MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 1 (2020), 6–16
March 2020

research paper
оригинални научни рад

ČECH ROUGH PROXIMITY SPACES

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Abstract. Topology is a strong root for constructs that can be helpful to enrich the original model of approximation spaces. This paper introduces closure spaces on rough sets via a proximity relation on approximation spaces. We have used rough proximity to define the nearness between rough sets. Some results have been proved in this advanced nearness structure named Čech rough proximity. Examples are given to illustrate the proposed approach. Finally, an application of the theory is presented to demonstrate the fruitfulness of this new structure.

1. Introduction

The theory of rough sets is motivated by practical needs in classification, concept formation, and data analysis with insufficient and incomplete information. Pawlak [12] introduced rough set theory in 1982. The idea of the theory turned out to be extremely useful in practice and an excellent tool to handle the granularity of data. Many real-life applications of this concept have been implemented in various fields such as civil engineering [2], pharmacology [7], medical data analysis [13], image processing [24], and many more. In recent years, there has been a fast growing interest in this emerging theory.

There is a close homogeneity between rough set theory and general topology. Topology is a rich source for constructs that can be helpful to enrich the original model of approximation spaces. So the combined study of rough set theory and topology becomes essential (see [20, 23]). The central idea of the rough set theory is given by two forms of approximation, namely lower and upper approximation, which correspond to the interior and closure operators, respectively.

Topological spaces are axiomatized using the concept "a point is near to a set". Proximity structures [10] are finer than topology and are based on the concept "one set is near to another set". Every proximity induces a unique topology. Proximity

2010 Mathematics Subject Classification: 54A05, 54A10, 54C60, 54E05, 54E17
Keywords and phrases: Rough sets; proximity spaces; closure space.

structures and rough sets together have many applications in the field of image analysis [18]. Peters et al. [14] studied the nearness structures and their properties in the context of rough set theory and used these structures to recognize similar objects based on the detection of patterns in pairs of images. The authors established an extension of approximation spaces introduced by Pawlak as well as generalized approximation spaces based on the introduction of a nearness relation. In [5], Henry and Peters introduced a new form of a classifier based on approximation spaces in the context of near sets for use in pattern recognition. In [16], Peters used proximity structures in forgery detection and the study of microfossils, while in [17] he utilizes the feature values of objects to define the nearness of objects and, consequently, classify the universal set with respect to the available information of the objects. The author studied the generalized approximation spaces based on such nearness relations, which provides the possibility of measuring our knowledge about objects based on the perception of the nearness of objects classified using attributes or features. In another article [15], Peters et al. discussed that two sets might also be treated as near to each other if their upper approximations intersect. That is, the nearness of two sets may depend on the nearness relation between their respective upper approximations. This motivates us to redefine proximity between two sets using the concept of rough distance between their respective upper approximations in an approximation space. Biswas [3] gave the idea of rough distance.

Proximity spaces [6, 10] and Čech closure spaces [4] are closely related and together form the field of interest of many researchers. Both of them are extensions of topological spaces. Every basic proximity structure induces a Čech closure operator (see also [8, 10]). In [21], Thron used the theory of convergence of grills to discuss the compactness of closure spaces. In this article, we axiomatize the rough proximity relation on approximation spaces and call them Čech rough proximity spaces. The co-relation between Čech rough proximity spaces and Čech rough closure spaces is studied. These structures are very helpful in describing and comparing visual objects such as paintings or digital images. An example in support of this approach is also discussed in this paper. We show that the category of Čech rough proximity spaces and rough proximal maps is a super category of the category of pseudo metric spaces and nonexpansive maps and the category of rough pseudo metric spaces and rough nonexpansive maps.

This paper is organized as follows. In Section 2, there are some basic definitions and results on proximity spaces, closure spaces and rough set theory, necessary for the development of further sections of the paper. In Section 3, we define the Čech rough proximity relation on an approximation space. Some properties of Čech rough proximity spaces and their relation with Čech rough closure spaces are studied. Examples are given to support the theory. In the last section, we discuss an application of the theory developed in this paper.

2. Preliminary and basic results

When a set of attributes portrays objects of a universe, one may characterize the indiscernibility of objects dependent on their attribute values. If the same values characterize two objects on specific attributes, they are said to be indistinguishable or equivalent. Set of all objects with the same description form an equivalence class. Concerning equivalence classes, a subset of the universe may be approximated by two subsets: Upper and lower approximations of the subset. They can be formally described by a pair of unary set-theoretic operators [12]. The classical rough set theory is based on equivalence relations.

It is quite interesting to comprehend these concepts to the case of more general relations because sometimes it is difficult to use Pawlak's rough set theory due to limitations of equivalence relations in generating a granule base (neighborhood base). Various generalized rough set models have been established, and their properties or structures have been investigated intensively. So by applying the basic approach in a more extensive setting, one may sum up the idea of approximation operators by utilizing binary relations, or more simply non-equivalence relations [25]. It is very pertinent to consider a similarity or tolerance relation, instead of an equivalence relation for approximation space [9, 11]. In this article, our focus is towards the study of topological aspects of approximation spaces using tolerance (reflexive and symmetric) relations.

Throughout this paper, U denotes a non-empty set called the universe, and $\mathcal{P}(U)$ denotes the power set of U. If R denotes an arbitrary relation on U then the pair (U, R) will be called an approximation space. We will consider rough sets defined by Yao [25]. In this section, we collect some basic definitions and results on rough sets, Čech and Kuratowski closure operators and proximity spaces.

2.1 Rough set theory

Define a mapping $R : \mathcal{P}(U) \to \mathcal{P}(U)$ as $R(x) = \{y : xRy\}$; i.e., $R(x)$ consist of all elements of U which are related to the elements 'x'. We may define two unary set theoretic operators $\overline{R}$ (upper approximation) and $\underline{R}$ (lower approximation) as follows:

$$\overline{R}(A) = \{x : R(x) \cap A \neq \emptyset\}; \qquad \underline{R}(A) = \{x : R(x) \subseteq A\}.$$

A set $A \subseteq U$ is said to be crisp if $\overline{R}(A) = \underline{R}(A)$, otherwise A is rough. The table on the next page displays the properties of lower and upper approximation operators.

If we choose R to be a reflexive relation, then $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$. If R is a symmetric relation, then $X \subseteq \underline{R}(\overline{R}(X))$; $\overline{R}(\underline{R}(X)) \subseteq X$. If R is a transitive relation, then $\underline{R}(X) \subseteq \underline{R}(\underline{R}(X))$; $\overline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$.

Remark 2.1. **Relation between $R(x)$ and $\overline{R}(\{x\})$**: $R(x) = \{y : xRy\}$, $\overline{R}(\{x\}) = \{y : R(y) \cap \{x\} \neq \emptyset\}$. If R is reflexive then $A \subseteq \overline{R}(A)$. So we have $\{x\} \subseteq \overline{R}(\{x\})$. Now let $z \in \overline{R}(\{x\})$. Then $R(z) \cap \{x\} \neq \emptyset$ which yields that zRx. If R is also symmetric, then xRz means $\overline{R}(\{x\}) \subseteq R(x)$. Thus we have $R(x) = \overline{R}(\{x\})$, if R is symmetric and reflexive.

lower approximation	upper approximation
$\underline{R}(X) = (\overline{R}(X^c))^c$.	$\overline{R}(X) = (\underline{R}(X^c))^c$.
$\underline{R}(U) = U$.	$\overline{R}(\emptyset) = \emptyset$.
$\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$.	$\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$.
$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$.	$\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$.
$X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$.	$X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$.
$\underline{R}(X) = \bigcap_{x \notin X} \underline{R}(\{x\}^c)$.	$\overline{R}(X) = \bigcup_{x \in X} \overline{R}(\{x\})$.

Table 1: Properties of upper and lower approximation operators

Throughout this paper, we will assume R to be a tolerance relation (reflexive and symmetric) on U. And for convenience, we will use $\overline{R}(x)$ in place of $\overline{R}(\{x\})$.

2.2 Proximity space

Let X be a non-empty set; a binary relation δ on $\mathcal{P}(X)$ is called a basic proximity if the following axioms are satisfied, for $A, B \subset U$:
(i) $A\delta B \Rightarrow B\delta A$; (ii) $(A \cup B)\delta C \Leftrightarrow A\delta C$ or $B\delta C$;

(iii) $A\delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$; (iv) $A \cap B \neq \emptyset \Rightarrow A\delta B$.

A binary relation δ is called EF-proximity, if it is a basic proximity and additionally satisfies the following axiom:

(v) $A\bar{\delta}B \Rightarrow \exists\ E \subset X$ such that $A\bar{\delta}E$ and $B\bar{\delta}E^c$.

Further, δ is called separated if the following axiom holds in addition:

(vi) $\{x\}\delta\{y\} \Rightarrow x = y$.

The pair (X, δ) is called a basic (EF, separated) proximity space. Sometimes the axiom (v) is referred as "strong axiom". In this study, we will not consider the strong axiom.

2.3 Closure space

A function $Cl : \mathcal{P}(U) \to \mathcal{P}(U)$ is said to be Čech closure operator on U if it satisfies the following axioms, for $A, B \subset U$:
(i) $Cl(\emptyset) = \emptyset$; (ii) $A \subseteq Cl(A)$; (iii) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

The pair (U, Cl) is said to be Čech closure space. Further, Cl is a Kuratowaski closure operator or topological closure operator on U if it satisfies following, in addition,

(iv) $Cl(Cl(A)) = Cl(A)$.

3. Čech rough proximity spaces

In this section, we define a rough proximity space. Some results on rough proximity spaces are proved. Examples are constructed to support the theory of rough proximity spaces. Let (U, R) be a Yao's approximation space and R be a tolerance relation on U. For category theory, we refer to [1].

Definition 3.1 ([22]). Let us consider an approximation space (U, R), where U is a non-empty universe of discourse and R is an arbitrary tolerance relation. Let δ_R be a relation on $\mathcal{P}(U)$ satisfying the following axioms, for $A, B \subset U$:
(P.1) $\overline{R}(A)\delta_R\overline{R}(B) \Leftrightarrow \overline{R}(B)\delta_R\overline{R}(A)$;

(P.2) $(\overline{R}(A) \cup \overline{R}(B))\delta_R\overline{R}(C) \Leftrightarrow \overline{R}(A)\delta_R\overline{R}(C)$ or $\overline{R}(B)\delta_R\overline{R}(C)$;

(P.3) $\overline{R}(A)\delta_R\overline{R}(B) \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$;

(P.4) $\overline{R}(A) \cap \overline{R}(B) \neq \emptyset \Rightarrow A\delta_R B$.

Then δ_R is called the Čech rough proximity on U and the pair (U, δ_R) is known as Čech rough proximity space.

If δ_R satisfies the property
(P.5) If $\overline{R}(A)\delta_R\overline{R}(x)$ and $\overline{R}(x)\delta_R\overline{R}(B)$, then $\overline{R}(A)\delta_R\overline{R}(B)$,
then we will call this proximity δ_1-type rough proximity on U.

And if δ_R satisfies property
(P.6) If $\overline{R}(x)\delta_R\overline{R}(y)$, then $\overline{R}(x) = \overline{R}(y)$,
then we will call this proximity R_0-type rough proximity on U.

We will use $\bar{\delta}_R$ to represent the relation "not near", i.e., $A\bar{\delta}_R B$ means A is not near B or A is far from B.

Definition 3.2. Let (U, R) be an approximation space. A function $Cl_R : \mathcal{P}(U) \to \mathcal{P}(U)$ is said to be Čech rough closure operator on U if it satisfies the following axioms, for $A, B \subseteq U$:
(i) $Cl_R(\emptyset) = \emptyset$; (ii) $Cl_R(A) \supseteq \overline{R}(A)$; (iii) $Cl_R((A) \cup (B)) = Cl_R(A) \cup Cl_R(B)$.
The pair (U, Cl_R) is said to be Čech rough closure space.

Remark 3.3. Let (U, Cl_R) be a Čech rough closure space and let $A \subseteq U$. The function $Cl_R|_A : \mathcal{P}(A) \to \mathcal{P}(A)$, defined as $Cl_R|_A(B) = \overline{R}(A) \cap Cl_R(B)$, is rough closure operators on A. The operator $Cl_R|_A$ is called relative Čech rough closure operator on A induced by Cl_R. The pair $(A, Cl_R|_A)$ is said to be Čech rough closure subspace of (U, Cl_R). Further, $(A, Cl_R|_A)$ is a closed subspace of (U, Cl_R) on U, if $Cl_R(A) = A$.

Let us establish the relation between the Čech proximity relation and Čech closure operator.

Theorem 3.4. *Define the operator $Cl_{\delta_R} : \mathcal{P}(U) \to \mathcal{P}(U)$ by: $Cl_{\delta_R}(A) := \{x \in U : R(x)\delta_R\overline{R}(A)\}$. Then operator Cl_{δ_R} is a Čech rough closure operator on U.*

Proof. Let (U, δ_R) be a Čech rough proximity space, and let $A, B \subseteq U$. Then $Cl_{\delta_R}(\emptyset) = \emptyset$ clearly. To prove $\overline{R}(A) \subseteq Cl_{\delta_R}(A)$, let $x \in \overline{R}(A)$. Then $\overline{R}(R(x)) \cap \overline{R}(\overline{R}(A)) \neq \emptyset$. By (P.4), $x \in Cl_{\delta_R}(A)$. Next, $x \in Cl_{\delta_R}(A \cup B)$ iff $R(x)\delta_R\overline{R}(A \cup B)$ iff $R(x)\delta_R(\overline{R}(A) \cup \overline{R}(B))$ iff $R(x)\delta_R\overline{R}(A)$ or $R(x)\delta_R\overline{R}(B)$ iff $x \in Cl_{\delta_R}(A) \cup Cl_{\delta_R}(B)$. Hence Cl_{δ_R} is a Čech rough closure operator. $\square$

Remark 3.5. (i) A Čech rough closure space is also a Čech closure space.

(ii) If δ_R also satisfies: $\overline{R}(Cl_{\delta_R}(A))\ \delta_R\ \overline{R}(Cl_{\delta_R}(B)) \Rightarrow \overline{R}(A)\delta_R\overline{R}(B)$, then Cl_{δ_R} (defined in the above theorem) becomes a Kuratowaski closure (or a topological closure) operator on U.

Example 3.6. Let (U, Cl_R) be a Čech rough closure space. Define a relation δ_R on $\mathcal{P}(U)$ such that $A\delta_R B$ iff $Cl_R(A) \cap Cl_R(B) \neq \emptyset$. Then (U, δ_R) is a Čech rough proximity space.

Example 3.7. Let $U = \{a_1, a_2, \ldots, a_{21}\}$. Define a relation R on U such that every element has neighborhood as shown in the following table.

x (element)	$R(x)$ (neighborhood)
a_1	$\{a_1, a_2, a_3, a_6, a_{12}, a_{18}, a_{19}\}$
a_2	$\{a_1, a_2, a_3, a_6, a_7, a_8, a_{18}, a_{19}, a_{21}\}$
a_3	$\{a_1, a_2, a_3, a_8, a_9, a_{15}, a_{19}, a_{20}\}$
a_4	$\{a_4, a_5, a_6, a_{10}\}$
a_5	$\{a_4, a_5\}$
a_6	$\{a_1, a_2, a_4, a_6, a_7, a_8, a_{12}, a_{18}, a_{20}\}$
a_7	$\{a_2, a_6, a_7, a_{18}, a_{21}\}$
a_8	$\{a_2, a_3, a_6, a_8, a_9, a_{10}, a_{15}, a_{19}\}$
a_9	$\{a_3, a_8, a_9, a_{11}\}$
a_{10}	$\{a_4, a_8, a_{10}, a_{12}, a_{13}, a_{14}\}$
a_{11}	$\{a_9, a_{11}, a_{19}, a_{20}\}$
a_{12}	$\{a_1, a_6, a_{10}, a_{12}, a_{13}, a_{14}, a_{19}\}$
a_{13}	$\{a_{10}, a_{12}, a_{13}, a_{14}, a_{20}\}$
a_{14}	$\{a_{10}, a_{12}, a_{13}, a_{14}, a_{20}\}$
a_{15}	$\{a_3, a_8, a_{15}, a_{16}, a_{19}\}$
a_{16}	$\{a_{15}, a_{16}, a_{19}, a_{20}\}$
a_{17}	$\{a_{17}, a_{18}, a_{19}, a_{21}\}$
a_{18}	$\{a_1, a_2, a_6, a_7, a_{17}, a_{18}, a_{19}, a_{21}\}$
a_{19}	$\{a_1, a_2, a_3, a_8, a_{11}, a_{12}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}\}$
a_{20}	$\{a_3, a_6, a_{11}, a_{13}, a_{14}, a_{16}, a_{20}, a_{21}\}$
a_{21}	$\{a_2, a_7, a_{17}, a_{18}, a_{20}, a_{21}\}$

Table 2: Neighborhoods of elements

Now define δ_R on $\mathcal{P}(U)$ as follows: $A\delta_R B$ iff $\overline{R}(A) \cap \overline{R}(B) \neq \emptyset$, $A, B \subseteq U$.

We can easily show that δ_R is a Čech rough proximity on U. Further if $A = \{a_5, a_9\}$, then by definition, $\overline{R}(A) = \{a_3, a_4, a_5, a_8, a_9, a_{11}\}$ and $Cl_{\delta_R}(A) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{15}, a_{19}, a_{20}\}$.

Then we have $\overline{R}(Cl_{\delta_R}(A)) = U$ and $\underline{R}(Cl_{\delta_R}(A)) = \{a_3, a_4, a_5, a_8, a_9, a_{11}, a_{15}\}$. Thus $\overline{R}(Cl_{\delta_R}(A)) \neq \underline{R}(Cl_{\delta_R}(A))$ and closure of an arbitrary set need not to be crisp, in general. Thus, the rough closure space on U consists of rough sets.

Example 3.8. Let (U, R) be an approximation space and Cl_R be a Čech rough closure operator on U and μ^* be an outer measure [19] on U. For $A, B \subset U$, define $A\delta_R B$ iff $Cl_R(A) \cap Cl_R(B) \neq \emptyset$ or $\mu^*(Cl_R(A)) \wedge \mu^*(Cl_R(B)) > 0$. Then δ_R is a Čech rough proximity on U.

Biswas [3] defined rough metric on Pawlak's approximation spaces [12] and discussed the properties of rough metric spaces. Equivalently, a rough pseudo metric on a Yao's approximation space can be defined as follows:

Definition 3.9 ([3]). Let U be a non-empty set and R be a tolerance relation defined on U. Then the function $d_R : U \times U \to \mathbb{R}$ is called a rough pseudo metric on U if the following are true, for all $x, y, z \in U$:
(i) $d_R(x, y) \geq 0$; (ii) $d_R(x, y) = 0$ if $R(x) = R(y)$;

(iii) $d_R(x, y) = d_R(y, x)$; (iv) $d_R(x, y) + d_R(y, z) \geq d_R(x, z)$.
The pair (U, d_R) is called a rough pseudo metric space.

Definition 3.10. Let (U, d_{R_1}) and (V, d_{R_2}) be two pseudo rough metric spaces. A function $f : U \to V$ is said to be a rough nonexpansive continuous mapping if $d_{R_2}(f(x), f(y)) \leq d_{R_1}(x, y)$, for all $x, y \in U$.

Definition 3.11. Let (U, δ_{R_1}) and (V, δ_{R_2}) be two rough proximity spaces. A function $f : U \to V$ is said to be a rough proximal mapping if $\overline{R}_1(x)\delta_{R_1}\overline{R}_1(y) \Rightarrow \overline{R}_2(\{f(x)\})\delta_{R_2}\overline{R}_2(\{f(y)\})$, $x, y \in U$.

Proposition 3.12. *Let (U, R) be an approximation space, where R is a tolerance relation and let 'd' be a rough pseudo metric on U. Define δ_R as follows:*
(i) $A\delta_R B$ iff $\inf\{d(a, b) : a \in \overline{R}(A)$ and $b \in \overline{R}(B)\} = 0, A, B \subset U$.

(ii) $A\delta_R B$ iff $\inf\{d(a, b) : a \in A$ and $b \in B\} = 0, A, B \subset U$.
Then δ_R is a Čech rough proximity on U.

Remark 3.13. (1) The category of rough pseudo metric spaces and rough nonexpansive maps is embedded into the category of Čech rough proximity spaces and rough proximal maps.

(2) The category of pseudo metric spaces and nonexapnsive maps is embedded into the category of rough pseudo metric spaces and rough nonexpansive maps (when R is an equality relation on U, classical case). Thus, the category of Čech rough proximity spaces and rough proximal maps is a super category of pseudo metric spaces and nonexpansive maps.

Example 3.14. Let $\mathbb{R}$ be a set of all real numbers and $\varepsilon > 0$ be a given fixed number. Define a binary relation R on $\mathbb{R}$ such that xRy iff $|x - y| < \varepsilon$. Clearly R is a tolerance relation. Define δ_R on $\mathbb{R}$ as: $A\delta_R B$ iff $\inf\{|x - y| : x \in A, y \in B\} \leq \varepsilon$. Then the pair $(\mathbb{R}, \delta_R)$ is a Čech rough proximity space. The closure Cl_{δ_R} induced by this Čech rough proximity space is a Čech rough closure operator and Cl_{δ_R} does not satisfies Kuratowski closure axiom. For example, let $A = (0, 1)$. Then $Cl_{\delta_R}(A) = (-3\varepsilon, 1+3\varepsilon)$ and $Cl_{\delta_R}Cl_{\delta_R}(A) = (-6\varepsilon, 1 + 6\varepsilon)$. Thus $Cl_{\delta_R}(A) \neq Cl_{\delta_R}Cl_{\delta_R}(A)$, in general.

Example 3.15. Let (U, Cl_{R_1}) and (V, Cl_{R_2}) be Čech rough closure spaces. Consider a continuous and closed function $f : U \to V$ such that $y \in Cl_{R_2}(f(x)) \Rightarrow f(x) \in Cl_{R_2}(\{y\})$, for all $y \in V$ and $x \in U$. Define $A\delta_R B$ iff $Cl_{R_2}(f(A)) \cap Cl_{R_2}(f(B)) \neq \emptyset, A, B \subset U$. Then δ_R is a Čech rough proximity on U.

Example 3.16. Let (U, R) be an approximation space and (U, Cl_R) be a Čech rough closure space. Define $A\delta_R B$ iff $\overline{R}(A) \cap Cl_R(B) \neq \emptyset$ or $Cl_R(A) \cap \overline{R}(B) \neq \emptyset, A, B \subset U$. Then (U, δ_R) is a Čech rough proximity space. If $Cl_R(A) = Cl_R(\overline{R}(A))$, then $Cl_R(A) \subset Cl_{\delta_R}(A)$, for all $A \subseteq U$.

Lemma 3.17. *Let (U, δ_R) be a Čech rough proximity space. If $\overline{R}(A)\delta_R\overline{R}(B)$ and $\overline{R}(B) \subseteq \overline{R}(C)$, then $\overline{R}(A)\delta_R\overline{R}(C)$.*

Proof. Given that $\overline{R}(A)\delta_R\overline{R}(B)$ and $\overline{R}(B) \subseteq \overline{R}(C)$. We may write $\overline{R}(C) = \overline{R}(B) \cup (\overline{R}(C) - \overline{R}(B))$. Suppose, if possible, $\overline{R}(A)\bar{\delta}_R\overline{R}(C)$ that is, $\overline{R}(A)\bar{\delta}_R(\overline{R}(B) \cup (\overline{R}(C) - \overline{R}(B)))$ which implies $\overline{R}(A)\bar{\delta}_R\overline{R}(B)$ and $\overline{R}(A)\bar{\delta}_R(\overline{R}(C) - \overline{R}(B))$ by (P.2), which contradicts the fact that $\overline{R}(A)\delta_R\overline{R}(B)$. Thus $\overline{R}(A)\delta_R\overline{R}(C)$. $\square$

By using the above lemma, we can prove the following result, which resembles the linearity property of Čech closure operator on the set U.

Theorem 3.18. *If δ_R is Čech rough proximity and $\overline{R}(A) \subseteq \overline{R}(B)$, then $Cl_{\delta_R}(A) \subseteq Cl_{\delta_R}(B)$.*

Proof. Let $y \in Cl_{\delta_R}(A)$. Then $\overline{R}(y)\delta_R\overline{R}(A)$. Suppose $y \notin A$. Therefore, $\overline{R}(y)\delta_R\overline{R}(B)$, that is, $y \in Cl_{\delta_R}(B)$. Hence $Cl_{\delta_R}(A) \subseteq Cl_{\delta_R}(B)$. $\square$

The following result finds the condition under which a Čech rough proximity space will induce a topological space, i.e., Čech closure operator induced by a Čech rough proximity becomes a Kuratowski (topological) closure operator.

Theorem 3.19. *Let (U, δ_R) is an approximation space and for all $A, B, C \subseteq U$, $\overline{R}(A)\delta_R\overline{R}(B)$ and $\overline{R}(C)\delta_R\overline{R}(\{x\})$ for all $x \in B \Rightarrow \overline{R}(C)\delta_R\overline{R}(A)$. Then δ_R satisfies the following property: $\overline{R}(Cl_{\delta_R}(A))\delta_R\overline{R}(Cl_{\delta_R}(B)) \Rightarrow \overline{R}(A)\delta_R\overline{R}(B)$. Thus Cl_{δ_R} is a Kuratowski closure operator on U.*

Proof. Note that for $A, B, C \subseteq U$, $\overline{R}(Cl_{\delta_R}(A))\delta_R\overline{R}(Cl_{\delta_R}(B))$ iff $\overline{R}((Cl_{\delta_R}(A) - \overline{R}(A)) \cup \overline{R}(A))\delta_R\overline{R}((Cl_{\delta_R}(B) - \overline{R}(B)) \cup \overline{R}(B))$ iff $(\overline{R}(Cl_{\delta_R}(A) - \overline{R}(A)) \cup \overline{R}\overline{R}(A))\delta_R(\overline{R}(Cl_{\delta_R}(B) - \overline{R}(B)) \cup \overline{R}\overline{R}(B))$.

Let $A^{\sim} = Cl_{\delta_R}(A) - \overline{R}(A)$ and $B^{\sim} = Cl_{\delta_R}(B) - \overline{R}(B)$ for convenience. So, $(\overline{R}(A^{\sim}) \cup \overline{R}\overline{R}(A))\delta_R(\overline{R}(B^{\sim}) \cup \overline{R}\overline{R}(B))$ implies $(\overline{R}(A^{\sim})\delta_R\overline{R}(B^{\sim})$ or $\overline{R}(A^{\sim})\delta_R\overline{R}\overline{R}(B))$ or $(\overline{R}\overline{R}(A)\delta_R\overline{R}(B^{\sim})$ or $\overline{R}\overline{R}(A)\delta_R\overline{R}\overline{R}(B))$. Suppose that $\overline{R}(A^{\sim})\delta_R\overline{R}(B^{\sim})$. Since $\overline{R}(A)\delta_R R(x)$ for all $x \in A^{\sim}$, therefore $\overline{R}(A)\delta_R\overline{R}(B^{\sim})$. Similarly, $\overline{R}(B)\delta_R R(y)$ for all $y \in B^{\sim}$. Thus $\overline{R}(A)\delta_R\overline{R}(B)$. Further, if $\overline{R}(A^{\sim})\delta_R\overline{R}\overline{R}(B)$, then $\overline{R}(A)\delta_R\overline{R}\overline{R}(B)$. If $y \in \overline{R}\overline{R}(B)$, then $\overline{R}(\{y\})\delta_R\overline{R}(B)$. Hence $\overline{R}(A)\delta_R\overline{R}(B)$. Similarly, $\overline{R}\overline{R}(A)\delta_R\overline{R}(B^{\sim})$ and $\overline{R}\overline{R}(A)\delta_R\overline{R}\overline{R}(B)$ also yield to $\overline{R}(A)\delta_R\overline{R}(B)$. As a consequence, we have $\overline{R}(Cl_{\delta_R}(A))\delta_R\overline{R}(Cl_{\delta_R}(B)) \Rightarrow \overline{R}(A)\delta_R\overline{R}(B)$.

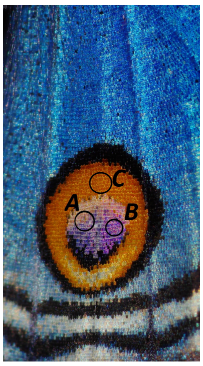
Further, we will only prove that $Cl_{\delta_R}Cl_{\delta_R}(A) = Cl_{\delta_R}(A)$. Let $x \in Cl_{\delta_R}(Cl_{\delta_R}(A))$. Then $R(x)\delta_R\overline{R}(Cl_{\delta_R}(A))$. Also for all $y \in Cl_{\delta_R}(A)$, $R(y)\delta_R\overline{R}(A)$. So, $R(x)\delta_R\overline{R}(A)$

and hence $x \in Cl_{\delta_R}(A)$. That is $Cl_{\delta_R}Cl_{\delta_R}(A) \subseteq Cl_{\delta_R}(A)$. Thus Cl_{δ_R} is a Kuratowski (topological) closure operator on U. □

4. Application of Čech rough proximity in comparing digital images

Here, we will discuss a visual application of of the Čech rough proximity structures. Consider the image of a butterfly in Figure 1b. An extracted part of the using is shown in Figure 1a, which we will consider as the universe U. Define a Čech rough proximity on U as follows:

Example 4.1. Let U be the set of pixels in Figure 1a. Colour strength of each pixel p can be represented by the triplet $p := (p_r, p_g, p_b)$, where p_r, p_g, p_b represents the red, green and blue intensity values of the pixel p, respectively. Each intensity value is on a scale of 0 to 255. We have coordinates of each pixel according to its RGB value.



(a) Set of Pixels.

(b) Original Picture

Figure 1: Digital Image of a Butterfly

Define a map $d : U \times U \to \mathbb{R}$ as: $d(p, q) = \max\,\{|p_r - q_r|, |p_g - q_g|, |p_b - q_b|\}$. Define a relation R on U as: $p_1 R p_2$ iff $d(p_1, p_2) \leq 5$. Thus the neighborhood $R(p)$ of a pixel p is the set of all pixels which have visual distance d less than or equal to value 5. Define $A \delta_R B$ iff $\inf\{d(p, q) : p \in A, q \in B \leq 10\}$. Then δ_R is a Čech rough proximity on U.

Let A and B denote two sets of pixels as shown in Figure 1a. So the colour strength of each pixel, say p, can be represented by the tuple value (p_r, p_g, p_b), where $p_r, p_g, p_b \in \{0, 1, 2, 3, \ldots, 255\}$. As we choose RGB values as a feature value of elements (pixels), so neighborhood of a given element (pixel) 'p' is $R(p) = \{x \in U : d(p, x) \leq 5\}$. The pixel p is very similar to each element in $R(p)$, because the difference between corresponding RGB values of p and any element in $R(p)$ is less than 5. In Figure 1a, we may see the set A and B are near as there are some elements (pixel) which look like same, i.e., their RGB values are very close or within the difference of 10. Also, the sets B and C are far from each other as RGB values of every element in C have difference more than 10 from RGB values of each element of B. So, by using this kind of structure (or similar to this) in a given picture, we may re-design the picture on lower resolution visual output devices. And also by using far (not near) relation $\bar{\delta}_R$, we may also distinguish different kind of objects in the picture on a digital platform.

4.1 Concluding remarks

The emergence of topology in the rough set theory is very helpful to get substantial results which yield hidden relations between data. Here, we have given a brief account of Čech closure spaces and Čech proximity spaces in the rough set theory. Čech closure spaces are the extension of topological spaces. Čech rough proximity spaces are the extension of rough pseudo metric spaces and pseudo metric spaces. Since this approach to the Čech rough proximity discerns the nearness between sets in an approximation space, therefore it may be considered as a contrivance for such studies in the fields of information science, artificial intelligence, computer science, pattern recognition, image processing, etc. One of the applications of the theory is also in the syllabus of this paper. Our approach connects rough sets, closure spaces, proximity spaces and topological spaces. There are still a number of fields which can be explored using Čech rough proximity spaces, like, uniform spaces, merotopic spaces, etc. This is part of our future research.

Acknowledgement. One of the authors, Pankaj Kumar Singh, gratefully acknowledge the University Grants Commission, New Delhi, India, for providing Senior Research Fellowship, Student ID. 201516NETJRF-10459-1, to pursue research work.

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(received 13.11.2018; in revised form 06.03.2019; available online 17.09.2019)

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