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APPROXIMATE BAHADUR EFFICIENCY OF HENZE-MEINTANIS EXPONENTIALITY TESTS WITH COMPARISON

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Abstract. In this paper, we present a class of tests proposed by Henze and Meintanis which is derived from the empirical characteristic function, and determine the asymptotic Bahadur efficiencies for two tests from the class. We compare those tests in Bahadur sense with the likelihood ratio tests and some other recent tests.

1. Introduction

Exponential distribution, besides normal distribution, is one of the commonly used distributions in practice. As a result, a multitude of goodness-of-fit tests have been developed for it. Therefore, the comparison of nonparametric tests on the basis of some quantitative characteristic becomes important in determining the proper test which is optimal for a given problem. The most widely used method for the comparison of these tests is via the calculation of their empirical powers.

Henze and Meintanis in [6] proposed a class of consistent tests for exponentiality based on a characterization involving the characteristic function. They proved that suitable test statistics have a nondegenerate limit null distribution and calculated empirical powers for two tests from the class. Those powers were compared with the empirical powers of previous tests. Obtained results indicate that Henze and Meintanis tests are serious competitors to the other tests.

Another method to compare tests is to calculate the relative efficiency between them. Suppose that Θ is the set of all continuous distribution functions on the real line. We assume that the parametric set Θ_0 consists of a single distribution function from null hypothesis and denote $\Theta_1 = \Theta\backslash\Theta_0$. Let $\{T_n\}$ and $\{V_n\}$ be two sequences of statistics based on n observations and assigned for testing the same null and alternative hypotheses, $H_0(\theta \in \Theta_0)$ and $H_1(\theta \in \Theta_1)$. Denote by $N_T(\alpha, \beta, \theta)$ the sample size necessary for the sequence T_n in order to attain the power β under

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the level α and the alternative value of parameter θ. Similary is defined $N_V(\alpha, \beta, \theta)$. The relative efficiency of the sequence $\{T_n\}$ with respect to the sequence $\{V_n\}$ is specified as the quantity $e_{T,V}(\alpha, \beta, \theta) = \frac{N_V(\alpha,\beta,\theta)}{N_T(\alpha,\beta,\theta)}$. However, explicit computation of this quantity is often difficult, or even impossible, in practice. Because of that, we make conclusions about the quality of tests based on the asymptotic relative efficiency (ARE) which is the limiting value of $e_{T,V}(\alpha, \beta, \theta)$ as $\alpha \to 0$, as $\beta \to 1$ or as $\theta \to \theta_0$, while keeping the two other parameters fixed. Depending on whether we are observing relative efficiencies for the low significant level, high powers or close alternatives, we distinguish the Bahadur, Hodges-Lehmann, and Pitman ARE, respectively. These three types are the best known types of ARE. Apart from these three methods, ARE can be obtained when two out of three parameters converge to their limiting values. For more details about this topic, we refer to [16].

The Bahadur efficiency has become very popular lately and it is often used as a measure of test quality [3,9,12,13,15,17,19,20]. One of the reasons is that it does not require the asymptotic normality of test statistics. In addition, Bahadur and Pitman efficiency very often locally coincide (see [22]).

Suppose that T_n is a sequence of test statistics whose large values are significant, i.e. the null hypothesis is rejected whenever $T_n > s$. Let the following convergence in P_θ be valid for $\theta \in \Theta_1$

$$\lim_{n\to\infty} \frac{1}{n} \log P_\theta\{T_n > s\} = -\frac{1}{2} c_T(\theta), \tag{1}$$

where $c_T(\theta)$ is the function describing the rate of exponential decrease of the attained level under the alternative called the Bahadur exact slope of the sequence $\{T_n\}$. Then $N_T(\alpha, \beta, \theta) \approx \frac{2\log\frac{1}{\alpha}}{c_T(\theta)}$, as $\alpha \to 0$. If it is not possible to find the limit value from (1), usually the approximate Bahadur efficiency is used. In that case, the exact distribution of $\{T_n\}$ is replaced by its limiting distribution and the approximate Bahadur slope can be obtained.

Let F be the limit in the distribution of a sequence of the distribution functions of the test statistic T_n, and that there exists a constant $a_T \in (0, \infty)$ so that, as $s \to \infty$, $\log(1 - F(s)) = -\frac{a_T s^2}{2}(1 + o(1))$. Suppose that there is a nonnegative function $b(\theta)$ on Θ such that $b(\theta) > 0$ for $\theta \in \Theta_1$ and $\frac{T_n}{\sqrt{n}} \to b_T(\theta)$ in P_θ-probability for $\theta \in \Theta_1$. In such case, the approximate Bahadur slope can be evaluated as (see [5])

$$c^*(\theta) = a_T b_T^2(\theta). \tag{2}$$

The approximate Bahadur ARE of a sequence of statistics $\{T_n\}$ with respect to another sequence $\{V_n\}$ is defined by $e^*_{T,V}(\theta) = \frac{c^*_T(\theta)}{c^*_V(\theta)}$.

Often the approximate Bahadur ARE is uncomputable for any alternative depending on θ, but it is possible to calculate the local approximate Bahadur ARE as $\theta \in \Theta_1$ approaches the null hypothesis, i.e. $\lim\limits_{\theta\to\theta_0} e^*_{T,V}(\theta), \theta_0 \in \Theta_0$.

The approximate and exact slopes are often locally (as $\theta \to \theta_0 \in \Theta_0$) equivalent, and in that case the approximate ARE gives a notion of the local exact ARE (see [2]).

The aim of this article is to derive the local approximate Bahadur slope of tests from [6], for various alternatives, and to compare them in Bahadur sense with other more recent tests of the exponentiality. Henze and Meintanis in [6] proved that the limit in the distribution of test statistics is an infinite linear combination of random variables with chi-square distribution. That is the reason why it is appropriate to use Bahadur method in this case.

The paper is organized as follows. In Section 2 we introduce test statistics and present their properties. In Section 3 we obtain the approximate Bahadur slope of this class of tests for different close alternatives and calculate the approximate Bahadur ARE against the likelihood ratio tests for these tests. Finally, we compare these tests with some other tests via approximate Bahadur efficiency.

2. Test statistics

Let $\psi(t) = E(\exp(itX)) = C(t) + iS(t)$ be the characteristic function of a non-negative random variable X, with real part $C(t) = E(\cos(tX))$ and imaginary part $S(t) = E(\sin(tX))$.

Theorem 2.1 ([10]). *Among all continuous non-negative random variables which possess smooth densities with a finite limit as $x \to 0^+$ and absolutely integral derivatives, the exponential law with parameter λ is the only one for which $S(t) = \lambda t C(t), t \in \mathcal{R}$ holds.*

Let $X_1, \ldots, X_n$ be the sample from distribution $F(x)$, and define the scaled data $Y_j = \frac{X_j}{\overline{X}_n}$, with $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ denoting the sample mean. For testing the null hypothesis $H_0 : F(x) = 1 - e^{-\lambda t}, \lambda > 0$, against general alternatives based on the characterization above, Henze and Meintanis in [6] proposed the test statistic

$$W_n = \int_0^\infty (s_n(t) - tc_n(t))^2 \omega(t) dt, \qquad (3)$$

where $s_n = \frac{1}{n}\sum_{j=1}^n \sin(tY_j)$ and $c_n = \frac{1}{n}\sum_{j=1}^n \cos(tY_j)$ are empirical counterparts of real and imaginary part of characteristic function $\psi(t)$ of the unit exponential distribution. Here, $\omega(\cdot)$ denotes a non-negative weight function satisfying $\int_0^\infty t^2\omega(t)dt < \infty$. We examine the efficiency of W_n for weight functions $\omega_1(t) = e^{-at}$ and $\omega_2(t) = e^{-at^2}$, where a is a positive tuning parameter.

The statistic W_n can be rewritten as

$$\begin{aligned} W_n &= \int_0^\infty (s_n(t) - tc_n(t))^2 \omega(t)\, dt = \int_0^\infty \left(\frac{1}{n}\sum_{j=1}^n \sin(tY_j) - t\frac{1}{n}\sum_{j=1}^n \cos(tY_j) \right)^2 \omega(t)\, dt \\ &= \frac{1}{n^2}\sum_{j,k=1}^n \int_0^\infty \left(\frac{e^{itY_j} - e^{-itY_j}}{2i} - t\frac{e^{itY_j} + e^{-itY_j}}{2} \right) \\ &\qquad \times \left(\frac{e^{itY_k} - e^{-itY_k}}{2i} - t\frac{e^{itY_k} + e^{-itY_k}}{2} \right) \omega(t)\, dt \end{aligned}$$

$$= \frac{1}{n^2}\sum_{j,k=1}^{n}\int_0^{\infty} g(X_j,t;\hat{\lambda})g(X_k,t;\hat{\lambda})\omega(t)\,dt = \frac{1}{n^2}\sum_{j,k=1}^{n}\Phi(X_j,X_k;\hat{\lambda})$$

where $\hat{\lambda} = \frac{1}{\overline{X}_n}$ is a consistent estimator of λ based on $X_1,\ldots,X_n$. The role of the sample mean is to scale the data and make the statistic scale invariant under the null hypothesis. Therefore we can assume that the parameter of distribution under H_0 is one.

A statistic which is represented as

$$V_n = \frac{1}{n^m}\sum_{1\le i_1,\ldots,i_m\le n}\Phi(X_{i_1},\ldots,X_{i_m}),$$

where Φ is a symmetric kernel function, is called *$V-$statistic of degree m*. If kernel satisfies $E(\Phi(X_1,X_2,\ldots,X_m)|X_1 = x) \equiv 0$, for every $x \in \mathbf{R}$, the statistic is called *degenerate*.

One can show that statistic W_n is degenerate V-statistic of degree two with an estimated parameter.

3. Approximate Bahadur efficiency

Let $\mathcal{G} = \{G(x,\theta), 0 < \theta < C\}$ be a family of alternative distribution functions such that $G(x,0)$ is exponential distribution and the regularity conditions from [17, Assumptions WD] are satisfied. The family $\mathcal{G}$ is called the family of close alternatives. The role of T_n, from the introduction, in this case, is played by the statistic $\sqrt{nW_n}$. The logarithmic tail behaviour of the limiting distribution of W_n, under the null hypothesis, is derived in the following lemma.

LEMMA 3.1. *For the statistic W_n and the given alternative density $g(x;\theta)$ from $\mathcal{G}$ the Bahadur approximate slope satisfies the relation $c^*_W(\theta) = \frac{b_W(\theta)}{\delta_1}$, where $b_W(\theta)$ is the limit in $P_\theta-$probability of W_n, and δ_1 is the largest eigenvalue of the operator A defined as $Aq(x) = \int_0^{+\infty}\Phi(x,y;1)q(y)dF(y)$, where $\Phi^*(x,y;\lambda) = \int_0^{\infty}(g(x,t;\lambda) + \mu'(t;\lambda)(x-\lambda))(g(x,t;\lambda)+\mu'(t;\lambda)(y-\lambda))\omega(t)dt$ and $\mu'(t;\lambda) = \frac{\partial E_\lambda(g(X,t;\gamma))}{\partial\lambda}|_{\gamma=\lambda}$.*

Proof. Using the result of Zolotarev [23], we have that the logarithmic tail behavior of limiting distribution function of $\widetilde{W}_n = \sqrt{nW_n}$ is $\log(1-F_{\widetilde{W}}(s)) = -\frac{s^2}{2\delta_1} + o(s^2)$, $s\to\infty$. The limit in probability of $\widetilde{W}_n/\sqrt{n}$ is $b_{\widetilde{W}}(\theta) = \sqrt{b_W(\theta)}$. By inserting the expressions into (2), we obtain the statement of the lemma. □

The limit in the probability of the sequence W_n under the close alternative from G is given in the following lemma.

LEMMA 3.2. *For a given close alternative density $g(x;\theta)$ whose distribution belongs to $\mathcal{G}$, we have that the limit in the probability of statistic W_n is*

$$b_W(\theta) = \Bigg(\int_0^{\infty}\int_0^{\infty}\Phi(x,y;1)h(x)h(y)dx\,dy + 2\mu'(0)\int_0^{\infty}\int_0^{\infty}\Phi'(x,y;1)g(x;0)h(y)dx\,dy$$

$$+\frac{(\mu'(0))^2}{2}\int_0^\infty\int_0^\infty \Phi''(x,y;1)g(x;0)g(y;0)dx\,dy\Bigg)\cdot\theta^2+o(\theta^2),\ \theta\to 0,$$

where $h(x)=\frac{\partial}{\partial\theta}g(x;0)$.

Proof. Since the sample mean converges almost surely to its expected value, by using the Law of large numbers for V-statistics with estimated parameters (see [7]), we can conclude that the limit in the probability of statistic W_n is equal to the one of the $W_n^*=\frac{1}{n^2}\sum_{k,j=1}^n\Phi(X_k,X_j;\mu(\theta))$. Without the loss of generality we may assume that $\mu(0)=1$ because the statistic is scale free under the null hypothesis. Then

$$b_W(\theta)=E_\theta(\Phi(X_1,X_2;\mu(\theta)))=\int_0^\infty\int_0^\infty\Phi(x,y;\mu(\theta))g(x;\theta)g(y;\theta)dx\,dy.$$

Due to the characterization and degeneracy of the statistic, after some calculation, we get that $b'_W(0)=0$. Since this term vanishes, it is necessary to determine the second derivative of the function. We obtain that

$$\begin{aligned}b''_W(\theta)=&\int_0^\infty\int_0^\infty\frac{\partial^2}{\partial\theta^2}\Phi(x,y;\mu(\theta))\left(\frac{\partial\mu(\theta)}{\partial\theta}\right)^2g(x;\theta)g(y;\theta)dx\,dy\\&+2\int_0^\infty\int_0^\infty\frac{\partial}{\partial\theta}\Phi(x,y;\mu(\theta))\frac{\partial\mu(\theta)}{\partial\theta}\frac{\partial(g(x;\theta)g(y;\theta))}{\partial\theta}dx\,dy\\&+\int_0^\infty\int_0^\infty\frac{\partial}{\partial\theta}\Phi(x,y;\mu(\theta))\frac{\partial^2\mu(\theta)}{\partial\theta^2}g(x;\theta)g(y;\theta)dx\,dy\\&+\int_0^\infty\int_0^\infty\Phi(x,y;\mu(\theta))\frac{\partial^2(g(x;\theta)g(y;\theta))}{\partial\theta^2}dx\,dy.\end{aligned}$$

When we restrict to $\theta=0$, we have

$$\begin{aligned}b''_W(0)=&2\int_0^\infty\int_0^\infty\Phi(x,y;1)h(x)h(y)dx\,dy+4\mu'(0)\int_0^\infty\int_0^\infty\Phi'(x,y;1)g(x;0)h(y)dx\,dy\\&+(\mu'(0))^2\int_0^\infty\int_0^\infty\Phi''(x,y;1)g(x;0)g(y;0)dx\,dy,\end{aligned}$$

where the property of kernel symmetry was used. By expanding $b_W(\theta)$ into Maclaurin series $b_W(\theta)=b_W(0)+b'_W(0)\theta+\frac{1}{2}b''_W(0)\theta^2+o(\theta^2)$, we complete the proof. □

Remark 3.3. The previous lemma shows that for statistic W_n estimated parameter influences expectation, so the result is not the same as the result of Nikitin and Peaucelle in [17] for $V-$statistic without an estimated parameter. This happens because the limiting mean function $\mu(\theta)$ of the statistic has a nonzero differential at zero.

The only value left to be determined before calculating the efficiency is the largest eigenvalue δ_1 of the operator A defined in Lemma 3.1. Since we cannot obtain it analytically, we use the method introduced in [3] to obtain the approximate value of δ_1.

Now, the local approximate Bahadur ARE may be calculated using Lemmas 3.1 and 3.2.

The exact Bahadur slopes always satisfy the Bahadur-Raghavachari inequality $c(\theta) \leq 2K(\theta)$, where $K(\theta)$ is the Kullback-Leibler information number which measures the statistical distance between the alternative and the null hypothesis. In contrast to the exact Bahadur slopes, there is no upper bound for approximate slopes. In most cases, these slopes are compared with approximate Bahadur slopes of likelihood ratio tests (see [11]). Under very general conditions the likelihood ratio test (LRT) has an approximate Bahadur slope equivalent to the double Kullback-Leibler distance between the alternative and the null hypothesis. Hence, we may consider the local approximate Bahadur ARE against the LRT. The common alternatives we are going to consider are:

(i) a Weibull distribution with density $g(x,\theta) = e^{-x^{1+\theta}}(1+\theta)x^{\theta},\ \theta > 0,\ x \geq 0;$

(ii) a Gamma distribution with density $g(x,\theta) = \frac{x^{\theta}e^{-x}}{\Gamma(\theta+1)},\ \theta > 0,\ x \geq 0;$

(iii) a Linear failure rate distribution with density $g(x,\theta) = e^{-x-\theta\frac{x^2}{2}}(1+\theta x)$, $\theta > 0$, $x \geq 0$;

(iv) a mixture of exponential distributions with negative weights (EMNW(β)) with density (see [8]) $g(x,\theta) = (1+\theta)e^{-x} - \theta\beta e^{-\beta x},\ \theta \in \left(0, \frac{1}{\beta-1}\right], x \geq 0.$

Notice that all alternatives belong to $\mathcal{G}$.

For comparison purposes we also calculate the local Bahadur ARE with respect to LTR of test W_n and of some newer, characterization based tests for the previously mentioned alternatives. We will consider the following tests:

(i) $K^{(1)}_{n,k}$ and $I^{(1)}_{n,k}$, proposed in [9] (originally denoted by $D^{(k)}_n$ and $I^{(k)}_n$, respectively);

(ii) $K^{(2)}_n$ and $I^{(2)}_n$, proposed in [12] (originally denoted by K_n and I_n, respectively);

(iii) $K^{(3)}_n$ and $I^{(3)}_n$, proposed in [15] (originally denoted by K_n and I_n, respectively);

(iv) $I^{P}_{n,a}$ and $I^{D}_{n,a}$, proposed in [13] (originally denoted by $J^{P}_{n,a}$ and $J^{D}_{n,a}$, respectively).

These tests are based on characterizations presented in [1, 18, 14, 21, 4], respectively. Some of the tests have limited normal distribution, while others converge in distribution to a supremum of a certain centered Gaussian process whose distribution is unknown. This is the prime reason to use Bahadur method in the present context. For the first six tests local Bahadur ARE against LRT were obtained, unlike the last two, where local approximate Bahadur ARE against LRT were obtained. The exact and approximate efficiencies coincide locally for all statistics we are considering in this work.

3.1 Statistic with weight function $\omega_1(t) = e^{-at}$

From the formula (3), with weight function $\omega_1(t) = e^{-at}$, we obtain $V-$statistic with estimated parameter and kernel

$$\begin{aligned}\Phi(x,y;\hat{\lambda}) =& \frac{a}{2(a^2+\hat{\lambda}^2(x-y)^2)} - \frac{a}{2(a^2+\hat{\lambda}^2(x+y)^2)} + a\hat{\lambda}^4\frac{a^2-3\hat{\lambda}^2(x-y)^2}{(a^2+\hat{\lambda}^2(x-y)^2)^3} \\ &+ a\hat{\lambda}^4\frac{a^2-3\hat{\lambda}^2(x+y)^2}{(a^2+\hat{\lambda}^2(x+y)^2)^3} - \frac{2a\hat{\lambda}(x+y)}{(a^2+\hat{\lambda}^2(x+y)^2)^2},\end{aligned}$$

where $\hat{\lambda} = \frac{1}{\bar{x}_n}$. This kernel is symmetric, its first projection is zero, and the limiting mean function of statistic has a nonzero differential at zero. We can use Lemma 3.2. Special case of W_n with this kernel is denoted with $T_{n,a}$.

In Table 1, we present the largest eigenvalues for the special values of tuning parameter a which were used by Henze and Meintanis in [6], obtained by using approximation method from [3].

a	0.5	0.75	1	1.5	2.5
δ_1	2.37	0.96	0.50	0.20	0.06

Table 1: Approximate eigenvalues

3.2 Statistic with weight function $\omega_1(t) = e^{-at^2}$

From the formula (3), with weight function $\omega_1(t) = e^{-at^2}$, we obtain $V-$statistic with estimated parameter and kernel

$$\begin{aligned}\Xi(x,y;\hat{\lambda}) = \frac{\sqrt{\pi}}{4\sqrt{a}}\Bigg(&\left(\frac{1}{2a} - \hat{\lambda}\frac{x+y}{a} - \hat{\lambda}^2\frac{(x-y)^2}{4a^2} - 1\right)e^{-\hat{\lambda}^2\frac{(x+y)^2}{4a}} \\ &+ \left(1 + \frac{1}{2a} - \hat{\lambda}^2\frac{(x-y)^2}{4a^2}\right)e^{-\hat{\lambda}^2\frac{(x-y)^2}{4a}}\Bigg),\end{aligned}$$

where $\hat{\lambda} = \frac{1}{\bar{x}_n}$. The same conditions as for statistic $T_{n,a}$ are true and we can use the same lemma. Special case of W_n with this kernel is denoted with $D_{n,a}$.

In Table 2, we present the largest eigenvalues for the special values of tuning parameter a which were used in Table 1 and obtained in the same way.

a	0.5	0.75	1	1.5	2.5
δ_1	0.50	0.31	0.22	0.13	0.07

Table 2: Approximate eigenvalues

4. Discussion

Figures 1–4 show the dependence of the local approximate Bahadur ARE with respect to LRT of statistic $T_{n,a}$ and $D_{n,a}$ on the parameter $a \in (0, 5)$. We can notice that local efficiencies are reasonable and significantly influenced by the value of the tuning parameter in both cases. For both statistics and almost all alternatives, the efficiencies increase up to a certain point and then decrease. This is not the case only for statistic $D_{n,a}$ and Gamma alternative, where the function is decreasing. The figures also illustrate that statistic $T_{n,a}$ is more efficient than $D_{n,a}$ in all cases, except for small values of a.

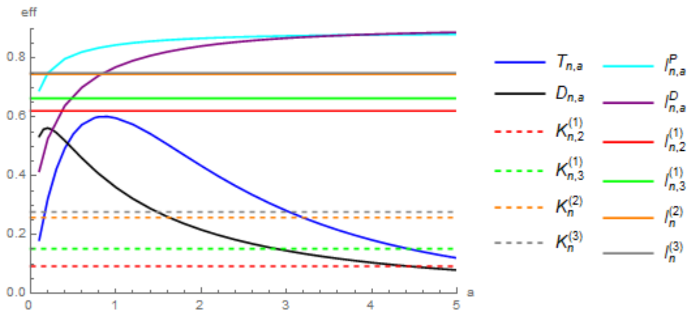

Figure 1: Local approximate Bahadur efficiencies w.r.t. LRT for a Weibull alternative

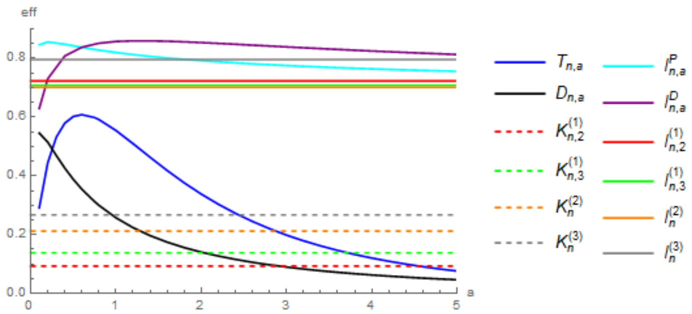

Figure 2: Local approximate Bahadur efficiencies w.r.t. LRT for a gamma alternative

Regarding the comparison with other tests from the figures, we notice that for all alternatives and some values of a, tests $T_{n,a}$ and $D_{n,a}$ have smaller local asymptotic efficiency than their competitors labeled with I_n, but larger than competitors labeled with K_n. Tests labeled with K_n are consistent, unlike the tests labeled with I_n. Hence, we may conclude that tests $T_{n,a}$ and $D_{n,a}$ have the largest local asymptotic efficiency of all consistent tests, for some values of a.

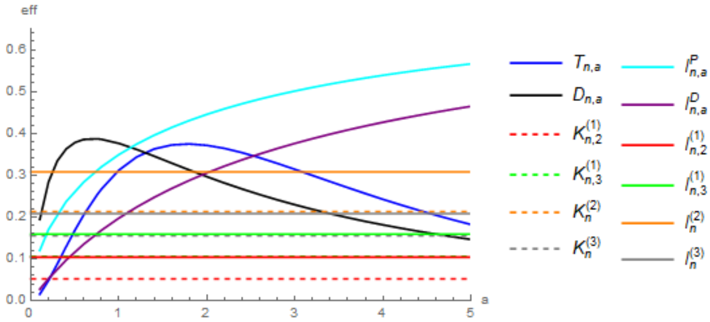

Figure 3: Local approximate Bahadur efficiencies w.r.t. LRT for a LFR alternative

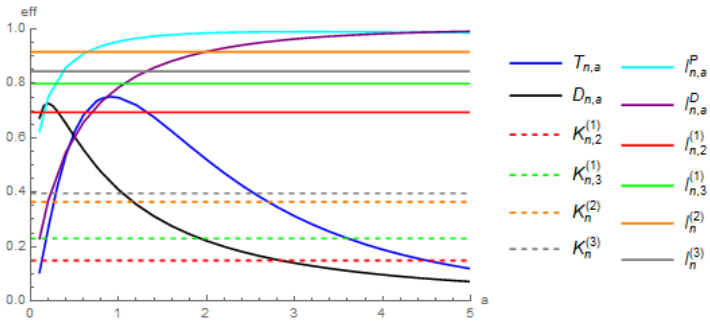

Figure 4: Local approximate Bahadur efficiencies w.r.t. LRT for an EMNW(3) alternative

Additionally, values of local approximate Bahadur ARE with respect to LRT for all tests which are used previously and for some values of a are presented in Table 3.

5. Conclusion

In theory, the ordering of tests by power is linked more closely to Hodges–Lehmann efficiency, and should not necessarily coincide with the ordering by local Bahadur efficiency. That is the reason why, apart from test power, relative Bahadur efficiency should be used as a measure of test quality. In this paper, we calculated the local approximate Bahadur ARE of two tests proposed in [6], for some choice of common alternatives. Among the two tests, we can see that the maximum values of efficiency for the same alternative are of similar value. However, there is no uniformly better test between these two tests for all alternatives and all values of a. Comparison with newer tests has shown that these tests should be taken into consideration when testing exponentiallity.

Alt.	Weibull	Gamma	LFR	EMNV(3)
$K_{n,2}^{(1)}$	0.092	0.093	0.052	0.149
$K_{n,3}^{(1)}$	0.152	0.138	0.106	0.230
$I_{n,2}^{(1)}$	0.621	0.723	0.104	0.694
$I_{n,3}^{(1)}$	0.664	0.708	0.159	0.799
$K_{n}^{(2)}$	0.258	0.212	0.213	0.364
$I_{n}^{(2)}$	0.746	0.701	0.308	0.916
$K_{n}^{(3)}$	0.277	0.267	0.155	0.396
$I_{n}^{(3)}$	0.750	0.796	0.208	0.844
$I_{n,0.5}^{P}$	0.812	0.843	0.262	0.888
$I_{n,0.75}^{P}$	0.833	0.830	0.312	0.931
$I_{n,1}^{P}$	0.846	0.820	0.349	0.954
$I_{n,1.5}^{P}$	0.860	0.804	0.405	0.976
$I_{n,2.5}^{P}$	0.873	0.783	0.476	0.989
$I_{n,0.5}^{D}$	0.674	0.826	0.117	0.608
$I_{n,0.75}^{D}$	0.733	0.849	0.160	0.716
$I_{n,1}^{D}$	0.771	0.857	0.198	0.786
$I_{n,1.5}^{D}$	0.816	0.859	0.258	0.870
$I_{n,2.5}^{D}$	0.858	0.846	0.344	0.945
$T_{n,0.5}$	0.542	0.602	0.169	0.624
$T_{n,0.75}$	0.598	0.598	0.253	0.735
$T_{n,1}$	0.601	0.554	0.312	0.748
$T_{n,1.5}$	0.526	0.440	0.368	0.651
$T_{n,2.5}$	0.350	0.258	0.348	0.404
$D_{n,0.5}$	0.490	0.388	0.378	0.626
$D_{n,0.75}$	0.419	0.312	0.387	0.555
$D_{n,1}$	0.361	0.257	0.376	0.486
$D_{n,1.5}$	0.275	0.185	0.337	0.293
$D_{n,2.5}$	0.175	0.111	0.260	0.173

Table 3: Local approximate Bahadur efficiencies w.r.t. LRT

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