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CAT^1-HYPERGROUPS AND PULLBACK CAT^1-HYPERGROUPS

Bijan Davvaz and Murat Alp

Abstract. Loday's 1-cat group definition plays very powerful role in making some new applications to crossed module due to Whitehead. There are many applications of cat^1-groups such as cat^1-polygroups and pullback cat^1-polygroups. The importance of hypergroups come from the properties of hypergroups such that hypergroups in the sense of Marty do not have identity element, inverse element and they are generalization of the well known groups. In this paper, we introduce the concept of cat^1-hypergroups, their examples and some related properties. Also, we investigate pullback cat^1-hypergroups and properties such as: every cat^1-group is a cat^1-hypergroup; construction of a cat^1-group from a crossed module of hypergroups and vice versa. Finally, we present the definition of pullback cat^1-hypergroups and some of their properties.

1. Introduction

Crossed modules have been used widely, and in various contexts, since their definition by Whitehead [16] in his investigation of the algebraic structure of second relative homotopy groups. Loday in [12] showed that the category of crossed modules is equivalent to that of cat^1-groups. After the definition of cat^1-groups many applications were given such as pullback cat^1-group [2]. The importance of hypergroups come from the properties of hypergroups such that a hypergroup in the sense of Marty does not have an identity element and inverse elements in general case. The notion of a hypergroup is a generalization of the well known notion of a group.

Hypergroups have many applications, in areas such as geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [5, 6]).

In this paper, we introduce the notion of cat^1-hypergroups and prove that crossed modules of hypergroups [3] is equivalent to cat^1-hypergroups by Loday's way. The rest of the paper is organized as follows. In the second section we review basic concepts

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regarding the crossed modules and cat^1-groups in the known format. In Section 3, some brief introduction about hypergroups and crossed modules of hypergroups are given. In Section 4, the definition of cat^1-hypergroups, their examples and some properties are presented. These properties are: a cat^1-group is a cat^1-hypergroup, construction of a cat^1-group from a crossed module of hypergroups and vice versa. Finally, in the last section, we present the definition of pullback cat^1-hypergroups and their properties. In order to show the second axiom of pullback cat^1-hypergroups we use that $[x,y] = [x,y]_r \cup [x,y]_l$ (see [1]).

2. Crossed modules and cat^{1}groups

Let G be a group and Ω be a non-empty set. A *(left) group action* is a binary operator $\tau : G \times \Omega \to \Omega$ that satisfies the following two axioms:
(i) $\tau(gh,\omega) = \tau(g,\tau(h,\omega))$, for all $g,h \in G$, $\omega \in \Omega$, (ii) $\tau(e,\omega) = \omega$, for all $\omega \in \Omega$.
For $\omega \in \Omega$ and $g \in G$, we write ${}^{g}\omega := \tau(g,\omega)$. A *crossed module* $X = (M,N,\partial,\tau)$ consists of groups M and N together with a homomorphism $\partial : M \to N$ and a (left) action $\tau : N \times M \to M$ on M, satisfying the conditions:
(i) $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$ and $g \in N$,
(ii) ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m,m' \in M$.
The crossed module X is also denoted by $X = (\partial : M \to N)$. Let M be a group and take $G = Aut(M)$. Then, ∂ sends x to the inner automorphism $x(-)x^{-1}$. This is obviously a crossed module with the respect to the action of $Aut(M)$ on M.

An *1-categorical group* or *cat^1-group* is a group G together with a subgroup N and two homomorphisms $s,b : G \to N$ satisfying $s|_N = b|_N = id_N$ and $[\ker s, \ker b] = e$. This cat^1-group is denoted by $C = (G;N)$ if no confusion can arise. A morphism of cat^1-groups $C \to C'$ is a group homomorphism $f : G \to G'$ such $f(N) \subseteq N'$ and $s'f = f|_N s$, $b'f = f|_N b$.

Lemma 2.1 ([12]). *The following data are equivalent:*
(i) a crossed module $\partial : M \to N$, *(ii) a cat^1 group* $C = (G;N)$.

3. Hypergroups and crossed modules of hypergroups

Let H be a non-empty set and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H,\star)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \star B = \bigcup_{\substack{a\in A\\ b\in B}} a \star b,$$

and $\{x\}$ is shown by x. A hypergroupoid $(H,\star)$ is called a *semihypergroup* if for all a,b,c of H we have $(a\star b)\star c = a \star (b \star c)$, which means that $\bigcup_{u\in a\star b} u \star c = \bigcup_{v\in b\star c} a \star v$.

A hypergroupoid $(H, \star)$ is called a *quasihypergroup* if for all a of H we have $a \star H = H \star a = H$. This condition is also called the *reproduction axiom.*

Definition 3.1. A hypergroupoid $(H, \star)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup.*

Remark 3.2. Every group is a hypergroup.

In a hypergroup $(H, \star)$, an element $e \in H$ is called a *scalar identity element* if $e \star x = x \star e = \{x\} := x$, for all $x \in H$.

We refer the readers to [4, 5, 9, 14, 15] for more details about hypergroups. In [4, 6] many examples of hypergroups are given. Here, we present one.

Example 3.3. Let S_3 be the symmetric group of order 6 and let $H = \langle(1\ 2)\rangle$. We consider the following hyperoperation on H: $x \star y = xHy = \{xy,\ x(1\ 2)y\}$ for all $x, y \in S_3$. This hyperoperation is a P-hyperoperation.

$\star$	i	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
i	i $(1\ 2)$	$(1\ 2)$ i	$(1\ 3)$ $(1\ 2\ 3)$	$(2\ 3)$ $(1\ 3\ 2)$	$(1\ 2\ 3)$ $(1\ 3)$	$(1\ 3\ 2)$ $(2\ 3)$
$(1\ 2)$	$(1\ 2)$ i	i $(1\ 2)$	$(1\ 2\ 3)$ $(1\ 3)$	$(1\ 2\ 3)$ $(2\ 3)$	$(1\ 3)$ $(1\ 2\ 3)$	$(2\ 3)$ $(1\ 3\ 2)$
$(1\ 3)$	$(1\ 3)$ $(1\ 3\ 2)$	$(1\ 3\ 2)$ $(1\ 3)$	i $(2\ 3)$	$(1\ 2\ 3)$ $(1\ 2)$	$(2\ 3)$ i	$(1\ 2)$ $(1\ 2\ 3)$
$(2\ 3)$	$(2\ 3)$ $(1\ 2\ 3)$	$(1\ 2\ 3)$ $(2\ 3)$	$(1\ 3\ 2)$ $(1\ 2)$	i $(1\ 3)$	$(1\ 2)$ $(1\ 3\ 2)$	$(1\ 3)$ i
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Table 1: Hyperoperation on H

It is easy to see that $(S_3, \star)$ is a non-commutative hypergroup. Indeed, for all $x, y, z \in S_3$ we have

$$\begin{aligned}
(x \star y) \star z =& \{xy,\ x(1\ 2)y\} \star z = xy \star z \cup x(1\ 2)y \star z \\
=& \{xyz,\ xy(1\ 2)z\} \cup \{x(1\ 2)yz,\ x(1\ 2)y(1\ 2)z\} \\
=& \{xyz,\ xy(1\ 2)z,\ x(1\ 2)yz,\ x(1\ 2)yz,\ x(1\ 2)y(1\ 2)z\}, \\
x \star (y \star z) =& x \star \{yz,\ y(1\ 2)z\} = x \star yz \cup x \star y(1\ 2)z \\
=& \{xyz,\ x(1\ 2)yz\} \cup \{xy(1\ 2)z,\ x(1\ 2)y(1\ 2)z\} \\
=& \{xyz, x(1\ 2)yz,\ xy(1\ 2)z,\ x(1\ 2)yz,\ x(1\ 2)y(1\ 2)z\}.
\end{aligned}$$

Thus, $(x \star y) \star z = x \star (y \star z)$. Moreover, we have

$$x \star S_3 = x \star \bigcup_{g \in S_3} x \star g = \bigcup_{g \in S_3} \{xg,\ x(1\ 2)g\} = S_3 = \bigcup_{g \in S_3} \{gx,\ g(1\ 2)x\} = S_3 \star x.$$

Definition 3.4. Let $(C,\star)$ and $(H,\circ)$ be two hypergroups. Let ∂ be a map from C into H. Then, ∂ is called a *strong homomorphism* if $\partial(x\star y)=\partial(x)\circ\partial(y)$, for all $x,y\in C$, where $\partial(x\star y)=\bigcup\limits_{z\in x\star y}\partial(z)$.

Several mathematicians considered actions of algebraic hyperstructures (see for example [13,17]). In [13], Madanshekaf and Ashrafi considered a generalized action of a hypergroup H on a non-empty set X and obtained some results in this respect. For the definition of crossed modules of hypergroups, we need the notion of hypergroup action.

Definition 3.5 ([13]). Let $(H,\circ)$ be a hypergroup and X be a non-empty set. A map $\alpha : H\times X\to\mathcal{P}^*(X)$ is called a *generalized action* of H on X, if the following axioms hold:

(i) $\alpha(g\star h,x)\subseteq\alpha(g,\alpha(h,x))$, for all $g,h\in H$ and $x\in X$, where $\alpha(g\star h,x)=\bigcup\limits_{k\in g\star h}\alpha(k,x)$.

(ii) For all $h\in H$, $\alpha(h,X)=X$, where $\alpha(h,X)=\bigcup\limits_{x\in X}\alpha(h,x)$.

If the equality in Definition 3.5 (i) holds, the action is called *strong generalized action*. Moreover, if H has the scaler identity element e, then the following condition must hold too:

(iii) $\alpha(e,x)=\{x\}:=x$, for all $x\in X$.

Example 3.6 ([13]). (i) For any hypergroup $(H,\star)$ and any non-empty set X, the map $\alpha : H\times X\to\mathcal{P}^*(X)$, given by $\alpha(h,x)=X$ is a strong generalized action of H on X. If we define $\alpha(h,x)=\{x\}$, then this map is also a strong generalized action of H on X.

(ii) Let $(H,\star)$ be a hypergroup. Then, the map $\alpha : H\times H\to\mathcal{P}^*(H)$, given by $\alpha(h,x)=h\star x$ is a strong generalized action of H on H.

Example 3.7 ([13]). Let X be a non-empty set, $f\in M_\theta$ and $H=M_f$. Then, the map $\alpha : H\times X\to\mathcal{P}^*(X)$, defined by $\alpha(h,x)=h(x)$ is a strong generalized action of H on X.

For $x\in X$, we put ${}^hx:=\alpha(h,x)$. Then, for a strong generalized action, we have
(i) ${}^g({}^hx)={}^{g\star h}x$, for all $g,h\in H$ and $x\in X$, (ii) $\bigcup\limits_{x\in X}{}^hx=X$, for all $h\in H$.

Definition 3.8. A *crossed module of hypergroups* $\mathcal{X}=(C,H,\partial,\alpha)$ consists of hypergroups $(C,\star)$ and $(H,\circ)$ together with a strong homomorphism $\partial : C\to H$ and a strong generalized action $\alpha : H\times C\to\mathcal{P}^*(C)$ on C, satisfying the conditions:

(i) $h\circ\partial(c)\subseteq\partial({}^hc)\circ h$, for all $c\in C$ and $h\in H$,

(ii) $c\star c'\subseteq{}^{\partial(c)}c'\star c$, for all $c,c'\in C$.

We say the action of H on C is *productive*, if for all $c \in C$ and $h \in H$ there exist $c_1, \ldots, c_n$ in C such that ${}^h c = c_1 \star \ldots \star c_n$.

Let $(H, \circ)$ be a hypergroup. We define the relation β^*_H as the smallest equivalence relation on H such that the quotient H/β^*_H, the set of all equivalence classes, is a group. In this case β^*_H is called the *fundamental equivalence relation* on H and H/β^*_H is called the *fundamental group*. The product $\odot$ in H/β^*_H is defined as follows: $\beta^*_H(x) \odot \beta^*_H(y) = \beta^*_H(z)$, for all $z \in \beta^*_H(x) \circ \beta^*_H(y)$. This relation was introduced by Koskas [10] and studied mainly by Corsini [4], Leoreanu-Fotea [11] and Freni [8] concerning hypergroups, Vougiouklis [15] concerning H_v-groups, Davvaz concerning polygroups [6], and many others. We consider the relation β_H as follows:

$$x \;\beta_H\; y \;\Leftrightarrow\; \text{there exist } z_1, \ldots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i.$$

Freni proved that for hypergroups $\beta_H = \beta^*_H$ in [8]. The kernel of the *canonical map* $\varphi_H : H \longrightarrow H/\beta^*_H$ is called the *core* of H and is denoted by ω_H. Here we also denote by ω_H the unit of H/β^*_H.

Throughout the paper, we denote the binary operations of the fundamental groups H/β^*_H and C/β^*_C by $\odot$ and $\otimes$, respectively.

Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\alpha : H \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : H/\beta^*_H \times H/\beta^*_C \to \mathcal{P}^*(H/\beta^*_C)$ in usual manner:

$$\psi(\beta^*_H(h), \beta^*_C(c)) = \{\beta^*_C(x) \mid x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_H(h)}} {}^z y\}.$$

By the definition of β^*_C, since the action of H on C is productive, we conclude that $\psi(\beta^*_H(h), \beta^*_C(c))$ is singleton, i.e., we have

$$\psi : H/\beta^*_H \times H/\beta^*_C \to H/\beta^*_C, \quad \psi(\beta^*_H(h), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y \in \beta^*_C(c) \\ z \in \beta^*_H(h)}} {}^z y.$$

We denote $\psi(\beta^*_H(h), \beta^*_C(c)) = {}^{[\beta^*_H(h)]}[\beta^*_C(c)]$.

Proposition 3.9 ([3]). *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\partial : C \to H$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta^*_C \to H/\beta^*_H$ by setting $\mathcal{D}(\beta^*_C(c)) = \beta^*_H(\partial(c))$, for all $c \in C$.*

Theorem 3.10. *Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crosed module of hypergroups such that the action of H on C is productive. Then, $\mathcal{X}_{\beta^*} = (C/\beta^*_C, H/\beta^*_H, \mathcal{D}, \psi)$ is a crossed module.*

Definition 3.11 ([3]). Let $\mathcal{X} = (C, H, \partial, \alpha)$ be a crossed module of hypergroups and $\iota : Q \to H$ be a strong homomorphism of hypergroups. Then, $\iota^\bullet \mathcal{X} = (\iota^\bullet C, Q, \partial^\bullet, \alpha^\bullet)$ is the pullback of $\mathcal{X}$ by ι, where $\iota^\bullet C = \{(q, c) \in Q \times C \mid \iota(q) = \partial(c)\}$ and $\partial^\bullet(q, c) = q$. The hypergroup action of Q on $\iota^\bullet C$ is given by

$${}^q(q_1, c) = \{(x, y) \;\mid \beta^*_H(x) = \beta^*_H(q) \odot \beta^*_H(q_1) \odot \beta^*_H(q)^{-1},\; y \in {}^{\iota(q)}c\}.$$

Theorem 3.12 ([3]). *$\iota^\bullet \mathcal{X} = (\iota^\bullet C, Q, \partial^\bullet, \alpha^\bullet)$ is a crossed module of hypergroups.*

4. Cat^1-hypergroups

In this section, we introduce the concept of cat^1-hypergroups. In order to do so, we need the definition of commutator of elements in hypergroups. Fortunately, Aghabozorgi, Davvaz and Jafarpour [1] have recently introduced the notion of commutator of elements in hypergroups. Since in hypergroups, the inverse element does not exist in general, their definition is so important. We recall the following definition from [1].

DEFINITION 4.1. Let $(H, \circ)$ be a hypergroup. We define the following
(i) $[x, y]_r = \{h \in H \mid x \circ y \cap y \circ x \circ h \neq \emptyset\}$;

(ii) $[x, y]_l = \{h \in H \mid x \circ y \cap h \circ y \circ x \neq \emptyset\}$;

(iii) $[x, y] = [x, y]_r \cup [x, y]_l$.
From now on we call $[x, y]_r$, $[x, y]_l$ and $[x, y]$ *right commutator of* x and y, *left commutator of* x and y, and *commutator of* x and y, respectively. Also, we will denote by $[H, H]_r$, $[H, H]_l$ and $[H, H]$ the set of all right commutators, left commutators and commutators, respectively.

PROPOSITION 4.2 ([1]). *If H is a group then $[y, x]_r^{-1} = [x, y]_r = [x^{-1}, y^{-1}]_l = [y^{-1}, x^{-1}]_l^{-1}$, for every x, y in H.*

EXAMPLE 4.3 ([1]). Suppose that $H = \{e, a, b\}$. Consider the hypergroup $(H, \circ)$ where $\circ$ is defined on H as follows:

$\circ$	e	a	b
e	a, b	e	e
a	e	a	b
b	e	a, b	a, b

It is easy to see that $\{a\} = [a, a]_r \neq [a, a]_l = \{a, b\} = [a^{-1}, b^{-1}]_l$, where a^{-1} is the inverse of a in H.

PROPOSITION 4.4 ([1]). *If H is a commutative hypergroup, then $[x, y]_r = [x, y]_l = [x, y]$, for all $(x, y) \in H^2$.*

LEMMA 4.5. *Let $(C, \star)$ and $(H, \circ)$ be two hypergroups and let $\partial : C \to H$ be a strong homomorphism. Then, $\partial(\omega_C) \subseteq \omega_H$.*

Proof. Suppose that $y \in \partial(\omega_C)$ is an arbitrary element. Then, we have

$$\beta_H^*(y) = \beta_H^*(\partial(\omega_C)) = \mathcal{D}(\beta_C^*(\omega_C)) = \mathcal{D}(\omega_C) = \omega_H,$$

(since $\mathcal{D}$ is a strong homomorphism). Thus, $y \in \omega_H$. $\square$

Now, we consider the notion of kernel of a strong homomorphism of hypergroups.

DEFINITION 4.6. Let $(H, \circ)$ and $(C, \star)$ be two hypergroups and $\partial : C \to H$ be a strong homomorphism. The *core-kernel* of ∂ is defined by $\ker^* \partial = \{x \in C \mid \partial(x) \in \omega_H\}$.

Theorem 4.7 ([3]). *$\ker^* \partial$ is a subhypergroup of C.*

Now, by applying the above definitions, we are in a situation to define the concept of cat^1-hypergroup.

Definition 4.8. A cat^1-hypergroup $\mathcal{C} = (k; t, h : H \to C)$ consists of hypergroups H and C, two strong epimorphisms $t, h : H \to C$ and an embedding $k : C \to H$ satisfying:

(CAT-H-1) $k = hk = Id_C$, (CAT-H-2) $[x, y] \subseteq \omega_H, \forall x \in \ker^* t, \forall y \in \ker^* h$.

The maps t, h are called the *source* and the *target*.

Proposition 4.9. *Condition (CAT-H-2) is equivalent to $[\beta^*_H(x), \beta^*_H(y)]{=}\omega_H{=}1_{P/\beta^*_H}$.*

Proof. Suppose that $[x, y] \subseteq \omega_H$. Then, by Definition 4.1, we have $[x, y]_r \cup [x, y]_l \subseteq \omega_H$. This implies that $\{h \in H \mid x \circ y \cap y \circ x \circ h \neq \emptyset\} \subseteq \omega_H$. Thus, we obtain $\beta^*_H(h) = \omega_H$, for all $h \in H$ such that $x \circ y \cap y \circ x \circ h \neq \emptyset$. Consider $h \in H$ such that $x \circ y \cap y \circ x \circ h \neq \emptyset$. Then, there exists $z \in x \circ y$ and $z \in y \circ x \circ h$. By applying the fundamental relation β^*_H, we obtain

$$\beta^*_H(z) = \beta^*_H(x) \odot \beta^*_H(y) \tag{1}$$

and

$$\begin{aligned}\beta^*_H(z) &= \beta^*_H(y) \odot \beta^*_H(x) \odot \beta^*_H(h) = \beta^*_H(y) \odot \beta^*_H(x) \odot \omega_H \\ &= \beta^*_H(y) \odot \beta^*_H(x) \ (\text{since } \omega_H = 1_{H/\beta^*_H})\end{aligned} \tag{2}$$

Thus, by the equations (1) and (2) we conclude that $\beta^*_H(x) \odot \beta^*_H(y) = \beta^*_H(y) \odot \beta^*_H(x)$. Therefore, $[\beta^*_H(x), \beta^*_H(y)] = \beta^*_H(x) \odot \beta^*_H(y) \odot \beta^*_H(x)^{-1} \odot \beta^*_H(y)^{-1} = \omega_H = 1_{H/\beta^*_H}$. The proof of the the converse is similar. □

Theorem 4.10. *A cat^1-group is a cat^1-hypergroup.*

Proof. If H and C are groups, then $\omega_H = \{e\}$, $\ker^* t = \ker t$ and $\ker^* h = \ker h$. □

The following theorem and lemma are noted in [7] regarding crossed polymodules. The proof for crossed module of hypergroups is similar. A proof is included for completeness. In the proof we use the notion of semi-direct product of fundamental groups.

Theorem 4.11. *From a crossed module of hypergroups $\mathcal{X} = (C, H, \partial, \alpha)$ we can construct a cat^1-group.*

Proof. According to Theorem 3.10, we know $(C/_{\beta^*_C}, H/_{\beta^*_H}, \mathcal{D}, \psi)$ is a crossed module. Now, we can consider

$$\begin{tikzcd} H/_{\beta^*_H} \ltimes C/_{\beta^*_C} \arrow[r, shift left, "h"] \arrow[r, shift right, "t"'] & H/_{\beta^*_H} \arrow[l, bend left=60, "k"] \end{tikzcd}$$

where $h(\beta^*_H(a), \beta^*_C(c)) = \mathcal{D}(\beta^*_C(c)) \odot \beta^*_H(a)$, $t(\beta^*_H(a), \beta^*_C(c)) = \beta^*_H(a)$

and $k(\beta^*_H(a)) = (\beta^*_H(a), w_C)$.

Then $h_{|H/_{\beta^*_H}} = t_{|H/_{\beta^*_H}} = Id_H$ and $[\ker h, \ker t] = 1_{H/_{\beta^*_H} \ltimes C/_{\beta^*_C}}$.

Therefore, we obtain a cat^1-group. □

Lemma 4.12. *For a cat*[1]*-hypergroup* $\mathcal{C} = (k; t, h : H \to C)$, $H/_{\beta^*_H} \cong \ker t^* \ltimes C/_{\beta^*_C}$, *where* $t^* : H/_{\beta^*_H} \to C/_{\beta^*_C}$, $t^*(\beta^*_H(a)) = \beta^*_C(t(a))$ *and* $k^* : C/_{\beta^*_C} \to H/_{\beta^*_H}$, $k^*(\beta^*_C(c)) = \beta^*_H(k(c))$.

Proof. We define $f : H/_{\beta^*_H} \to \ker t^* \ltimes C/_{\beta^*_C}$ by $f(\beta^*_H(a)) = (k^*t^*(\beta^*_H(a)) \otimes \beta^*_H(a), t^*(\beta^*_H(a)))$ and $g : \ker t^* \ltimes C/_{\beta^*_C} \to H/_{\beta^*_H}$ by $g(\beta^*_H(a), \beta^*_C(c)) = k^*(\beta^*_H(a)) \otimes \beta^*_C(c))$. It is not difficult to see that f, g are homomorphisms and f is the inverse of g. $\square$

Note that in the previous lemma, since $\ker t^* \trianglelefteq H/_{\beta^*_H}$ and $k^*(C/_{\beta^*_C}) \leq H/_{\beta^*_H}$ there is an action of $k^*(C/_{\beta^*_C})$ on $\ker t^*$ by conjugation. Hence, the semi-direct product $\ker t^* \ltimes C/_{\beta^*_C}$ is defined. Similarly to the proof of [7, Theorem 3.6], we can prove the following theorem.

Theorem 4.13. *From a cat*[1]*-hypergroup* $\mathcal{C} = (k; t, h : H \to C)$ *we can construct a crossed module.*

5. Pullback cat¹-hypergroups

In this section, we define the notion of pullback cat[1]-hypergroups and we obtain some results in this respect. In particular, we present the universal property of induced cat[1]-hypergroups.

Definition 5.1. A pullback cat[1]-hypergroup is defined as follows:

$$\begin{tikzcd}
\iota^{\bullet\bullet}H \arrow[r, "h^{\bullet\bullet}"] \arrow[d, "t^{\bullet\bullet}"] \arrow[dr, "\pi"] & Q \arrow[l, bend right, "k^{\bullet\bullet}"'] \arrow[dr, "\iota"] & \\
Q \arrow[u, bend left, "k^{\bullet\bullet}"] \arrow[dr, "\iota"] & H \arrow[r, "h"] \arrow[d, "t"] & C \arrow[l, bend right, "k"'] \\
 & C \arrow[u, bend left, "k"] &
\end{tikzcd}$$

Let $\mathcal{C} = (k; t, h : H \to C)$ be a cat[1]-hypergroup and let $\iota : Q \to C$ be a strong homomorphism. Define $\iota^{\bullet\bullet}\mathcal{C} = (k^{\bullet\bullet}; t^{\bullet\bullet}, h^{\bullet\bullet} : \iota^{\bullet\bullet}H \to Q)$ to be the pullback of H, where $\iota^{\bullet\bullet}H = \{(q_1, a, q_2) \in Q \times H \times Q \mid \iota(q_1) = t(a),\ \iota(q_2) = h(a)\}$, $t^{\bullet\bullet}(q_1, p, q_2) = q_1$, $h^{\bullet\bullet}(q_1, p, q_2) = q_2$ and $k^{\bullet\bullet}(q) = (q, k\iota(q), q)$. Multiplication in $\iota^{\bullet\bullet}P$ is componentwise. The pair (π, ι) is a morphism of cat[1]-hypergroups, where $\pi : \iota^{\bullet\bullet}H \to H,\ (q_1, a, q_2) \mapsto a$.

Theorem 5.2. *By a pullback cat*[1]*-hypergroup, we have a cat*[1]*-hypergroup.*

Proof. We verify the cat¹-hypergroup axioms. For the first axiom, we have

$$t^{\bullet\bullet}k^{\bullet\bullet}(q) = t^{\bullet\bullet}(q, k\iota(q), q) = q, \quad h^{\bullet\bullet}k^{\bullet\bullet}(q) = h^{\bullet\bullet}(q, k\iota(q), q) = q.$$

Thus, $t^{\bullet\bullet}k^{\bullet\bullet} = h^{\bullet\bullet}k^{\bullet\bullet} = Id_Q$ and (CAT-H-1) is satisfied.

In order to prove the second condition, suppose that $x = (q'_1, a_1, q_1) \in \ker^* t^{\bullet\bullet}$, $y = (q_2, a_2, q'_2) \in \ker^* h^{\bullet\bullet}$. We have $[x, y] = [x, y]_r \cup [x, y]_l$, where

$$[x,y]_r = \{(q,a,q') | (q'_1,a_1,q_1) \times (q_2,a_2,q'_2) \cap (q_2,a_2,q'_2) \times (q'_1,a_1,q_1) \times (q,a,q') \neq \emptyset\}$$
$$= \{(q,a,q') | q'_1 \cdot q_2 \cap q_2 \cdot q'_1 \cdot q \neq \emptyset, a_1 \circ a_2 \cap a_2 \circ a_1 \circ a \neq \emptyset, q_1 \cdot q'_2 \cap q'_2 \cdot q_1 \cdot q' \neq \emptyset\},$$
$$[x,y]_l = \{(q,a,q') \mid (q'_1,a_1,q_1) \times (q_2,a_2,q'_2) \cap (q,a,q') \times (q_2,a_2,q'_2) \times (q'_1,a_1,q_1) \neq \emptyset\}$$
$$= \{(q,a,q') | q'_1 \cdot q_2 \cap q \cdot q_2 \cdot q'_1 \neq \emptyset, a_1 \circ a_2 \cap a \circ a_2 \circ a_1 \neq \emptyset, q_1 \cdot q'_2 \cap q' \cdot q'_2 \cdot q_1 \neq \emptyset\}$$

Suppose that (q, a, q') is an arbitrary element of $[x, y]_r$. Then, by the above equations we obtain

$$q'_1 \cdot q_2 \cap q \cdot q_2 \cdot q'_1 \neq \emptyset, \tag{3}$$
$$a_1 \circ a_2 \cap a \circ a_2 \circ a_1 \neq \emptyset,$$
$$q_1 \cdot q'_2 \cap q' \cdot q'_2 \cdot q_1 \neq \emptyset. \tag{4}$$

Similarly as in the proof of Proposition 4.9, from the equations (3) and (4) we conclude that $[\beta^*_Q(q'_1), \beta^*_Q(q_2)] = 1_{Q/\beta^*_Q}$ and $[\beta^*_Q(q_1), \beta^*_Q(q'_2)] = 1_{Q/\beta^*_Q}$. This implies that $[q'_1, q_2] \subseteq \omega_Q$ and $[q_1, q'_2] \subseteq \omega_Q$. On the other hand, by the definition of $\iota^{\bullet\bullet}$, we obtain $\iota(q'_1) = t(a_1) \in \iota(\omega_Q)$ and $\iota(q'_2) = h(a_2) \in \iota(\omega_Q)$. Now, according to Lemma 4.5, we have $\iota(\omega_Q) \subseteq \omega_C$. Hence, $t(a_1) \in \omega_C$ and $h(a_2) \in \omega_C$. Thus, $a_1 \in \ker^* t$ and $a_2 \in \ker^* h$. Now, we obtain $[x, y]_r \subseteq \omega_Q \times \omega_Q \times \omega_Q$. In a similar way, we obtain $[x, y]_l \subseteq \omega_Q \times \omega_Q \times \omega_Q$. Therefore, (CAT-H-2) is satisfied, too. □

The universal property of induced cat¹-hypergroup is the following.

COROLLARY 5.3. *Let $\mathcal{C} = (k; t, h : H \to C)$ be a cat¹-hypergroup and let $\iota^{\bullet\bullet}\mathcal{C} = (k^{\bullet\bullet}; t^{\bullet\bullet}, h^{\bullet\bullet} : \iota^{\bullet\bullet}H \to Q)$ be induced by the strong homomorphism $\iota : Q \to C$. The corresponding diagram is given as follows:*

$$\begin{tikzcd}
G \arrow[drr, bend left, "\psi"] \arrow[dr, dotted, "\psi'"] \arrow[ddr, shift left, Rightarrow, "t'"] \arrow[ddr, shift right, Rightarrow, "h'"'] & & \\
& \iota^{\bullet\bullet}H \arrow[r, "\pi"] \arrow[d, shift right, Rightarrow, "h^{\bullet\bullet}"'] \arrow[d, shift left, Rightarrow, "t^{\bullet\bullet}"] & H \arrow[d, shift right, Rightarrow, "h"'] \arrow[d, shift left, Rightarrow, "t"] \\
& Q \arrow[r, "\iota"'] & C
\end{tikzcd}$$

The pair (π, ι) is a morphism of cat¹-hypergroups such that for any cat¹-hypergroups $\mathcal{H} = (k'; t', h' : G \to Q)$ and any morphism of cat¹-hypergroups $(\psi, \iota) : \mathcal{C} \to \mathcal{H}$ there is a unique morphism $((\psi', 1) : \iota^{\bullet\bullet}\mathcal{C} \to \mathcal{H}))$ of cat¹-hypergroups such that $\pi\psi' = \psi$.

The proof of the following theorem is similar to the proof of [7, Theorem 4.3].

THEOREM 5.4. *If $\iota^{\bullet}\mathcal{X}$ is the pullback of the crossed module of hypergroups $\mathcal{X}$ over $\iota : Q \to H$ and if $\mathcal{A}, \mathcal{B}$ are the cat¹-groups obtained from $\mathcal{X}, \iota^{\bullet}\mathcal{X}$ respectively, then $\mathcal{B} \cong \iota^{**}\mathcal{A}$.*

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Department of Mathematics, Yazd University, Yazd, Iran

E-mail: davvaz@yazd.ac.ir

Department of Mathematics, College of Engineering and Technology, American University of the Middle East, Kuwait

E-mail: Murat.alp@aum.edu.kw