by part (a). Hence, $(K :_R S) \subseteq Ann_R(S)$. If $r \in (K :_R (S :_M I))$, then $(K :_R S) \subseteq (K :_R (S :_M I))$. Therefore, $(K :_R S) \subseteq Ann_R(S) \cup (K :_R (S :_M I))$. The other inclusion always holds.

(c) $\Rightarrow$ (d) This follows from the fact that if an ideal is a union of two ideals, then it is equal to one of them.

(d) $\Rightarrow$ (a) This is clear. □

Definition 2.4. Let $I$ be an ideal of $R$. We say that a non-zero submodule $S$ of an $R$-module $M$ is an *$I$-second submodule* of $M$ if satisfies the equivalent conditions of Theorem 2.3. This can be regarded as a dual notion of $I$-prime submodule. In case, $I = 0$ we say that $S$ is a *weak second submodule* of $M$.

Let $I$ be an ideal of $R$. Clearly every second submodule is an $I$-second submodule. But the converse is not true in general as we see in the following example.

Example 2.5. (a) If $I = 0$, then every module is an $I$-second submodule of itself but every module is not a second module. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is weak second which is not second.

(b) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. Take $I = 4\mathbb{Z}$ as an ideal of $\mathbb{Z}$ and $S = \bar{3}\mathbb{Z}_{12}$ as a submodule of $\mathbb{Z}_{12}$. Then $S$ is an $I$-second submodule of $\mathbb{Z}_{12}$. But $S$ is not a second submodule.

Example 2.6. Let $I$ be an ideal of $R$ and $S$ a non-zero submodule of an $R$-module $M$. If for each $r \in R$, a completely irreducible submodule $L$ of $M$, $r \in (L :_R S) \setminus (L :_R (S :_M I))$ implies that $S \subseteq L$ or $r \in Ann_R(S)$ we cannot conclude that (similar to Lemma 2.2 (c) $\Rightarrow$ (a)), $S$ is an $I$-second submodule of $M$. For example, consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $2\mathbb{Z}$ satisfies the mentioned condition above but it is not an $I$-second submodule of $\mathbb{Z}$ for ideal $I = 4\mathbb{Z}$ of $\mathbb{Z}$.

Let $I$ be an ideal of $R$ and $M$ be an $R$-module. If $I = R$, then every submodule is an $I$-second submodule. So in the rest of this paper we can assume that $I \neq R$.

Theorem 2.7. *Let $M$ be an $R$-module. Then we have the following.*
*(a) Let $I, J$ be ideals of $R$ such that $I \subseteq J$. If $S$ is an $I$-second submodule of $M$, then $S$ is an $J$-second submodule of $M$. In particular, every weak second submodule is an $I$-second submodule for each ideal $I$ of $R$.*

*(b) If $S$ an $I$-second submodule of $M$ which is not second, then $Ann_R(S)(S :_M I) \subseteq S$.*

*Proof.* (a) The result follows from the fact that $I \subseteq J$ implies that $(rS :_R S) \setminus (rS :_R (S :_M J)) \subseteq (rS :_R S) \setminus (rS :_R (S :_M I))$, for each $r \in R$.

(b) Assume on the contrary that $Ann_R(S)(S :_M I) \not\subseteq S$. We show that $S$ is second. Let $rS \subseteq K$ for some $r \in R$ and a submodule $K$ of $M$. If $r \notin (K :_R (S :_M I))$, then $S$ is a $I$-second submodule implies that $S \subseteq K$ or $r \in Ann_R(S)$ as needed. So assume that $r \in (K :_R (S :_M I))$. First, suppose that $r(S :_M I) \not\subseteq S$. Then there exists a submodule $L$ of $M$ such that $S \subseteq L$ but $r(S :_M I) \not\subseteq L$. Then $r \in (K \cap L :_R S) \setminus (K \cap L :_R (S :_M I))$. So $S \subseteq K \cap L$ or $r \in Ann_R(S)$ and hence $S \subseteq K$ or $r \in Ann_R(S)$. So we can assume that $r(S :_M I) \subseteq S$. On the other hand, if $Ann_R(S)(S :_M I) \not\subseteq K$, then there exists $t \in Ann_R(S)$ such that $t \notin (K :_R (S :_M I))$. Then $t + r \in (K :_R S) \setminus (K :_R (S :_M I))$. Thus $S \subseteq K$ or $t + r \in Ann_R(S)$ and hence $S \subseteq K$ or $r \in Ann_R(S)$. So we can assume that $Ann_R(S)(S :_M I) \subseteq K$. Since $Ann_R(S)(S :_M I) \not\subseteq S$, there exist $t \in Ann_R(S)$, a submodule $T$ of $M$ such that $S \subseteq T$ and $t(S :_M I) \not\subseteq T$. Now we have $r + t \in (K \cap T :_R S) \setminus (K \cap T :_R (S :_M I))$. So $S$ is an $I$-second submodule gives $S \subseteq K \cap T$ or $r + t \in Ann_R(S)$. Hence $S \subseteq K$ or $r \in Ann_R(S)$, as requested. $\square$

An $R$-module $M$ is said to be a *comultiplication module* if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$ [5].

THEOREM 2.8. *Let $I$ be an ideal of $R$, $M$ an $R$-module, and $S$ be a submodule of $M$. Then we have the following.*
*(a) If $S$ is an $I$-second submodule of $M$ such that $Ann_R((S :_M I)) \subseteq IAnn_R(S)$, then $Ann_R(S)$ is an $I$-prime ideal of $R$.*

*(b) If $M$ is a comultiplication $R$-module and $Ann_R(S)$ is an $I$-prime ideal of $R$, then $S$ is an $I$-second submodule of $M$.*

*Proof.* (a) Let $ab \in Ann_R(S) \setminus IAnn_R(S)$ for some $a, b \in R$. Then $aS \subseteq (0 :_M b)$. As $ab \notin IAnn_R(S)$ and $Ann_R((S :_M I)) \subseteq IAnn_R(S)$, we have $ab \notin Ann_R((S :_M I))$. This implies that $a \notin ((0 :_M b) :_R (S :_M I))$. Thus $a \in Ann_R(S)$ or $S \subseteq (0 :_M b)$. Hence $a \in Ann_R(S)$ or $b \in Ann_R(S)$, as needed.

(b) Let $r \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r \in R$ and submodule $K$ of $M$. As $M$ is a comultiplication $R$-module, there exists an ideal $J$ of $R$ such that $K = (0 :_M J)$. Thus $rJ \subseteq Ann_R(S)$. Since $r \notin (K :_R (S :_M I))$, we have $Jr(S :_M I) \neq 0$. This implies that $Jr \not\subseteq Ann_R((S :_M I)$. Since always $IAnn_R(S \subseteq Ann_R((S :_M I))$, we have $rJ \not\subseteq IAnn_R(S)$. Thus by assumption, $r \in Ann_R(S)$ or $J \subseteq Ann_R(S)$ and so $S \subseteq (0 :_M J) = K$. □

The next corollary follows from Theorem 2.11, by setting $I = 0$.

COROLLARY 2.9. *Let $M$ an $R$-module and $S$ be a submodule of $M$. Then we have the following.*
*(a) If $M$ is faithful and $S$ is a weak second submodule of $M$, then $Ann_R(S)$ is a weakly prime ideal of $R$.*

*(b) If $M$ is a comultiplication $R$-module and $Ann_R(S)$ is a weakly prime ideal of $R$, then $S$ is a weak second submodule of $M$.*

The following example shows that the condition "$M$ is a comultiplication $R$-module" in Corollary 2.9 (b) cannot be omitted.

EXAMPLE 2.10. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, and $S = 2\mathbb{Z} \oplus 0$. Then $M$ is not a comultiplication $R$-module. Clearly, $Ann_R(S) = 0$ is a weakly prime ideal of $R$. But $S$ is not a weak second submodule of $M$.

PROPOSITION 2.11. *Let $I$ be an ideal of $R$ and $M$ be an $R$-module. Let $N$ be an $I$-second submodule of $M$. Then we have the following statements.*
*(a) If $K$ is a submodule of $M$ with $K \subset N$, then $N/K$ is an $I$-second submodule of $M/K$.*

*(b) Let $N$ be a finitely generated submodule of $M$ and $S$ be a multiplicatively closed subset of $R$ with $Ann_R(N) \cap S = \emptyset$. Then $S^{-1}N$ is an $S^{-1}I$-second submodule of $S^{-1}M$.*

*Proof.* (a) This follows from the fact that $r \notin (r(S/K) :_R (S/K :_{M/K} I))$ implies that $r \notin (rS :_R (S :_M I))$.

(b) As $Ann_R(N) \cap S = \emptyset$ and $N$ is finitely generated, $S^{-1}N \neq 0$ by using [8, P. 43, Exe. 1]. Now the claim follows from the fact that $r/s \notin ((r/s)S^{-1}N :_{S^{-1}R} (S^{-1}N :_{S^{-1}M} S^{-1}I)$ implies that $r \notin (rN :_R (N :_M I))$. □

THEOREM 2.12. *Let $M$ be a primary $R$-module. Then every proper weak second submodule of $M$ is a primary submodule of $M$.*

*Proof.* Let $N$ be a proper weak second submodule of $M$ and $rx \in N$ for some $r \in R$ and $x \in M$. If $r \notin (rN :_R M)$, then $rN = 0$ or $rN = N$ since $N$ is weak second. In the first case, $r^2x \in rN = 0$. Now as $M$ is primary, $x = 0$ or $r \in \sqrt{Ann_R(M)}$. This implies that $x \in N$ or $r \in \sqrt{Ann_R(M/N)}$, as needed. If $rN = N$, then $rx = rn$ for some $n \in N$. This implies that $x = n \in N$ or $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(M/N)}$ since $M$ is primary. Now suppose that $r \in (rN :_R M)$. Then $rx \in rM \subseteq rN$. Therefore, similarly to the pervious case we are done. □

PROPOSITION 2.13. *Let $I$ be an ideal of $R$, $M$ and $\acute{M}$ be $R$-modules, and let $f : M \to \acute{M}$ be an $R$-monomorphism. If $\acute{N}$ is an $I$-second submodule of $\acute{M}$ such that $\acute{N} \subseteq Im(f)$, then $f^{-1}(\acute{N})$ is an $I$-second submodule of $M$.*

*Proof.* As $\acute{N} \neq 0$ and $\acute{N} \subseteq Im(f)$, we have $f^{-1}(\acute{N}) \neq 0$. Let $r \notin (rf^{-1}(\acute{N}) :_R (f^{-1}(\acute{N}) :_M I))$; then one can see that $r \notin (r\acute{N} :_R (\acute{N} :_{\acute{M}} I))$ by using assumptions. Thus $r\acute{N} = 0$ or $r\acute{N} = \acute{N}$. This implies that $rf^{-1}(\acute{N}) = 0$ or $rf^{-1}(\acute{N}) = f^{-1}(\acute{N})$ as requested. □

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is of the form $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$.

LEMMA 2.14. *Let $R = R_1 \times R_2$ be a decomposable ring, $I = I_1 \times I_2$ an ideal of $R$, and $M = M_1 \times M_2$ be an $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. If $(0 :_{M_2} I_2) \neq 0$ and $S_1$ is a non-zero $R_1$-submodule of $M_1$, then the following statements are equivalent:*
*(a) $S_1$ is a second $R_1$-submodule of $M_1$;*

*(b) $S_1 \times 0$ is a second $R$-submodule of $M = M_1 \times M_2$;*

*(c) $S_1 \times 0$ is an $I$-second $R$-submodule of $M = M_1 \times M_2$.*

*Proof.* (a) $\Rightarrow$ (b) follows from [7, 2.23] and (b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) Let $r \in R_1$. Then $(0 :_{M_2} I_2) \neq 0$ implies that $(r_1, 1)(S_1 \times 0 :_{M_1 \times M_2} I) \not\subseteq (r_1, 1)(S_1 \times 0)$. Thus by part (c), $(r_1, 1)(S_1 \times 0) = (S_1 \times 0)$ or $(r_1, 1)(S_1 \times 0) = 0 \times 0$. Hence $r_1S_1 = S_1$ or $r_1S_1 = 0$, as needed. □

THEOREM 2.15. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Let $I$ be an ideal of $R$ such that $(0 :_{M_1} I_1) \neq 0$ and $(0 :_{M_2} I_2) \neq 0$. If $S = S_1 \times S_2$ is an $I$-second $R$-submodule of $M = M_1 \times M_2$, then either $(S :_M I) = S$ or $S$ is a second submodule of $M$.*

*Proof.* Let $(S :_M I) \neq S$. Then either $(S_1 :_{M_1} I_1) \neq S_1$ or $(S_2 :_{M_2} I_2) \neq S_2$. Suppose that $(S_2 :_{M_2} I_2) \neq S_2$. Then $(S_2 :_{R_2} (S_2 :_{M_2} I_2)) \neq R$. Hence $1 \notin (S_2 :_{R_2} (S_2 :_{M_2} I_2))$. If $S_1 = 0$, then the result follows from Lemma 2.14. So suppose that $S_1 \neq 0$. As $(0,1) \notin ((0,1)(S_1 \times S_2) :_R (S_1 \times S_2 :_M I))$, we have $(0,1)(S_1 \times S_2) = S_1 \times S_2$ or $(0,1)(S_1 \times S_2) = 0 \times S_2$ since $S = S_1 \times S_2$ is an $I$-second $R$-submodule of $M$. Therefore, $S_2 = 0$. Hence by Lemma 2.14, $S = S_1 \times 0$ is a second $R$-submodule of $M$. □

Example 2.16. Let $R_1 = R_2 = M_1 = M_2 = S_1 = \mathbb{Z}_6$. Then by Theorem 2.15, $S_1 \times 0$ is not a weak second submodule of $M_1 \times M_2$.

Theorem 2.17. *Let $I$ be an ideal of $R$, $M_1$, $M_2$ be $R$-modules, and let $N$ be a submodule of $M_1$. Then $N \oplus 0$ is an $I$-second submodule of $M_1 \oplus M_2$ if and only if $N$ is an $I$-second submodule of $M_1$ and for $r \in (rN :_R (N :_{M_1} I))$, $rN \neq 0$, and $rN \neq N$, we have $r \in Ann_R((0 :_{M_2} I))$.*

*Proof.* $(\Rightarrow)$ Let $r \notin (rN :_R (N :_{M_1} I))$. Then $r \notin (r(N \oplus 0) :_R (N \oplus 0 :_M I))$. Since $N \oplus 0$ is an $I$-second submodule, either $r(N \oplus 0) = N \oplus 0$ or $r(N \oplus 0) = 0 \oplus 0$. Thus either $rN = N$ or $rN = 0$, so $N$ is $I$-second. Now, let $r \in (rN :_R (N :_{M_1} I))$, $rN \neq 0$, and $rN \neq N$. Assume on the contrary that $r \notin Ann_R((0 :_{M_2} I))$. Then there exists $x_2 \in M_2$ such that $Ix_2 = 0$ and $rx_2 \neq 0$. This implies that $r(0, x_2) \in r(N \oplus 0 :_M I) \setminus r(N \oplus 0)$. So since $N \oplus 0$ is an $I$-second submodule, either $r(N \oplus 0) = N \oplus 0$ or $r(N \oplus 0) = 0 \oplus 0$. Thus either $rN = N$ or $rN = 0$, which is a contradiction. Therefore, $r \in Ann_R((0 :_{M_2} I))$.

$(\Leftarrow)$ Let $r \notin (r(N \oplus 0) :_R (N \oplus 0 :_M I))$. Then if $rN = N$ or $rN = 0$, the result is clear. So suppose that $rN \neq N$ and $rN \neq 0$. We show that $r \notin (rN :_R (N :_{M_1} I))$ and this contradiction proves the result because $N$ is an $I$-second submodule of $M_1$. Assume on the contrary that $r \in (rN :_R (N :_{M_1} I))$. Then by assumption, $r \in Ann_R((0 :_{M_2} I))$. This implies that if $(x_1, x_2) \in N \oplus (0 :_M I)$, then $r(x_1, x_2) \in r(N \oplus 0)$. Therefore, $r \in (r(N \oplus 0) :_R (N \oplus 0 :_M I))$, which is a desired contradiction. □

A non-zero $R$-module $M$ is said to be *secondary* if for each $a \in R$ the endomorphism of $M$ given by multiplication by $a$ is either surjective or nilpotent [13].

Corollary 2.18. *Let $I$ and $P$ be ideals of $R$, $M_1$, $M_2$ be $R$-modules, and let $N$ be a submodule of $M_1$. Let $S_i$ $(1 \leq i \leq n)$ be $P$-secondary submodules of $M_1$ with $\sum_{i=1}^n S_i = (N :_{M_1} I)$. If $N$ is an $I$-second submodule of $M_1$ and $P \subseteq Ann_R((0 :_{M_2} I))$, then $N \oplus 0$ is an $I$-second submodule of $M_1 \oplus M_2$.*

*Proof.* Let $r \in (rN :_R (N :_{M_1} I))$, $rN \neq 0$, and $rN \neq N$. Then we will prove that $r \in Ann_R((0 :_{M_2} I))$ and hence the result is obtained by Theorem 2.17. Assume on the contrary that $r \notin Ann_R((0 :_{M_2} I))$. Hence $r \notin P$. On the other hand, $r(\sum_{i=1}^n S_i) = r(N :_{M_1} I) \subseteq rN$. But $\sum_{i=1}^n S_i$ is a $P$-secondary submodule by [13, 2.1], so either $r(\sum_{i=1}^n S_i) = \sum_{i=1}^n S_i$ or $r \in P$. This implies that $rN = N$ or $r \in P$, which is a contradiction. Thus $r \in Ann_R((0 :_{M_2} I))$. □

Theorem 2.19. *Let $I$ be an ideal of $R$ and $M$ be an $R$-module. Then we have the following.*

*(a) If $\bigcap_{n=1}^{\infty} I^n M = 0$ and every proper submodule of $M$ is $I$-prime, then every non-zero submodule of $M$ is $I$-second.*

*(b) If $\sum_{n=1}^{\infty}(0 :_M I^n) = M$ and every non-zero submodule of $M$ is $I$-second, then every proper submodule of $M$ is $I$-prime.*

*Proof.* (a) Let $S$ be a non-zero submodule of $M$, $r \in (K :_R S) \setminus (K :_R (S :_M I))$ for some $r \in R$ and a submodule $K$ of $M$ and $rS \neq 0$. If $rS \not\subseteq IK$, then as $K$ is $I$-prime, we have $rM \subseteq K$ or $S \subseteq K$. If $rM \subseteq K$, then $r(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq K$ and we are done. Now suppose that $rS \subseteq IK$. As $rS \neq 0$ and $\bigcap_{n=1}^{\infty} I^n K = 0$, there exists a positive integer $t$ such that $rS \not\subseteq I^t K$. Therefore, there is a positive integer $h$ such that $rS \subseteq I^{h-1}K$ but $rS \not\subseteq I^h K$, where $2 \leq h \leq t$. Thus since $I^{h-1}K$ is $I$-prime, $S \subseteq I^{h-1}K$ or $rM \subseteq I^{h-1}K$. If $rM \subseteq I^{h-1}K$, then $r(S :_M I) \subseteq K$ which is a contradiction. So $S \subseteq I^{h-1} \subseteq K$ as needed.

(b) Let $P$ be a proper submodule of $M$, $rK \subseteq P \setminus IP$ for some $r \in R$ and a submodule $K$ of $M$ and $rM \not\subseteq P$. If $r(K :_M I) \not\subseteq P$, then as $K$ is $I$-second, we have $rK = 0$ or $K \subseteq P$. If $rK = 0$, then $rK \subseteq IP$ which is a contradiction. So $K \subseteq P$ and we are done. Now suppose that $r(K :_M I) \subseteq P$. As $rM \not\subseteq P$ and $\sum_{n=1}^{\infty}(K :_M I^n) = M$, there exists a positive integer $t$ such that $r(K :_M I^t) \not\subseteq P$. Therefore, there is a positive integer $h$ such that $r(K :_M I^{h-1}) \subseteq P$ but $r(K :_M I^h) \not\subseteq P$, where $2 \leq h \leq t$. Thus since $(K :_M I^{h-1})$ is $I$-second, $(K :_M I^{h-1}) \subseteq P$ or $r(K :_M I^{h-1}) = 0$. If $r(K :_M I^{h-1}) = 0$, then $0 = rK \subseteq IP$ which is a contradiction. So $K \subseteq (K :_M I^{h-1}) \subseteq P$ as needed. □

By setting $I = 0$ in the previous theorem we get the following.

Corollary 2.20. *Let $I$ be an ideal of $R$ and $M$ be an $R$-module. Then every proper submodule of $M$ is weakly prime if and only if every non-zero submodule of $M$ is weak second.*

Corollary 2.21. *Let $(R, m)$ be a local ring and $M$ be an $R$-module. Then we have the following.*
*(a) If $M$ is a Noetherian $R$-module and every proper submodule of $M$ is $I$-prime, then every non-zero submodule of $M$ is $I$-second.*

*(b) If $M$ is an Artinian $R$-module and every non-zero submodule of $M$ is $I$-second, then every proper submodule of $M$ is $I$-prime.*

*Proof.* Part (a) follows from [12, 4.6] and Theorem 2.19, while (b) follows from [6, 3.2] and Theorem 2.19. □

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