MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
71, 3 (2019), 196–206
September 2019

research paper
оригинални научни рад

ON LIE-YAMAGUTI COLOR ALGEBRAS

A. Nourou Issa and Patricia L. Zoungrana

Abstract. Lie-Yamaguti color algebras are defined and some examples are provided. It is shown that any Leibniz color algebra has a natural Lie-Yamaguti structure. For a given Lie-Yamaguti color algebra, an enveloping Lie color algebra is constructed and it is proved that any Lie color algebra with reductive decomposition induces a Lie-Yamaguti structure on some of its subspaces.

1. Introduction

Lie-Yamaguti algebras (first called "generalized Lie triple systems") were introduced in [18] as an algebraic treatment of tangent spaces of homogeneous spaces with invariant affine connections [12]. Lie-Yamaguti algebras were also called "Lie triple algebras" in [9] and the recent terminology is introduced in [10].

In the framework of G-graded algebras (see [4, 5, 8]), a $\mathbb{Z}_2$-graded generalization of Lie-Yamaguti algebras is considered in [13] (see also [22] for some aspects of Lie-Yamaguti superalgebras and their connections with Lie superalgebras; in [23] Killing forms and invariant forms of Lie-Yamaguti superalgebras are defined and studied). In [16] Lie color algebras (called ε-Lie algebras) were considered. However, a particular type of ε-Lie algebras were called Lie color algebras in [15]. For an earlier work on Lie color algebras, one refers to [14]. The interest of researchers in Lie color algebras increased since the work [8] on the classification of Lie superalgebras. The important role in theoretical physics played by Lie color algebras (and particularly Lie superalgebras) is well-known. Commutative color algebras were defined and used in [11] in the development of a color quantization theory.

In this paper we consider a color generalization of Lie-Yamaguti algebras that we call Lie-Yamaguti color algebras. They contain usual Lie-Yamaguti algebras and Lie-Yamaguti superalgebras as special cases.

2010 Mathematics Subject Classification: 17B55, 17B60, 17B99
Keywords and phrases: Lie color algebra; Akivis color algebra; Leibniz color algebra; Lie-Yamaguti color algebra.

In Section 2 useful basic notions are collected and the definition of our basic object is given. The Section 3 is devoted to the construction of a natural Lie-Yamaguti structure on any Leibniz color algebra and, as a specific example, we point out a Lie-Yamaguti structure on the direct sum $\mathfrak{gl}(V) \oplus V$, where V is a G-graded vector space and $\mathfrak{gl}(V)$ the Lie color algebra of endomorphisms of V. In fact, $\mathfrak{gl}(V) \oplus V$ has a Leibniz color algebra structure [20]. For an application of Leibniz color algebras one may refer to [21]. In Section 4, for a given Lie-Yamaguti color algebra T, we construct an enveloping Lie color algebra $L(T)$ containing as a subalgebra the algebra of inner derivations of T. It turns out that $L(T)$ has a reductive decomposition. Conversely, any Lie color algebra with reductive decomposition induces a Lie-Yamaguti structure on some of its subspace.

All vector spaces and algebras are finite-dimensional over a fixed ground field $\mathbb{K}$ of characteristic zero. The group of nonzero elements of $\mathbb{K}$ is denoted by $\mathbb{K}^*$ and G is an abelian group.

2. Definitions and examples

In this section we recall some useful and basic notions that could be found in [4, 5] and we define the basic object of this paper.

Definition 2.1 ([4,5]). (i) A $\mathbb{K}$-vector space V is said to be *G-graded* whenever we are given a family $(V_g)_{g\in G}$ of subspaces of V such that $V = \bigoplus_{g\in G} V_g$ (direct sum). An element $v \in V$ is said to be *homogeneous of degree* $g \in G$ if $v \in V_g$.

(ii) If $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ are two G-graded vector spaces, a linear mapping $\phi : V \to W$ is said to be *homogeneous of degree* $\delta \in G$ if $\phi(V_g) \subseteq W_{g+\delta}$ for all $g \in G$.

Definition 2.2 ([4, 5]). An algebra A is called a (binary) *G-graded algebra* if it is a G-graded vector space $A = \bigoplus_{g\in G} A_g$ and if, moreover, $A_g A_{g'} \subseteq A_{g+g'}$ for all $g, g' \in G$.

Definition 2.3 ([4, 5]). A mapping $\varepsilon : G \times G \to \mathbb{K}^*$ is called a *bicharacter on G* if the following identities:

$$\varepsilon(i, j + k) = \varepsilon(i, j)\varepsilon(i, k), \quad \varepsilon(i + j, k) = \varepsilon(i, k)\varepsilon(j, k), \quad \varepsilon(i, j)\varepsilon(j, i) = 1$$

hold for all $i, j, k \in G$.

We assume throughout this paper that ε is a fixed bicharacter on G. All elements in a graded algebra A are assumed to be homogeneous. If $x \in A_i$ $(A_i \subset A)$, then we write $\bar{x}$ for the degree i of x, $\bar{x} := i$ and for $x \in A_i$, $y \in A_j$, we set $\varepsilon(\bar{x}, \bar{y}) := \varepsilon(i, j)$.

Definition 2.4 ([14,16]). A *Lie color algebra* is a G-graded vector space $L = \bigoplus_{i\in G} L_i$ along with a bracket $[,] : L \times L \to L$ such that

(i) $[L_i, L_j] \subseteq L_{i+j}$, $\forall i, j \in G$,

(ii) $[x, y] = -\varepsilon(\bar{x}, \bar{y})[y, x]$ (*ε-skew symmetry*),

(iii) $[[x,y],z]+\varepsilon(\bar{x},\bar{y}+\bar{z})[[y,z],x]+\varepsilon(\bar{x}+\bar{y},\bar{z})[[z,x],y]=0$ (ε-*Jacobi identity*) for all $x\in L_i$, $y\in L_j$, $z\in L_k$.

Observe that the ε-Jacobi identity can be written in another equivalent form as $\varepsilon(\bar{z},\bar{x})[[x,y],z]+\varepsilon(\bar{x},\bar{y})[[y,z],x]+\varepsilon(\bar{y},\bar{z})[[z,x],y]=0$.

Lie color algebras were first called ε-*Lie algebras* (see [16]). In [15] the ε-Lie algebras with the grading group G such that $\varepsilon(g,g)=1$ for all $g\in G$ were called *color algebras*.

Example 2.5. (i) If $G=\mathbb{Z}_2$ (the additive group of integers modulo 2) and if one defines ε as $\varepsilon(i,j):=(-1)^{ij}$ for all $i,j\in\mathbb{Z}_2$, then Lie color algebras are just *Lie superalgebras*.

(ii) If one chooses ε as $\varepsilon(i,j):=1$ for all $i,j\in G$, then a Lie color algebra is a G-graded Lie algebra [16].

(iii) Let A be any *associative* G-graded algebra and ε any bicharacter on G. If one defines on A the bracket $[,]$ as

$$[x,y]=xy-\varepsilon(\bar{x},\bar{y})yx \tag{1}$$

(the ε-commutator of x,y) for all $x\in A_i$, $y\in A_j$, $i,j\in G$, then $(A,[,])$ turns out to be a Lie color algebra [16].

Observe that Lie color algebras are examples of *nonassociative* G-graded algebras. Another variety of such algebras is the one of Leibniz color algebras [6] (see Section 3).

Definition 2.6. A G-graded vector subspace $H=\bigoplus_{i\in G}H_i$ of a Lie color algebra $L=\bigoplus_{i\in G}L_i$, where $H_i\subset L_i$, for all $i\in G$, is
(i) a *color subalgebra* of L if $[H_i,H_j]\subset H_{i+j}$ for all $i,j\in G$;

(ii) a *color ideal* of L if $[L_i,H_j]\subset H_{i+j}$ for all $i,j\in G$.

The notion of a (binary) G-graded algebra is extended to the one of a ternary G-graded algebra in [17] with the introduction of *Lie supertriple systems* (first called *G-Lie-graded triples*).

Definition 2.7. A *Lie color triple system* $(T,[,,])$ is a G-graded vector space $T=\bigoplus_{g\in G}T_g$ along with a ternary operation $[,,]:T\times T\times T\to T$ satisfying $[T_i,T_j,T_k]\subseteq T_{i+j+k}$, $i,j,k\in G$, such that, for all u,v,x,y,z in T,
(i) $[x,y,z]=-\varepsilon(\bar{x},\bar{y})[y,x,z]$,

(ii) $[x,y,z]+\varepsilon(\bar{x},\bar{y}+\bar{z})[y,z,x]+\varepsilon(\bar{x}+\bar{y},\bar{z})[z,x,y]=0$,

(iii) $[u,v,[x,y,z]]=[[u,v,x],y,z]+\varepsilon(\bar{u}+\bar{v},\bar{x},)[x,[u,v,y],z]+\varepsilon(\bar{u}+\bar{v},\bar{x}+\bar{y},)[x,y,[u,v,z]]$.

In view of [21], a Lie color triple system is a 3-Lie color algebra with additional conditions.

Example 2.8. Any Lie color algebra $(L,[,])$, $L=\bigoplus_{g\in G}L_g$, is turned into a Lie color triple system $(L,[,,])$ if one defines $[a,b,c]:=[[a,b],c]$ for all $a,b,c\in L$.

Observe that for $\varepsilon(i,j) := (-1)^{ij}$ with $i,j \in \mathbb{Z}_2$, a Lie color triple system is just a Lie super triple system.

The notion of a binary color algebra can also be extended to the one of *binary-ternary* color algebra (an example is an Akivis color algebra as defined below). First recall that *Akivis algebras* were introduced in [1,2] under the name "W-algebras" and it is shown [2] that any ordinary nonassociative algebra $(A,\cdot)$ has an Akivis algebra structure with respect to commutator "$[,]$" and associator "$as(,,)$" on A:

$$[x,y] := xy - yx, \quad as(x,y,z) := xy\cdot z - x\cdot yz$$

for all $x,y,z \in A$. Next, Akivis algebras were generalized to *Akivis superalgebras* in [3] and a color generalization is given by the following definition.

Definition 2.9. An *Akivis color algebra* $(A,[,],\{,,\})$ is a G-graded vector space $A = \bigoplus_{g\in G} A_g$ along with a binary operation $[,] : A\times A \to A$ and a ternary operation $\{,,\} : A\times A\times A \to A$ satisfying $[A_i,A_j] \subseteq A_{i+j}$, $\{A_i,A_j,A_k\} \subseteq A_{i+j+k}$, $i,j,k \in G$, such that $[x,y] = -\varepsilon(\bar{x},\bar{y})[y,x]$,

$$\begin{aligned}\varepsilon(\bar{z},\bar{x})[[x,y],z] &+ \varepsilon(\bar{x},\bar{y})[[y,z],x] + \varepsilon(\bar{y},\bar{z})[[z,x],y]\\ &= \varepsilon(\bar{z},\bar{x})\{x,y,z\} + \varepsilon(\bar{x},\bar{y})\{y,z,x\} + \varepsilon(\bar{y},\bar{z})\{z,x,y\}\\ &\quad - \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})\{y,x,z\} - \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{z})\{x,z,y\} - \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{z})\{z,y,x\} \quad (2)\end{aligned}$$

for all $x,y,z \in A$. The identity (2) is called the *ε-Akivis identity.*

Remark 2.10. (i) If $\{x,y,z\} = 0$ for all $x,y,z \in A$ in Definition 2.9 then one gets a Lie color algebra $(A,[,])$ (the identity (2) reduces to the ε-Jacobi identity).

(ii) For $\varepsilon(i,j) := (-1)^{ij}$ for all $i,j \in \mathbb{Z}_2$, an Akivis color algebra is just an Akivis superalgebra.

The construction described above is extended to the color algebra setting as it could be seen from the following.

Proposition 2.11. *Let $A = \bigoplus_{g\in G} A_g$ be a nonassociative color algebra. If one defines on A a binary operation by* (1) *and a ternary operation $\{,,\} := as(,,)$, then $(A,[,],\{,,\})$ is an Akivis color algebra.*

Proof. The ε-skew symmetry of "$[,]$" is obvious. For any $x,y,z \in A$, we have

$$\varepsilon(\bar{z},\bar{x})[[x,y],z] = \varepsilon(\bar{z},\bar{x})xy\cdot z - \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})yx\cdot z - \varepsilon(\bar{y},\bar{z})z\cdot xy + \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{z})z\cdot yx,$$

$$\varepsilon(\bar{x},\bar{y})[[y,z],x] = \varepsilon(\bar{x},\bar{y})yz\cdot x - \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{y},\bar{z})zy\cdot x - \varepsilon(\bar{z},\bar{x})x\cdot yz + \varepsilon(\bar{z},\bar{x})\varepsilon(\bar{y},\bar{z})x\cdot zy,$$

$$\varepsilon(\bar{y},\bar{z})[[z,x],y] = \varepsilon(\bar{y},\bar{z})zx\cdot y - \varepsilon(\bar{y},\bar{z})\varepsilon(\bar{z},\bar{x})xz\cdot y - \varepsilon(\bar{x},\bar{y})y\cdot zx + \varepsilon(\bar{x},\bar{y})\varepsilon(\bar{z},\bar{x})y\cdot xz.$$

Adding memberwise the equalities above and next rearranging terms in a suitable way, we come to the identity (2). □

An Akivis color algebra constructed from a given non-associative color algebra A as in Proposition 2.11 is said to be *associated to A*.

Another type of binary-ternary color algebras is the one of Lie-Yamaguti color algebras that is defined as follows.

Definition 2.12. A *Lie-Yamaguti color algebra* (*LY color algebra* for short) is a G-graded vector space $T=\bigoplus_{g\in G}T_g$ with a binary operation, denoted by juxtaposition, satisfying $T_iT_j\subseteq T_{i+j}$ and a ternary operation "$[,,]$" satisfying $[T_i,T_j,T_k]\subseteq T_{i+j+k}$, $i,j,k\in G$, such that

(LY1) $xy=-\varepsilon(\bar{x},\bar{y})yx$,

(LY2) $[x,y,z]=-\varepsilon(\bar{x},\bar{y})[y,x,z]$,

(LY3) $\varepsilon(\bar{z},\bar{x})[x,y,z]+\varepsilon(\bar{x},\bar{y})[y,z,x]+\varepsilon(\bar{y},\bar{z})[z,x,y]+\varepsilon(\bar{z},\bar{x})xy\cdot z+\varepsilon(\bar{x},\bar{y})yz\cdot x+\varepsilon(\bar{y},\bar{z})zx\cdot y=0$,

(LY4) $\varepsilon(\bar{z},\bar{x})[xy,z,u]+\varepsilon(\bar{x},\bar{y})[yz,x,u]+\varepsilon(\bar{y},\bar{z})[zx,y,u]=0$,

(LY5) $[u,v,xy]=[u,v,x]y+\varepsilon(\bar{u}+\bar{v},\bar{x})x[u,v,y]$,

(LY6) $[u,v,[x,y,z]]=[[u,v,x],y,z]+\varepsilon(\bar{u}+\bar{v},\bar{x})[x,[u,v,y],z]+\varepsilon(\bar{u}+\bar{v},\bar{x}+\bar{y})[x,y,[u,v,z]]$,
for all u,v,x,y,z in T.

Remark 2.13. If $[u,v,w]=0$ for all u,v,w in T, then T is a Lie color algebra. For $uv=0$ for all u,v in T, we get a Lie color triple system.

Example 2.14. (i) If $\varepsilon(i,j)=1$ for all i,j in G in a LY color algebra T, then T is a *G-graded LY algebra*.

(ii) Let $G=\mathbb{Z}_2$ and let $\varepsilon(i,j):=(-1)^{ij}$ for all $i,j\in\mathbb{Z}_2$. Then the LY color algebra T is just a Lie-Yamaguti superalgebra [13, 22].

(iii) Any Lie color algebra is a LY color algebra with respect to the operations $xy:=[x,y]$ and $[x,y,z]:=[[x,y],z]$.

Other examples of LY color algebras could be derived from the construction in Section 3 below.

3. Lie-Yamaguti color algebra structures on Leibniz color algebras

In this section we point out that Leibniz color algebras [6] have natural LY color algebra structures (this extends the result from [10] relating Leibniz algebras and Lie-Yamaguti algebras). For a proof of this fact, we shall use properties of the Akivis color algebra associated to a given Leibniz color algebra, as in [7] for Leibniz algebras.

Definition 3.1 ([6]). A (*left*) *Leibniz color algebra* is a G-graded vector space $L=\bigoplus_{g\in G}L_g$ with a binary operation, denoted by juxtaposition, satisfying $L_iL_j\subseteq L_{i+j}, i,j\in G$ and

$$x\,(yz)=(xy)\,z+\varepsilon(\bar{x},\bar{y})y\,(xz)\,. \quad (3)$$

The identity (3) is refered as to the *ε-Leibniz rule.* In particular, any Lie color algebra is a Leibniz color algebra.

Proposition 3.2. *Let L be a Leibniz color algebra and $(L, [,], \{,,\})$ its associated Akivis color algebra. Then, for any $x, y, z \in L$,*

$$\{x, y, z\} = -\varepsilon(\bar{x}, \bar{y}) y\,(xz)\,, \tag{4}$$

$$(xy + \varepsilon(\bar{x}, \bar{y}) yx)\, z = 0, \tag{5}$$

$$x\,[y, z] = [xy, z] + \varepsilon(\bar{x}, \bar{y})\,[y, xz]\,. \tag{6}$$

Proof. The identity (3) implies $(xy)\,z - x\,(yz) = -\varepsilon(\bar{x}, \bar{y}) y\,(xz)$ which is (4).

Next, in (3), switching x and y, we have $(yx)\,z = y\,(xz) - \varepsilon(\bar{y}, \bar{x}) x\,(yz)$, so that $\varepsilon(\bar{x}, \bar{y})(yx)z = \varepsilon(\bar{x}, \bar{y}) y\,(xz) - x\,(yz)$. Then $(xy + \varepsilon(\bar{x}, \bar{y}) yx)\, z = (xy)\, z + \varepsilon(\bar{x}, \bar{y})\,(yx)\, z = x\,(yz) - \varepsilon(\bar{x}, \bar{y}) y\,(xz) + \varepsilon(\bar{x}, \bar{y}) y\,(xz) - x\,(yz) = 0$, so we get (5).

It remains to prove (6). Again, by switching y and z in (3), we have $(xz)\,y = x\,(zy) - \varepsilon(\bar{x}, \bar{z}) z\,(xy)$ and $\varepsilon(\bar{y}, \bar{z})\,(xz)\,y = \varepsilon(\bar{y}, \bar{z}) x\,(zy) - \varepsilon(\bar{x} + \bar{y}, \bar{z}) z\,(xy)$. Hence, $(xy)\,z - \varepsilon(\bar{y}, \bar{z})\,(xz)\,y = x\,(yz) - \varepsilon(\bar{x}, \bar{y}) y\,(xz) - \varepsilon(\bar{y}, \bar{z}) x\,(zy) + \varepsilon(\bar{x} + \bar{y}, \bar{z}) z\,(xy)$ that is $x\,(yz - \varepsilon(\bar{y}, \bar{z}) zy) = (xy)\,z - \varepsilon(\bar{x} + \bar{y}, \bar{z}) z\,(xy) + \varepsilon(\bar{x}, \bar{y})\,[y\,(xz) - \varepsilon(\bar{y}, \bar{x} + \bar{z})\,(xz)\,y]$ or $x\,[y, z] = [xy, z] + \varepsilon(\bar{x}, \bar{y})\,[y, xz]$, which is (6). □

In an Akivis color algebra $(A, [,], \{,,\})$ consider the ternary product defined by $(x, y, z) = \varepsilon(\bar{x}, \bar{y})\{y, x, z\} - \{x, y, z\}$.

Proposition 3.3. *For any x, y, z in A,*

$$(x, y, z) = -\varepsilon(\bar{x}, \bar{y})\,(y, x, z)\,, \tag{7}$$

$$\begin{aligned} &[[x, y]\,, z] + \varepsilon(\bar{x}, \bar{y} + \bar{z})\,[[y, z]\,, x] + \varepsilon(\bar{x} + \bar{y}, \bar{z})\,[[z, x]\,, y] \\ &\quad + (x, y, z) + \varepsilon(\bar{x}, \bar{y} + \bar{z})\,(y, z, x) + \varepsilon(\bar{x} + \bar{y}, \bar{z})\,(z, x, y) = 0. \end{aligned} \tag{8}$$

Proof. We have, for any x, y, z in A,

$$\begin{aligned} (x, y, z) &= \varepsilon(\bar{x}, \bar{y})\{y, x, z\} - \{x, y, z\} \\ &= -\varepsilon(\bar{x}, \bar{y})\,[\varepsilon(\bar{y}, \bar{x})\{x, y, z\} - \{y, x, z\}] = -\varepsilon(\bar{x}, \bar{y})\{y, x, z\} \end{aligned}$$

and so (7) is verified.

In an Akivis color algebra $(A, [,], \{,,\})$, the identity (2) is transformed as

$$\begin{aligned} &[[x, y]\,, z] + \varepsilon(\bar{x}, \bar{y} + \bar{z})\,[[y, z]\,, x] + \varepsilon(\bar{x} + \bar{y}, \bar{z})\,[[z, x]\,, y] \\ &= \{x, y, z\} + \varepsilon(\bar{x}, \bar{y} + \bar{z})\{y, z, x\} + \varepsilon(\bar{x} + \bar{y}, \bar{z})\{z, x, y\} - \varepsilon(\bar{x}, \bar{y})\{y, x, z\} \\ &\quad - \varepsilon(\bar{y}, \bar{z})\{x, z, y\} - \varepsilon(\bar{x}, \bar{y} + \bar{z})\varepsilon(\bar{y}, \bar{z})\{z, y, x\} \\ &= -\,[\varepsilon(\bar{x}, \bar{y})\,\{y, x, z\} - \{x, y, z\}] - \varepsilon(\bar{x}, \bar{y} + \bar{z})\,[\varepsilon(\bar{y}, \bar{z})\,\{z, y, x\} - \{y, z, x\}] \\ &\quad - \varepsilon(\bar{x} + \bar{y}, \bar{z})\,[\varepsilon(\bar{z}, \bar{x})\,\{x, z, y\} - \{z, x, y\}] \\ &= -\,(x, y, z) - \varepsilon(\bar{x}, \bar{y} + \bar{z})\,(y, z, x) - \varepsilon(\bar{x} + \bar{y}, \bar{z})\,(z, x, y) \end{aligned}$$

and so (8) follows. □

Corollary 3.4. *Let L be a Leibniz color algebra and $(L, [,], \{,,\})$ its associated Akivis color algebra. Then, for any $x, y, z \in L$,*

$$(x, y, z) = -\,(xy)\,z = -\frac{1}{2}\,[x, y]\,z, \tag{9}$$

$$[[x, y]\,, z] + \varepsilon(\bar{x}, \bar{y} + \bar{z})\,[[y, z]\,, x] + \varepsilon(\bar{x} + \bar{y}, \bar{z})\,[[z, x]\,, y]$$

$$= (xy)\,z + \varepsilon(\bar{x}, \bar{y}+\bar{z})\,(yz)\,x + \varepsilon(\bar{x}+\bar{y}, \bar{z})\,(zx)\,y. \tag{10}$$

Proof. We have

$$(x,y,z) = \varepsilon(\bar{x},\bar{y})\{y,x,z\} - \{x,y,z\} \overset{(4)}{=} \varepsilon(\bar{x},\bar{y})\,(-\varepsilon(\bar{y},\bar{x})x\,(yz)) + \varepsilon(\bar{x},\bar{y})y\,(xz)$$
$$= -\left[x\,(yz) - \varepsilon(\bar{x},\bar{y})y\,(xz)\right] \overset{(3)}{=} -(xy)\,z$$
$$[x,y]\,z \overset{(1)}{=} (xy - \varepsilon(\bar{x},\bar{y})yx)\,z = (xy)\,z - \varepsilon(\bar{x},\bar{y})\,(yx)\,z \overset{(5)}{=} 2\,(xy)\,z.$$

So $(x,y,z) = -(xy)\,z = -\frac{1}{2}[x,y]\,z$, which proves (9). The identity (10) is a consequence of (8) and (9). □

COROLLARY 3.5. *Let L be a Leibniz color algebra and $(L, [,], \{,,\})$ its associated Akivis color algebra. Then, for any $a, x, y, z \in L$,*

$$a\,(x,y,z) = (ax,y,z) + \varepsilon(\bar{a},\bar{x})\,(x,ay,z) + \varepsilon(\bar{a},\bar{x}+\bar{y})\,(x,y,az)\,. \tag{11}$$

Proof. We have

$$a\,(x,y,z) \overset{(9)}{=} -a\,((xy)\,z) \overset{(3)}{=} -(a\,(xy))\,z - \varepsilon(\bar{a},\bar{x}+\bar{y})\,(xy)\,(az)$$
$$\overset{(3)}{=} -\left[(ax)\,y + \varepsilon(\bar{a},\bar{x})x\,(ay)\right]z - \varepsilon(\bar{a},\bar{x}+\bar{y})\,(xy)\,(az)$$
$$= -((ax)\,y)\,z - \varepsilon(\bar{a},\bar{x})\,(x\,(ay))\,z - \varepsilon(\bar{a},\bar{x}+\bar{y})\,(xy)\,(az)$$
$$\overset{(9)}{=} (ax,y,z) + \varepsilon(\bar{a},\bar{x})\,(x,ay,z) + \varepsilon(\bar{a},\bar{x}+\bar{y})\,(x,y,az)$$

so we get (11). □

We are now ready to prove the main result of this section.

THEOREM 3.6. *Let L be a Leibniz color algebra and $(L, [,], \{,,\})$ its associated Akivis color algebra. If define $(x,y,z) = \varepsilon(\bar{x},\bar{y})\{y,x,z\} - \{x,y,z\}$ for all $x,y,z \in L$, then $(L, [,], (,,))$ is a LY color algebra.*

Proof. (LY1) and (LY2) are obvious and (LY3) follows from (8). We have

$$\varepsilon(\bar{z},\bar{x})\,([x,y]\,,z,v) + \varepsilon(\bar{x},\bar{y})\,([y,z]\,,x,v) + \varepsilon(\bar{y},\bar{z})\,([z,x]\,,y,v)$$
$$\overset{(9)}{=} -\varepsilon(\bar{z},\bar{x})\,([x,y]\,z)\,v - \varepsilon(\bar{x},\bar{y})\,([y,z]\,x)\,v - \varepsilon(\bar{y},\bar{z})\,([z,x]\,y)\,v$$
$$\overset{(9)}{=} -2\varepsilon(\bar{z},\bar{x})\,((xy)\,z)\,v - 2\varepsilon(\bar{x},\bar{y})\,((yz)\,x)\,v - 2\varepsilon(\bar{y},\bar{z})\,((zx)\,y)\,v$$
$$\overset{(3)}{=} -2\varepsilon(\bar{z},\bar{x})\,[x\,(yz) - \varepsilon(\bar{x},\bar{y})y\,(xz)]\,v - 2\varepsilon(\bar{x},\bar{y})\,((yz)\,x)\,v - 2\varepsilon(\bar{y},\bar{z})\,((zx)\,y)\,v$$
$$\overset{(5)}{=} -2\varepsilon(\bar{z},\bar{x})[x(yz) - \varepsilon(\bar{x},\bar{y})y(xz)]v - 2\varepsilon(\bar{x},\bar{y})((yz)x)v - 2\varepsilon(\bar{y},\bar{z})(-\varepsilon(\bar{z},\bar{x})(xz)y)v$$
$$= -2\varepsilon(\bar{z},\bar{x})[x(yz) + \varepsilon(\bar{x},\bar{z})\varepsilon(\bar{x},\bar{y})(yz)x]v + 2\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})[y(xz) + \varepsilon(\bar{y},\bar{x})\varepsilon(\bar{y},\bar{z})(xz)y]v$$
$$= -2\varepsilon(\bar{z},\bar{x})[x(yz) + \varepsilon(\bar{x},\bar{y}+\bar{z})(yz)x]v + 2\varepsilon(\bar{z},\bar{x})\varepsilon(\bar{x},\bar{y})[y(xz) + \varepsilon(\bar{y},\bar{x}+\bar{z})(xz)y]v \overset{(5)}{=} 0$$

and so (LY4) holds. Next,

$$(x,y,[z,v]) \overset{(9)}{=} -\frac{1}{2}\,[x,y]\,[z,v] \overset{(6)}{=} -\frac{1}{2}\,([[x,y]\,z,v] + \varepsilon(\bar{x}+\bar{y},\bar{z})\,[z,[x,y]\,v])$$

$$\overset{(9)}{=} [(x,y,z),v] + \varepsilon(\bar{x}+\bar{y},\bar{z})\,[z,(x,y,v)],$$

which proves (LY5). Finally,

$$(x,y,(z,v,w)) \overset{(9)}{=} -(xy)\,(z,v,w)$$

$$\overset{(11)}{=} -((xy)\,z,v,w) - \varepsilon(\bar{x}+\bar{y},\bar{z})\,(z,(xy)\,v,w) - \varepsilon(\bar{x}+\bar{y},\bar{z}+\bar{v})\,(z,v,(xy)\,w)$$

$$= ((x,y,z),v,w) - \varepsilon(\bar{x}+\bar{y},\bar{z})\,(z,(x,y,v),w) + \varepsilon(\bar{x}+\bar{y},\bar{z}+\bar{v})\,(z,v,(x,y,w))$$

and so (LY6) holds. This completes the proof. □

Theorem 3.6 above is an interesting tool for constructing nontrivial examples of LY color algebras as we shall see below.

If $V = \bigoplus_{g\in G} V_g$ is a G-graded vector space, then the associative algebra $\mathrm{End}\,(V)$ is equipped with the induced G-grading $\mathrm{End}\,(V) = \bigoplus_{g\in G}\mathrm{End}\,(V)_g$, where $\mathrm{End}\,(V)_g := \{A \in \mathrm{End}\,(V) \mid A(V_\alpha) \subseteq V_{g+\alpha}\}$. By (1) we get a Lie color algebra $\mathfrak{gl}(V) := (\mathrm{End}\,(V), [,])$.

Consider the direct sum $\mathcal{E} := \mathfrak{gl}(V) \oplus V$ and define on $\mathcal{E}$ an operation "$\circ$" by $(A+x)\circ(B+y) = [A,B] + Ay$.

Proposition 3.7 ([20]). *$(\mathcal{E},\circ)$ is a Leibniz color algebra.*

Proof. Let $e_1 = A+x$, $e_2 = B+y$, $e_3 = C+z$. Since $\bar{e}_1 = \bar{A} = \bar{x}$, $\bar{e}_2 = \bar{B} = \bar{y}$ and $\mathfrak{gl}(V)$ is a Lie color algebra, we have

$$\{e_1 \circ e_2\} \circ e_3 + \varepsilon(\bar{e}_1,\bar{e}_2) e_2 \circ \{e_1 \circ e_3\} - e_1 \circ \{e_2 \circ e_3\}$$

$$= ([A,B] + Ay) + \varepsilon(\bar{A},\bar{B})(B+y) \circ ([A,C] + Az) - (A+x) \circ ([B,C] + Bz)$$

$$= [[A,B],C] - [A,[B,C]] + \varepsilon(\bar{A},\bar{B})[B,[A,C]] + ([A,B] - AB + \varepsilon(\bar{A},\bar{B})BA)z = 0,$$

so the ε-Leibniz rule (3) holds for $(\mathcal{E},\circ)$. □

Example 3.8. Define on the Leibniz color algebra $(\mathcal{E},\circ)$ the operations "$[,]$" and "$(,,)$" by $[X,Y] := X\circ Y - \varepsilon(\bar{X},\bar{Y}) Y \circ X$ (see (1)) and $(X,Y,Z) := -(X\circ Y)\circ Z$ (see (9)). Then Theorem 3.6 says that $(\mathcal{E},[,],(,,)) := (\mathfrak{gl}(V)\oplus V,[,],(,,))$ is a LY color algebra.

We observe that the construction of Example 3.8 could be extended to 3-Lie color algebras since any 3-Lie color algebra gives rise to a Leibniz color algebra [21].

In case when $G = \mathbb{Z}_2$ and $\varepsilon(i,j) = (-1)^{ij}$ for all $i,j \in \mathbb{Z}_2$, Theorem 3.6 yields the following:

Theorem 3.9. *Let $(L,\cdot)$ be a Leibniz superalgebra and $(L,[,],\{,,\})$ its associated Akivis superalgebra. If one defines $(x,y,z) := (-1)^{\bar{x}\bar{y}}\{y,x,z\} - \{x,y,z\}$ for all x,y,z in L, then $(L,[,],(,,))$ is a LY superalgebra.*

4. Enveloping Lie color algebras of Lie-Yamaguti color algebras

In this section we prove that, for each LY color algebra T, there exists an enveloping Lie color algebra $L(T)$ such that T is a subspace in $L(T)$. It turns out that $L(T)$

permits a reductive decomposition. Conversely, any Lie color algebra with reductive decomposition induces a LY color algebra structure on some of its subspace.

DEFINITION 4.1. Let $T = \bigoplus_{g\in G} T_g$ be a LY color algebra. An element $D \in \mathrm{End}\,(T)_g$ is called a *derivation of degree* g of T if, for any $x, y, z \in T$,

$$D\,(xy) = D\,(x)\,y + \varepsilon\,(g, \bar{x})\,xD\,(y)\,;$$
$$D\,([x,y,z]) = [D\,(x)\,,y,z] + \varepsilon\,(g,\bar{x})\,[x, D\,(y)\,,z] + \varepsilon\,(g, \bar{x}+\bar{y})\,[x,y,D\,(z)]\,.$$

PROPOSITION 4.2. *Let $T = \bigoplus_{g\in G} T_g$ be a LY color algebra. For any $x, y \in T$, consider the operators $D_{x,y}$ defined by*

$$D_{x,y}(z) := [x, y, z] \tag{12}$$

for all $z \in T$. Then $D_{x,y}$ are derivations of degree $\bar{x} + \bar{y}$ of T.

Proof. The fact that $D_{x,y} \in \mathrm{End}\,(T)_g$ for a given $g \in G$ follows from (12). The identities (LY5) and (LY6) imply that $D_{x,y}$ are derivations of degree $\bar{x} + \bar{y}$ of T. □

The derivations $D_{x,y}$ as defined by (12) will be called *inner derivations* of the LY color algebra T.

Denote by $D_g(T,T)$ the vector space spanned by all inner derivations of a given degree $g \in G$. Then we may consider the G-graded space $D(T,T) := \bigoplus_{g\in G} D_g(T,T)$ and we have the following color generalization of a result by K. Yamaguti [19] in the case of LY algebras.

THEOREM 4.3. *Let $T = \bigoplus_{g\in G} T_g$ be a LY color algebra. Then there exist a Lie color algebra $(L(T), [,])$ and a subalgebra $D(T,T)$ in $L(T)$ such that $L(T) = T \oplus D(T,T)$ and $[T, D(T,T)] \subseteq T$.*

Proof. For each $g \in G$ consider the external direct sum $L_g(T) := T_g \oplus D_g(T,T)$, and then we can form the G-graded space $L(T) = \bigoplus_{g\in G} L_g(T)$. Clearly, $L(T) = T \oplus D(T,T)$.

Now define on $L(T)$ a bracket "$[,]$" by setting

$$[x, y] := xy + D_{x,y}, \tag{13}$$
$$[D_{x,y}, z] := [x, y, z](= D_{x,y}(z)), \tag{14}$$
$$[D_{x,y}, D_{z,t}] := D_{[x,y,z],t} + \varepsilon(\bar{z}, \bar{x}+\bar{y})D_{z,[x,y,t]} \tag{15}$$

for all $x, y, z, t \in T$. The ε-skew symmetry of "$[,]$" as defined by (13)-(15) follows from the ε-skew symmetry (LY1) and (LY2) of the operations of T. Next, using the identities (LY3)–(LY6), one checks that

$$[[x,y],z] + \varepsilon(\bar{x}, \bar{y}+\bar{z})[[y,z],x] + \varepsilon(\bar{x}+\bar{y}, \bar{z})[[z,x],y] = 0,$$
$$[[D_{x,y}, z], t] + \varepsilon(\bar{x}+\bar{y}, \bar{z}+\bar{t})[[z,t], D_{x,y}] + \varepsilon(\bar{x}+\bar{y}+\bar{z}, \bar{t})[[t, D_{x,y}], z] = 0,$$
$$[[D_{x,y}, z], D_{u,v}] + \varepsilon(\bar{x}+\bar{y}, \bar{z}+\bar{u}+\bar{v})[[z, D_{u,v}], D_{x,y}] + \varepsilon(\bar{x}+\bar{y}+\bar{z}, \bar{u}+\bar{v})[[D_{u,v}, D_{x,y}], z] = 0,$$
$$[[D_{x,y}, D_{z,t}], D_{u,v}] + \varepsilon(\bar{x}+\bar{y}, \bar{z}+\bar{t}+\bar{u}+\bar{v})[[D_{z,t}, D_{u,v}], D_{x,y}] + \varepsilon(\bar{x}+\bar{y}+\bar{z}+\bar{t}, \bar{u}+\bar{v})[[D_{u,v}, D_{x,y}], D_{z,t}] = 0$$

for all $t, u, v, x, y, z \in T$, i.e. the ε-Jacobi identity holds for "$[,]$" and so $(L(T), [,])$ is a Lie color algebra.

By (15) we get that $D(T,T)$ is a subalgebra of $(L(T), [,])$ and (14) implies that $[T, D(T,T)] \subseteq T$. □

As for Lie algebras (see, e.g., [10]), a Lie color algebra satisfying the conditions of Theorem 4.3 will be said to have a *reductive decomposition*. It turns out that the converse of Theorem 4.3 is also true. Specifically we shall prove the following.

Theorem 4.4. *Let $\mathfrak{g}$ be a Lie color algebra with a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, where $\mathfrak{h}$ is a subalgebra and $\mathfrak{m}$ a subspace in $\mathfrak{g}$. Then $\mathfrak{m}$ has a LY color algebra structure.*

Proof. Let $(\mathfrak{g}, [,])$ be a Lie color algebra with a reductive decomposition. For any $x, y \in \mathfrak{g}$, the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ implies $[x,y] := x \cdot y + D_{x,y}$, where $x \cdot y \in \mathfrak{m}$ and $D_{x,y} \in \mathfrak{h}$ which means that "$[,]$" induces a binary operation "$\cdot$" on $\mathfrak{m}$. The ε-skew symmetry of "$[,]$" implies the one of "$\cdot$" and $D_{x,y}$. Now the inclusion $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ induces a ternary operation on $\mathfrak{m}$ if one sets $[u,v,w] := [D_{u,v}, w]$ for all $u, v, w \in \mathfrak{m}$ and the ε-skew symmetry of $D_{u,v}$ implies that $[u,v,w] = -\varepsilon(u,v)[v,u,w]$. Thus we obtain (LY1) and (LY2) for $(\mathfrak{m}, \cdot, [,,])$.

Next, considering in $(\mathfrak{g}, [,])$ the suitable ε-Jacobi identities with respect to $x \in \mathfrak{m}$ and $D_{u,v} \in \mathfrak{h}$ with $u, v \in \mathfrak{m}$, one gets the identities (LY3)-(LY6). Thus one proves the validity of the set of identities (LY1)–(LY6) in $(\mathfrak{m}, \cdot, [,,])$. □

We conclude this paper with some suggestions for further research. First, the classification problem of LY color algebras could be of interest. Next, as for some algebras (or their generalizations) studied recently by various researchers, a study of the cohomology theory for LY color algebras is topical.

Acknowledgement. The authors would like to thank the anonymous referee for valuable suggestions which helped us to improve the presentation of this paper.

References

[1] M. A. Akivis, *Three-webs of multidimensional surfaces*, (Russian), Trudy Geom. Sem., **2** (1969), 7–31.

[2] M. A. Akivis, *Local algebras of multidimensional three-webs*, Siberian Math. J., **17(1)** (1976), 3–8.

[3] H. Albuquerque, A. P. Santana, *Akivis superalgebras and speciality*, Contemporary Math., **483** (2009), 13–22.

[4] N. Bourbaki, *Algèbre, Chap. II*, Hermann, Paris, 1962.

[5] N. Bourbaki, *Algèbre, Chap. III*, Hermann, Paris, 1971.

[6] A. S. Dzumadil'daev, *Cohomologies of color Leibniz algebras. Presimplicial approach*, in: Lie Teory and its Applications in Physics III, Proceedings of the third international workshop (1999), 124–135.

[7] A. N. Issa, *Remarks on the construction of Lie-Yamaguti algebras from Leibniz algebras*, Int. J. Algebra **5(14)** (2011), 667–677.

[8] V. G. Kac, *Lie superalgebras*, Adv. Math., **26(1)** (1977), 8–96.

[9] M. Kikkawa, *Geometry of homogeneous Lie loops*, Hiroshima Math. J., **5** (1975), 141–179.

[10] M. K. Kinyon, A. Weinstein, *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*, Amer. J. Math., **123** (2001), 525–550.

[11] V. Lychagin, *Colour calculus and colour quantizations*, Acta Appl. Math., **41(1-3)** (1995), 193–226.

[12] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math., **70** (1954), 33–65.

[13] M. F. Ouédraogo, *Sur les superalgèbres triples de Lie*, Thèse de Doctorat de 3eme Cycle Mathématiques, Université de Ouagadougou, Ouagadougou, Burkina-Faso, 1999.

[14] R. Ree, *Generalized Lie elements*, Canadian J. Math., **12** (1960), 493–502.

[15] V. Rittenberg, D. Wyler, *Generalized superalgebras*, Nucl. Phys., **B 139(3)** (1978), 189–202.

[16] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys., **20(4)** (1979), 712–720.

[17] H. Tilgner, *A graded generalization of Lie triples*, J. Algebra, **47(1)** (1977), 190–196.

[18] K. Yamaguti, *On the Lie triple system and its generalization*, J. Sci. Hiroshima Univ. Ser. A **21** (1957-1958), 155–160.

[19] K. Yamaguti, *On the theory of Malcev algebras*, Kumamoto J. Sci. Ser. A, **6(1)** (1963), 9–45.

[20] T. Zhang, *Omni-Lie color algebras and Lie color* 2*-algebras*, arXiv:1305.0891v1.

[21] T. Zhang, *Cohomology and deformations of* 3*-Lie colour algebras*, Linear Multilinear A., **63**(4) (2015), 651-671.

[22] P. L. Zoungrana, *A note on Lie-Yamaguti superalgebras*, Far East J. Math. Sci., **100(1)** (2016), 1–18.

[23] P. L. Zoungrana, A. N. Issa, *On Killing forms and invariant forms of Lie-Yamaguti superalgebras*, Int. J. Math. Math. Sci., **2017** (2017), Art. ID 7842482, 9p.

(received 01.02.2018; in revised form 13.10.2018; available online 20.12.2018)

Département de Mathématiques, Université d'Abomey-Calavi, 01 BP 4521 Cotonou, Bénin

E-mail: woraniss@yahoo.fr

Département de Mathématiques de la Décision, Université Ouaga 2, 12 BP 412 Ouagadougou 12, Burkina-Faso

E-mail: patibffr@yahoo.fr