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# AN EXISTENCE RESULT FOR A CLASS OF $p$-BIHARMONIC PROBLEM INVOLVING CRITICAL NONLINEARITY

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**Abstract**. This paper is concerned with the following elliptic equation with Hardy potential and critical Sobolev exponent

$$\Delta(|\Delta u|^{p-2}\Delta u) - \lambda\frac{|u|^{p-2}u}{|x|^{2p}} = \mu h(x)|u|^{q-2}u + |u|^{p^*-2}u \quad \text{in } \Omega, \quad u \in W_0^{2,p}(\Omega).$$

By means of the variational approach, we prove that the above problem admits a nontrivial solution.

## 1. Introduction

In recent years, a large number of papers have dealt with the existence of solutions of nonlinear problems involving Sobolev critical and Hardy exponents. See [4,6,13,16,18] and the references therein.

The importance of $p$-biharmonic operator has been recognized by several authors, see, e.g., [7, 11]. Furthermore, this type of equation furnishes a model for studying traveling waves in suspension bridges [14]. In [19], the authors considered a $p$-biharmonic problem involving the Hardy term, and they proved the existence of infinitely many solutions for their problem. In the same spirit, the authors in [10] were interested in the existence of solutions for this type of singular elliptic problems. When $p = 2$, this kind of the problem was studied by several authors, we quote [2, 12, 17].

We cannot apply the standard variational arguments directly, because of the lack of compactness of the inclusion of $W^{2,p}(\Omega)$ into $L^{p^*}(\Omega)$, i.e., in general, the Palais-Smale condition is not satisfied.

In this note, we consider the problem

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u) - \lambda\frac{|u|^{p-2}u}{|x|^{2p}} = \mu h(x)|u|^{q-2}u + |u|^{p^*-2}u \quad \text{in } \Omega,\\ u \in W_0^{2,p}(\Omega),\end{gathered} \tag{1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p^* = Np/(N-2p)$ is the critical Sobolev exponent, $1 < p < \frac{N}{2}$, $N \geq 5$ and $h \in L^{p^*/[p^*-q]}(\Omega)$, $\lambda^* = [N(p-1)(N-2p)/p^2]^p > \lambda \geq 0$, $\mu \geq 0$.

Let $L^s(\Omega)$ be the Lebesque space equipped with the norm $|u|_s = \left(\int_\Omega |u|^s\,dx\right)^{1/s}$, $1 \leq s < \infty$ and let $W_0^{2,p}(\Omega)$ be the usual Sobolev space with respect to the norm $\|u\| = \left(\int_\Omega |\Delta u|^p\,dx\right)^{1/p}$.

Define the constant $S_\lambda = \inf_{u \in W_0^{2,p}(\Omega)} \frac{\int_\Omega |\Delta u|^p\,dx - \lambda \int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx}{|u|_{p^*}^p}$, with $\lambda \in [0, \lambda^*)$.

By the Hardy-Rellich inequality (see [17]), we denote the norm

$$\|u\|_1 = \left(\int_\Omega (|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}})\,dx\right)^{\frac{1}{p}},$$

which is equivalent to the standard norm $\|\cdot\|$, for $0 \leq \lambda < \lambda^*$.

Let $u \in W_0^{2,p}(\Omega)$ be a weak solution of (1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v\,dx - \lambda \int_\Omega \frac{|u|^{p-2}uv}{|x|^{2p}}\,dx - \mu \int_\Omega h(x)|u|^{q-2}uv\,dx - \int_\Omega |u|^{p^*-2}uv\,dx = 0,$$

$\forall v \in W_0^{2,p}(\Omega)$.

Now we state the main results:

Theorem 1.1. *Assume that $q \in (p, p^*)$ and $h$ is a nonnegative function with $h \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $\lambda^* > \lambda \geq 0$. Then there exists $\mu^* > 0$ such that the problem (1) has a nontrivial solution when $\mu \geq \mu^*$.*

In the sequel, one takes $h \not\equiv 0$, especially $h \equiv 1$.

Theorem 1.2. *Assume that $q < p$ and $\lambda^* > \lambda \geq 0$. Then there exists $\mu^* > 0$ such that the problem (1) has a nontrivial solution when $\mu \in (0, \mu^*)$.*

## 2. Proof of the result

To show the existence of solution, we shall use the Mountain Pass Theorem [3].

We consider the energy functional associated to the problem (1),

$$\phi(u) = \frac{1}{p}\left(\int_\Omega |\Delta u|^p\,dx\right) - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx - \frac{\mu}{q}\int_\Omega h(x)|u|^q\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx. \quad (2)$$

It is well known that the functional $\phi \in C^1(W_0^{2,p}(\Omega), \mathbb{R})$ and for any $\varphi \in W_0^{2,p}(\Omega)$, there holds

$$\phi'(u)\cdot\varphi = \left(\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\,dx\right) - \lambda \int_\Omega \frac{|u|^{p-2}u\varphi}{|x|^{2p}}\,dx - \mu \int_\Omega h(x)|u|^{q-2}u\varphi\,dx - \int_\Omega |u|^{p^*-2}u\varphi\,dx. \quad (3)$$

Lemma 2.1. *Under the assumptions of Theorem 1.1 we have the following assertions:*
*(i) There exist two positive constants $r$ and $\rho$, such that $\phi(u) \geq r$ for $\|u\| = \rho$.*
*(ii) There is $e \in W_0^{2,p}(\Omega)$ with $\phi(e) < 0$ and $\|e\| > 0$.*

*Proof.* (i) From the formula for $\phi$, there exist positive constants $C_0, C_1, C_2$ and $C_3$, such that

$$\begin{aligned}\phi(u) &\geq \frac{1}{p}\int_\Omega |\Delta u|^p\,dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx - C_0|h|_{\frac{p^*}{p^*-q}}\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{q}{p^*}} - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx\\ &\geq \frac{1}{p}\int_\Omega |\Delta u|^p\,dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx - C_0|h|_{\frac{p^*}{p^*-q}}\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{q}{p^*}} - \frac{1}{p^*}S_0^{\frac{-p^*}{p}}\|u\|^{p^*}\\ &\geq C_1\|u\|_1^p - C_2\|u\|_1^{p^*} - C_3\|u\|_1^q.\end{aligned}$$

Since $q \in (p, p^*)$ then for $\rho > 0$ sufficiently small, we may find $r > 0$ such that $\inf_{\|u\|=\rho}\phi(u) \geq r > 0$.

(ii) Taking $\omega \in C_0^\infty(\Omega)$, then for $t > 0$

$$\phi(t\omega) \leq \frac{t^p}{p}\|u\|^p - \frac{t^{p^*}}{p^*}\int_\Omega |\omega|^{p^*} - \mu t^q\int_\Omega h(x)|\omega|^q\,dx \to -\infty,$$

when $t \to \infty$. □

Lemma 2.2. *If $(u_n)_n$ is a Palais-Smale sequence $(PS)_c$ of the functional $\phi$, then $(u_n)_n$ is bounded and the functional $\phi$ satisfies $(PS)_c$ condition provided $c < \frac{1}{N}S^{\frac{N}{2p}}$.*

*Proof.* From the hypothesis, $(u_n)_n$ is bounded in $W_0^{2,p}(\Omega)$. In fact, from (2) and (3)

$$\phi(u_n) = c + o(1), \tag{4}$$

and

$$\phi'(u_n).u_n = o(1)\|u_n\|. \tag{5}$$

Combining (4) with (5) we get

$$o(1)(1 + \|u_n\|) + c \geq \phi(u_n) - \frac{1}{p^*}\phi'(u_n).u_n \geq (\frac{1}{p} - \frac{1}{p^*})\|u_n\|^p - C\|u_n\|^q.$$

It shows that $(u_n)_n$ is bounded in $W_0^{2,p}(\Omega)$. Therefore, there exists a subsequence, denoted also by $(u_n)_n$, satisfying

$$u_n \rightharpoonup u, \text{ in } W_0^{2,p}(\Omega), \qquad \frac{|u_n|^{p-2}u_n}{|x|^{2p}} \rightharpoonup \frac{|u|^{p-2}u}{|x|^{2p}}, \text{ in } L^p(\Omega),$$

$$|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u, \text{ in } L^{p^*}(\Omega), \qquad u_n \to u, \text{ a.e.in. } \Omega.$$

Furthermore, $u_n \to u$, in $L^q(\Omega)$, so by the Lebesgue dominated convergence theorem,

$$\int_\Omega h(x)|u_n|^q\,dx \to \int_\Omega h(x)|u|^q\,dx. \tag{6}$$

A standard argument shows that the weak limit $u$ of $(u_n)_n$ is a critical point of $\phi$ and then $\phi'(u) = 0$.

Meanwhile, let $\omega_n = u_n - u$. Then by Brezis-Lieb lemma in [5] we get

$$\|\omega_n\|^p = \|u_n\|^p + \|u\|^p + o_n(1), \tag{7}$$

$$|\omega_n|_{p^*}^{p^*} = |u_n|_{p^*}^{p^*} - |u|_{p^*}^{p^*} + o_n(1), \tag{8}$$

$$\int_\Omega \frac{|u_n|^p}{|x|^{2p}}\,dx = \int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx + \int_\Omega \frac{|\omega_n|^p}{|x|^{2p}}\,dx + o_n(1).$$

From (7), (8) and (6) we have

$$\|\omega_n\|^p = |\omega_n|_{p^*}^{p^*} + o_n(1) \tag{9}$$

and

$$\frac{1}{p}\|\omega_n\|^p - \frac{1}{p^*}|\omega_n|_{p^*}^{p^*} = c - \phi(u) + o_n(1). \tag{10}$$

In view of the boundedness of $(u_n)_n$ in $W_0^{2,p}(\Omega)$ we may assume that there exists $l \geq 0$ with

$$\|\omega_n\|^p \to l. \tag{11}$$

It follows from (9) and (11) that

$$|\omega_n|_{p^*}^{p^*} \to l, \tag{12}$$

and using the definition of $S_\lambda$, we have $\|\omega_n\|^p \geq S_\lambda \left(|\omega_n|^{p^*}\right)^{\frac{p}{p^*}}$; so we infer that $l \geq S_\lambda l^{\frac{p}{p^*}}$, and thus we claim that $l = 0$. Indeed, if $l > 0$ from the previous inequality we have $l \geq S_\lambda^{\frac{N}{2p}}$. From (10), (11) and (12), we have $\phi(u) + \frac{l}{N} = c < \frac{1}{N}S_\lambda^{\frac{N}{2p}}$, which implies that $\phi(u) < 0$.

Meanwhile, we know that $\phi'(u)\cdot\varphi = 0$, $\forall \varphi \in W_0^{2,p}(\Omega)$, hence

$$\|u\|_1^p = \mu \int_\Omega h(x)|u|^q\,dx - \int_\Omega |u|^{p^*}\,dx,$$

so it follows that

$$\phi(u) = \frac{1}{p}\|u\|_1^p - \frac{\mu}{q}\int_\Omega h(x)|u|^q\,dx - \int_\Omega \frac{1}{p^*}|u|^{p^*}\,dx \geq 0.$$

On the other hand, the assumption $c < \frac{1}{N}S_\lambda^{\frac{N}{2p}}$ implies that $\phi(u) < 0$.

This contradicts $\phi(u) \geq 0$. Hence $l = 0$ and it yields $u_n \to u$ in $W_0^{2,p}(\Omega)$. □

*Proof* (of Theorem 1.1). We will use the Mountain Pass Lemma to prove the existence of a solution for the problem (1). In our case, we have already checked the mountain pass geometry conditions included in Lemma 2.1. It remains to prove that $c < \frac{p}{N}S^{\frac{N}{2p}}$.

We choose $\omega \in C_0^\infty(\Omega)$ such that $|\omega|_{p^*} = 1$, $\lim_{t\to\infty}\phi(t\omega) = -\infty$, and thus $\sup_{t\geq 0}\phi(t\omega) = \phi(t_\mu\omega)$, for some $t_\mu > 0$. Further, $t_\mu$ satisfies

$$t_\mu^{p-1}\int_\Omega \left(|\Delta\omega|^p - \lambda\frac{|\omega|^p}{|x|^{2p}}\right)dx - t_\mu^{p^*-1}\int_\Omega |\omega|^{p^*}\,dx - \mu t_\mu^{q-1}\int_\Omega h(x)|\omega|^q\,dx = 0, \tag{13}$$

and so

$$t_\mu^{q-1}\left(t_\mu^{p-q}\int_\Omega \left(|\Delta\omega|^p - \frac{|\omega|^p}{|x|^{2p}}\right)dx - t_\mu^{p^*-q} - \mu\int_\Omega h(x)|\omega|^q\,dx\right) = 0.$$

Since $-t_\mu^{p^*-q} - \mu\int_\Omega h(x)|\omega|^q\,dx \to -\infty$ as $\mu \to \infty$, we have $t_\mu \to 0$ as $\mu \to \infty$. From the continuity of the functional $\phi$ we entail that $\sup_{t\geq 0}\phi(t\omega) \to 0$ as $\mu \to \infty$; so we may find $\mu^*$ such that for every $\mu \geq \mu^*$, we have $\sup_{t\geq 0}\phi(t\omega) < \frac{1}{N}S^{\frac{N}{2p}}$.

Putting $v = t\omega$, we have, for $t$ large enough, that $\phi(v) < 0$. By the definition of the minimax value in the Mountain Pass Lemma, if we take $\alpha(t) = tv$, then $c \leq \sup_{t\geq 0}\phi(tv) < \frac{1}{N}S^{\frac{N}{2p}}$. □

Remark 2.3. (i) In view of the Ekeland variational principle [8], we can prove that there exists a $(P.S)_c$ sequence $(u_n)_n \subset \overline{B_\rho(0)}$ with $c = \inf_{\overline{B_\rho(0)}} \phi < 0$. Hence we obtain a second solution of the problem (1).
(ii) Under the same conditions of Theorem 1.1, it is possible to prove the analogous result for the problem

$$\Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = \mu h(x)|u|^{q-2}u + |u|^{p^*-2}u + g \quad \text{in } \Omega,$$
$$u = \nabla u = 0 \quad \text{on } \partial\Omega,$$

$\lambda, \mu > 0$ and $g$ is small enough in the norm of $(W_0^{2,p}(\Omega))^*$. With this in mind, the proof is an adaptation of the above argument.

Lemma 2.4. *There exist $\mu^* > 0$, $\rho > 0$ and $r > 0$ such that for all $\mu \in (0, \mu^*)$ we have $\phi(u) \geq r > 0$, for $\|u\| = r$.*

*Proof.* From the Hölder's inequality and the compact embedding theorem, we have

$$\begin{aligned}\phi(u) &\geq \frac{1}{p}\int_\Omega |\Delta u|^p\,dx - \lambda \int_\Omega \frac{|u|^p}{p|x|^{2p}}\,dx - \frac{\mu}{q}\int_\Omega |u|^q\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx \\ &\geq C_1\|u\|_1^p - \frac{C_2\mu}{q}\|u\|^q - \frac{1}{p^* S_\lambda^{\frac{p^*}{p}}}\|u\|^{p^*} \\ &\geq C_3\|u\|^p - \frac{C_2\mu}{q}\|u\|^q - \frac{1}{p^* S_\lambda^{\frac{p^*}{p}}}\|u\|^{p^*}, \qquad (14)\end{aligned}$$

with $C_1, C_2, C_3 > 0$. Since $q < p$ then for $\|u\| = \rho > 0$, we may find $r > 0$ where $\inf_{\|u\|=\rho} \phi(u) \geq r > 0$. □

Lemma 2.5. *The weak limit $u_*$ of $(u_n)_n$ is a nontrivial solution to* (1) *for $\mu \in (0, \mu^*)$.*

*Proof.* It is clear that the functional $\phi$ is bounded from below in $\overline{B}_\rho(0) = \{u \in W_0^{1,p}(\Omega) \;:\; \|u\| \leq \rho\}$, with $\rho > 0$ given by Lemma 2.1. Hence, using the Ekeland's variational principle [4] with distance $d(u, v) = \|u - v\|$, a standard argument (see for instance [13]) shows the existence of a $(P.S)_{\widetilde{c}}$ sequence $(u_n)_n \subset \overline{B}_\rho(0)$ satisfying $\widetilde{c} = \inf_{\overline{B}_\rho(0)} \phi$. Moreover, $\widetilde{c} = \inf_{\overline{B}_{r(0)}} \phi < 0$ and

$$\widetilde{c} + o(1) = \phi(u_n) \geq C_1\|u_n\|_1^p - C_2\|u_n\|_1^{p^*} - C_3\|u_n\|_1^q.$$

Therefore, $C_2\|u\|_1^{p^*} + C_3\|u\|_1^q > -\widetilde{c} > 0$ and $u_* \neq 0$.

On the other hand,

$$\|u_n\|^p \int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\right)(\Delta u_n - \Delta u)\,dx =$$

$$\phi'(u_n)\cdot(u_n - u) + \mu\int_\Omega |u_n|^{q-2}u_n(u_n - u)\,dx =$$

$$\int_\Omega |u_n|^{p^*-2}u_n(u_n - u)\,dx - \|u_n\|^p \int_\Omega |\Delta u|^{p-2}\Delta u\,(\Delta u_n - \Delta u)\,dx.$$

In view of $u_n \rightharpoonup u$, arguing as in Leray-Lions [15] and in [9], it yields $\Delta u_n(x) \to \Delta u(x)$ a.e $x \in \Omega$, and $u_n(x) \to u(x)$ a.e. in $\Omega$. Then

$$\|u_n\|^p \int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\right)(\Delta u_n - \Delta u)\, dx \to 0.$$

Using the following inequalities

$$|x-y|^\gamma \le 2^\gamma(|x|^{\gamma-2}x - |y|^{\gamma-2}y).(x-y) \quad \text{if} \quad \gamma \ge 2,$$

$$|x-y|^2 \le \frac{1}{\gamma-1}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y).(x-y) \quad \text{if} \quad 1<\gamma<2,$$

$\forall x, y \in \mathbb{R}^N$, where $x.y$ is the inner product in $\mathbb{R}^N$, we get

$$\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\right)(\Delta u_n - \Delta u)\, dx \to 0.$$

Consequently, $\|u_n - u\| \to 0$, which implies that $u_n \to u$ *in* $W_0^{2,p}(\Omega)$. □

*Proof* (of Theorem 1.2). Theorem 1.2 is a direct corollary of Lemma 2.4 and 2.5. □

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References

**[1]** C.O. Alves, *Multiple positive solutions for equations involving critical Sobolev exponent in* $\mathbb{R}^N$, Electron. J. Differential Equations **13** (1997), 1–10.

**[2]** C.O. Alves, J. M. do Ó, *Positive solutions of a fourth-order semilinear problem involving critical growth*, Adv. Nonlinear Stud. **2** (2002), 437–458.

**[3]** J.P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

**[4]** J.G. Azorero, I.P. Alonso, *Mumtiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Am. Math. Soc. **323(2)** (1991), 877–895.

**[5]** H. Brezis, E. Lieb, *A relation between point wise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

**[6]** H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*. Comm. Pure Appl. Math. **36** (1983), 437–477.

**[7]** P. Drábek, *Solvability and bifurcations of nonlinear equations*, Pitman Res. Notes Math. Ser. **264**, Longman Scientific and Technical, Harlow, 1992, copublished in the United States with John Wiley and Sons, New York, 1992.

**[8]** I. Ekeland, *On the variational principle*, J. Math. Anal. App. **47** (1974), 324–353.

**[9]** A. El Hamidi, J.M. Rakotoson, *Compactness and quasilinear problems with critical exponents*, Diff. Int. Equ. **18** (2005), 1201–1220.

**[10]** R. Filippucci, P. Pucci, F. Robert, *On a p-Laplace equation with multiple critical nonlinearities*, J. Math. Pures Appl. **91** (2009), 156–177.

**[11]** S. Fučik, A. Kufner, *Nonlinear Differential Equations*, Elsevier, Amsterdam-Oxford-New York, 1980.

**[12]** F. Gazzola, H-Ch. Grunau, *Radial entire solutions for supercritical biharmonic equations*, Math. Ann. **334** (2006), 905–936.

**[13]** D. Kang and S. Peng, *Positive solutions for singular critical elliptic problems*, Appl. Math. Lett. **17(4)** (2004), 411–416.

**[14]** A. Lazer, P. McKenna, *Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. **32** (1990), 537–578 .

**[15]** J. Leray, J.L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, **93** (1965), 97–107.

**[16]** H. Liu, *Multiple positive solutions for a semilinear elliptic equation with critical Sobolev exponent*, J. Math. Anal. Appl. **354** (2009), 451–458.

**[17]** E. Mitidieri, *A simple approach to Hardy inequalities*, Mat. Zametki **67(4)** (2000), 563–572. translation in Math. Notes 67 (2000), no. 3-4, 479-486.

**[18]** A. Ourraoui, *On a p-Kirchhoff problem involving a critical nonlinearity*, C. R. Acad. Sci. Paris, Ser. I **352** (2014), 295–298.

**[19]** H. Xie, J. Wang, *Infinitely many solutions for $p-$harmonic equation with singular term*, J. Inequal. Appl. (2013), 9.

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