MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
71, 1–2 (2019), 162–168
March 2019

research paper
оригинални научни рад

MOMENT RELATIONS OF RECORD VALUES FROM A PENG-YAN EXTENDED WEIBULL DISTRIBUTION

Zoran Vidović

Abstract. We derive new recurrence relations between the single and double moments of record values from a Peng and Yan extension of the Weibull distribution. Also, we provide a series representation of a single record moment.

1. Introduction

An extended Weibull lifetime distribution with one scale parameter and two shape parameters was proposed recently by Peng and Yan in [11]. It is suitable to model complex system lifetimes with possible upside-down bathtub shape. Its behavior can be deduced directly from a characteristic shape of the Weibull probability plot.

We will say that a random variable follows the Peng-Yan extended Weibull (PYEW) distribution if its cumulative distribution function (cdf) $F(\cdot)$ and its probability density function (pdf) $f(\cdot)$ are

$$F(x;\alpha,\beta,\lambda) = 1 - e^{-\alpha x^{\beta} e^{-\frac{\lambda}{x}}}, \tag{1}$$

and

$$f(x;\alpha,\beta,\lambda) = \alpha(\lambda+\beta x)x^{\beta-2}e^{-\frac{\lambda}{x}-\alpha x^{\beta}e^{-\frac{\lambda}{x}}}, \tag{2}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\lambda \geq 0$. Parameter α is the scale parameter, while parameters β and λ are the shape parameters.

It is straightforward to show that there exists a functional relation

$$\frac{f(x)}{1-F(x)} = \frac{\lambda+\beta x}{x^2}\{-\log(1-F(x))\}. \tag{3}$$

between the pdf and cdf given by (1) and (2), respectively.

We point out that when $\lambda = 0$, the Peng-Yan model reduces to the standard two-parameter Weibull model with the scale parameter α and the shape parameter β, when $\lambda = 0$ and $\beta = 2$ it reduces to the Rayleigh distribution, while when $\lambda = 0$ and $\beta = 1$ it reduces to the exponential distribution.

2010 Mathematics Subject Classification: 62G30, 62G99
Keywords and phrases: Recurrence relations; moments; records.

Let $\{X_i\}$ be a sequence of independent and identically distributed (iid) random variables with common distribution function $F(\cdot)$ and probability density function $f(\cdot)$. Set $Y_n = \max\{X_1, X_2, \dots, X_n\}$, $n \geq 1$. The value of X_j is an upper record value of this sequence if it exceeds all previous observations, i.e. if $X_j > Y_{j-1}$ for $j > 1$. It is supposed, by definition, that X_1 is the first upper record value. Lower records are defined similarly. The time at which records occur is also of great importance. The record time sequence $\{T_n, n \geq 1\}$ is defined as $T_1 = 1$ with probability one and $T_n = \min\{j : X_j > X_{T_{n-1}}\}$ for $n \geq 2$. The upper record value sequence is therefore defined as $R_n = X_{T_n},\ n \geq 1$.

In real-life applications such as data from weather stations, sports data, economic data, strees-strength data, records play an important role. A pioneer work of Chandler [8] invokes possible applications of record values in statistical inference, while for more information readers are commonly referred to [1] and reference therein.

Let $R_1, R_2, \dots, R_n$ denote the first n records from $F(\cdot)$. Then, the pdf of the n-th record R_n and the joint pdf of R_m and R_n, where $1 \leq m < n$, are respectively (see [1])

$$f_{R_n}(x) = \frac{\{-\ln(1-F(x))\}^{n-1}}{\Gamma(n)} f(x), \text{ for } x > 0 \text{ and } n \geq 1, \tag{4}$$

and
$$f_{R_m,R_n}(x,y) = \frac{1}{\Gamma(m)\Gamma(n-m)} \left[-\ln(1-F(y)) + \ln(1-F(x))\right]^{n-m-1}$$
$$\times \left[-\ln(1-F(x))\right]^{m-1} \frac{f(x)f(y)}{1-F(x)}, \text{ for } 0 < x < y < \infty. \tag{5}$$

In this paper we will derive certain recurrence relations for the single and product moments of record values from the PYEW model. Similar recurrence relations for the single and double moments from the Weibull, Lomax, generalized Pareto, Normal and Gumbel distributions have been established in [2–6].

The interest of this paper is reflected in cases where large order moments of record values from (1) are considered. Then we could apply recurrence relations for their evaluations. Result of these evaluations could eventually lead to a smaller computational error than in cases where numerical integration was applied. One possible application is in estimating the appropriate best linear unbiased estimators (BLUE's).

This paper is structured as follows. Recurrence relations of upper record moments from PYEW distribution with a series presentation of upper record moments are derived in Section 2. Section 3 presents some applications.

2. Moments of Peng-Yan extended Weibull records

In this section, the main results of this paper are obtained. They consist of recurrence relations for moments of upper record values and a moment representation theorem for single upper record values from a PYEW model.

The single and double moments of the upper record values are given by

$$\mu_n^{(r)} = E(R_n^r) = \int_0^\infty x^r f_{R_n}(x)\, dx, \tag{6}$$

and $$\mu_{m,n}^{(r,s)} = E(R_m^r R_n^s) = \int_0^\infty \int_0^y x^r y^s f_{R_m,R_n}(x,y)\,dx\,dy, \text{ for } 0<x<y<\infty \tag{7}$$

for $1 \le m < n$, where $f_{R_n}(\cdot)$ and $f_{R_m,R_n}(\cdot,\cdot)$ are given by (4) and (5), respectively.

Additionally, single moments of upper record values from (1) have the following series representation.

Lemma 2.1. *Let $0<t<1$ and $t_0 \in [0,1]$. The inverse function of $G(x) = \alpha x^\beta e^{-\lambda/x}$ can be represented in power series as*

$$G^{-1}(t) = \sum_{i=0}^{\infty} a_i (t-t_0)^i \tag{8}$$

for parameters $\alpha>0$, $\beta>0$ and $\lambda \ge 0$.

Proof. The proof follows directly from the application of Bürmann-Lagrange expansion and from the properties of Lambert function, see e.g. [10, p. 35–36] and [9]. □

Theorem 2.2. *Let a random variable X have pdf (2). Let $\{c_k\}$ be a sequence given by $c_0 = a_0^l$ and $c_i = (ia_0)^{-1}\sum_{j=1}^{i}(lj-i+j)a_j c_{i-j}$ for $i \ge 1$, where $\{a_k\}$ are stated as in (8). Then the l-th moment of the m-th upper record R_m is given by*

$$\mu_m^{(l)} = \sum_{i=0}^{\infty} c_i \sum_{j=0}^{i} \binom{i}{j} (-t_0)^{i-j} \frac{\Gamma(m+j)}{\Gamma(m)}.$$

Proof. Using (4), the l-th moment of R_m is given by

$$\begin{aligned}\mu_m^{(l)} &= \int_0^\infty x^l \frac{\{-\log(1-F(x;\alpha,\beta,\lambda))\}^{m-1} f(x)}{\Gamma(m)}\,dx \\ &= \frac{1}{\Gamma(m)} \int_0^\infty x^l \{\alpha x^\beta e^{-\frac{\lambda}{x}}\}^{m-1} \alpha(\lambda+\beta x) x^{\beta-2} e^{-\frac{\lambda}{x} - \alpha x^\beta e^{-\frac{\lambda}{x}}}\,dx.\end{aligned}$$

Making the transformation $t = \alpha x^\beta e^{-\frac{\lambda}{x}}$, we obtain

$$\mu_m^{(l)} = \frac{1}{\Gamma(m)} \int_0^\infty \left\{G^{-1}(t)\right\}^l t^{m-1} e^{-t}\,dt.$$

Using [15, Formula (0.314)], binomial formula and Lemma 2.1 as our main tools we get

$$\begin{aligned}\mu_m^{(l)} &= \frac{1}{\Gamma(m)} \int_0^\infty \left\{G^{-1}(t)\right\}^l t^{m-1} e^{-t}\,dt = \frac{1}{\Gamma(m)} \int_0^\infty \left\{\sum_{i=0}^{\infty} a_i (t-t_0)^i\right\}^l t^{m-1} e^{-t}\,dt \\ &= \frac{1}{\Gamma(m)} \int_0^\infty \sum_{i=0}^{\infty} c_i (t-t_0)^i t^{m-1} e^{-t}\,dt = \frac{1}{\Gamma(m)} \int_0^\infty \sum_{i=0}^{\infty} c_i \sum_{j=0}^{i} \binom{i}{j} (-t_0)^{i-j} t^j t^{m-1} e^{-t}\,dt \\ &= \sum_{i=0}^{\infty} c_i \sum_{j=0}^{i} \binom{i}{j} (-t_0)^{i-j} \frac{\Gamma(m+j)}{\Gamma(m)}.\end{aligned}$$ □

In the next two theorems we present recurrent relations for single and double moments of the upper record values.

Theorem 2.3. *For $r \geq 2$ and $n \geq 1$, the single moments of upper record values from* (1) *satisfy the recurrence relation*

$$\mu_n^{(r)} = \frac{r}{r+\beta n}\left[\frac{\lambda n}{r-1}\left(\mu_{n+1}^{(r-1)} - \mu_n^{(r-1)}\right)\right] + \frac{\beta n}{r+\beta n}\mu_{n+1}^{(r)}. \tag{9}$$

Proof. From (3) and (6), we have $\mu_n^{(r)} = \frac{1}{\Gamma(n)}(\lambda I_{r-2} + \beta I_{r-1})$, where I_r is of the form (see [12, Equation (2.6)]) $I_r = \int_0^\infty x^r(-\ln(1-F(x)))^n(1-F(x))\,dx$. Then, we obtain $\mu_n^{(r)} = \frac{\lambda n}{r-1}\left(\mu_{n+1}^{(r-1)} - \mu_n^{(r-1)}\right) + \frac{\beta n}{r}\left(\mu_{n+1}^{(r)} - \mu_n^{(r)}\right)$, from which the proof follows. □

Corollary 2.4. *Setting $\lambda = 0$ in* (9)*, we obtain the recurrence relations for single moments of upper records from a two-parameter Weibull distribution with parameters α and β as*

$$\mu_n^{(r)} = \frac{\beta n}{r+\beta n}\mu_{n+1}^{(r)}, \quad \textit{for } r \geq 2 \textit{ and } n \geq 1.$$

Corollary 2.5. *Setting $\lambda = 0$ and $\beta = 2$ in* (9)*, we obtain the recurrence relations for single moments of upper records from a Rayleigh distribution as*

$$\mu_n^{(r)} = \frac{2n}{r+2n}\mu_{n+1}^{(r)}, \quad \textit{for } r \geq 2 \textit{ and } n \geq 1.$$

Theorem 2.6. *For $m \geq 1$, $r \geq 2$ and $s \geq 1$, the double moments of upper record values from* (1) *satisfy the recurrence relation*

$$\mu_{m,m+1}^{(r,s)} = \frac{\lambda m}{r-1}\left(\mu_{m+1}^{(s+r-1)} - \mu_{m,m+1}^{(r-1,s)}\right) + \frac{\beta m}{r}\left(\mu_{m+1}^{(s+r)} - \mu_{m,m+1}^{(r,s)}\right), \tag{10}$$

and for $1 \leq m \leq n-1$, we have

$$\mu_{m,n}^{(r,s)} = \frac{\lambda m}{r-1}\left(\mu_{m+1,n}^{(r-1,s)} - \mu_{m,n}^{(r-1,s)}\right) + \frac{\beta m}{r}\left(\mu_{m+1,n}^{(r,s)} - \mu_{m,n}^{(r,s)}\right). \tag{11}$$

Proof. From (5) and (7), we have that the double moments of record values from (1) can be written as

$$\mu_{m,n}^{(r,s)} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^y y^s f(y) I(y)\,dy, \tag{12}$$

where

$$I(y) = \int_0^y x^r(-\ln(1-F(x)))^{m-1} \times (-\ln(1-F(y)) + \ln(1-F(x)))^{n-m-1}\frac{f(x)}{1-F(x)}. \tag{13}$$

Case $n = m+1$. Here (13) reduces to

$$I(y) = \lambda T_{r-2} + \beta T_{r-1}, \tag{14}$$

where T_r is of the form (see [12, Equation (2.13)])

$$T_r = \frac{y^{r+1}}{r+1}(-\ln(1-F(x)))^m - \frac{m}{r+1}\int_0^y x^{r+1}(-\ln(1-F(x)))^{m-1}\frac{f(x)}{1-F(x)}. \tag{15}$$

Using (14), (15) and (12) we obtain the desired result.

Case when $1 \leq m \leq n-1$. For this case (13) can be written as

$$I(y) = \lambda Q_{r-2} + \beta Q_{r-1}, \tag{16}$$

where Q_r is of the form (see [12, Equation (2.16)])

$$\begin{aligned} Q_r =& \frac{n-m-1}{r+1} \int_0^y x^{r+1} (-\ln(1-F(x)))^m \\ & \times (-\ln(1-F(y)) + \ln(1-F(x)))^{n-m-2} \frac{f(x)}{1-F(x)}\, dx \\ & - \frac{m}{r+1} \int_0^y x^{r+1} (-\ln(1-F(x)))^{m-1} \\ & \times (-\ln(1-F(y)) + \ln(1-F(x)))^{n-m-1} \frac{f(x)}{1-F(x)}\, dx. \end{aligned} \tag{17}$$

Using (17),(16) and (12) we obtain the result. □

COROLLARY 2.7. *For $r \geq 2$, when setting $\lambda = 0$ in* (10) *and* (11)*, we obtain the recurrence relations for double moments of upper records from a two-parameter Weibull distribution with parameters α and β as*

$$\mu_{m,m+1}^{(r,s)} = \frac{\beta m}{r} \left(\mu_{m+1}^{(s+r)} - \mu_{m,m+1}^{(r,s)} \right),$$

and for $1 \leq m \leq n-1$ and $s \geq 1$, we have

$$\mu_{m,n}^{(r,s)} = \frac{\beta m}{r} \left(\mu_{m+1,n}^{(r,s)} - \mu_{m,n}^{(r,s)} \right).$$

COROLLARY 2.8. *For $r \geq 2$, when setting $\lambda = 0$ and $\beta = 2$ in* (10) *and* (11)*, we obtain the recurrence relations for double moments of upper records from a Rayleigh distribution as*

$$\mu_{m,m+1}^{(r,s)} = \frac{2m}{r} \left(\mu_{m+1}^{(s+r)} - \mu_{m,m+1}^{(r,s)} \right),$$

and for $1 \leq m \leq n-1$ and $s \geq 1$, we have

$$\mu_{m,n}^{(r,s)} = \frac{2m}{r} \left(\mu_{m+1,n}^{(r,s)} - \mu_{m,n}^{(r,s)} \right).$$

3. Applications

The recurrence relations presented in previous section can be used for deriving the BLUE for scale parameter σ implemented in the pdf

$$f(x;\sigma,\beta,\lambda) = \frac{1}{\sigma}\left(\frac{\lambda}{\sigma} + \beta\frac{x}{\sigma}\right) \left(\frac{x}{\sigma}\right)^{\beta-2} e^{-\frac{\lambda}{\sigma}\frac{1}{\frac{x}{\sigma}} - \left(\frac{x}{\sigma}\right)^{\beta} e^{-\frac{\lambda}{\sigma}\frac{1}{\frac{x}{\sigma}}}}, \tag{18}$$

where $x > 0$, $\sigma > 0, \beta > 0$ and $\lambda \geq 0$.

Let $R_1, R_2, \ldots, R_n$ be the upper record values from (18). Then, the BLUE of σ is of the form (see [7])

$$\sigma^* = \frac{\mu^T \mathbf{\Sigma^{-1}}}{\mu^T \mathbf{\Sigma^{-1}} \mu} \mathbf{R} = \sum_{i=1}^{n} a_i R_i, \tag{19}$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_n)$, $\mu = (\mu_1^{(1)}, \mu_2^{(1)}, \ldots, \mu_n^{(1)})$ and $\mathbf{\Sigma^{-1}}$ is the inverse of the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}. \end{pmatrix}$$

The quantities $\mu_i^{(1)}$, $1 \leq i \leq n$, are defined as in previous section, while σ_{ii} denotes $Var(R_i) = \mu_i^{(2)} - (\mu_i^{(1)})^2$, for $1 \leq i \leq n$, and $\sigma_{ij} = \sigma_{ji} = Cov(R_i, R_j) = \mu_{i,j}^{(1,1)} - \mu_i^{(1)}\mu_j^{(1)}$, for $1 \leq i < j \leq n$. Accordingly, the variance of σ^* is given by $Var(\sigma^*) = \frac{\sigma^2}{\mu^T \mathbf{\Sigma^{-1}} \mu}$.

We could refer to [6, 13] for more information on this topic.

3.1 Example

Let us consider a generated sample of size 20 from (18) with parameters $(\sigma, \beta, \lambda) = (2, 1, 1)$ presented in Table 1.

Table 1: Simulated data set.

2.9232	2.2642	1.5951	1.4524	4.3710	6.6785	1.0097	2.7564	1.5543	0.8859
2.2928	5.6076	4.3055	2.1090	5.2321	2.0422	1.5649	4.6087	1.0578	1.8453

The upper record values extracted from the simulated data set are: 2.9232, 4.371 and 6.6785.

Further, for this case we have that $\mu = (2.7697, 4.8726, 6.9197)$ and

$$\Sigma = \begin{pmatrix} 4.4673 & 4.3339 & 4.2948 \\ 4.3339 & 8.3382 & 8.2494 \\ 4.2948 & 8.2494 & 12.3038 \end{pmatrix}$$

Using (19), we can obtain the BLUE for σ as

$$\sigma^* = 0.0268 \cdot 2.9232 + 0.0072 \cdot 4.371 + 0.1287 \cdot 6.6785 = 0.9693,$$

with variance $Var(\sigma^*) = 4/3.9357 = 1.0163$. This result is justified by the low precision of parameter point estimators based on records.

We refer to [6, 14] for more illustrative examples.

References

[1] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, *Records*, Wiley, 1998.

[2] N. Balakrishnan, M. Ahsanullah, *Relations for single and product moments of record values from Lomax distribution*, Sankhya Ser. B, (1994), 140–146.

[3] N. Balakrishnan, M. Ahsanullah, *Recurrence relations for single and product moments of record values from generalized Pareto distribution*, Commun. Stat. Theory., **23(10)** (1994), 2841–2852.

[4] N. Balakrishnan, M. Ahsanullah, P. S. Chan, *Relations for single and product moments of record values from Gumbel distribution*, Stat. Probabil. Lett., **15(3)** (1992), 223–227.

[5] N. Balakrishnan, P. S. Chan, *Record values from Rayleigh and Weibull distributions and associated inference*, Nist. Spec. Publ., (1994), 41.

[6] N. Balakrishnan, P. S. Chan, *On the normal record values and associated inference*, Stat. Probabil. Lett., **39(1)** (1998), 73–80.

[7] N. Balakrishnan, A. C. Cohen, *Order Statistic and Inference*, Academic Press Inc, 1991.

[8] K. N. Chandler, *The distribution and frequency of record values*, J. R. Stat. Soc. B, **14** (1952), 220–228.

[9] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, *On the Lambert W function*, Adv. Comput. Math., **5(1)** (1996), 329–359.

[10] F. B. Hildebrand, *Introduction to Numerical Analysis*, Courier Corporation, 1987.

[11] X. Peng, Z. Yan, *Estimation and application for a new extended Weibull distribution*, Reliab. Eng. Syst. Safe., **121** (2014), 34–42.

[12] K. S. Sultan, *Record values from the modified Weibull distribution and applications*, International Mathematical Forum, **41(2)** (2007), 2045–2054.

[13] K. S. Sultan, M. E. Moshref, *Record values from generalized Pareto distribution and associated inference*, Metrika, **51(2)** (2000), 105–116.

[14] Z. Vidović, *On MLEs of the parameters of a modified Weibull distribution based on record values*, J. Appl. Stat., (2018), 1–10.

[15] D. Zwillinger, *Table of Integrals, Series and Products*, Academic Press, 2007.

(received 20.09.2018; in revised form 04.12.2018; available online 15.12.2018)

Teacher Education Faculty, University of Belgrade, 11000 Belgrade, Serbia

E-mail: zoran.vidovic@uf.bg.ac.rs