MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
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ON RATIONAL PONTRYAGIN HOMOLOGY RING OF THE BASED LOOP SPACE ON A FOUR-MANIFOLD

Dragana Borović and Svjetlana Terzić

Abstract. In this paper we consider the based loop space ΩM on a simply connected manifold M. We first prove, only by means of the rational homotopy theory, that the rational homotopy type of ΩM is determined by the second Betti number $b_2(M)$. We further consider the problem of computation of the rational Pontryagin homology ring $H_*(\Omega M)$ when $b_2(M) \leq 3$. We prove that $H_*(\Omega M)$ is up to degree 5 generated by the elements of degree 1 for $b_2(M) = 3$.

1. Introduction

A based loop space ΩX of a pointed topological space X is an H - space where one of the possible multiplication is given by loop concatenation. The ring structure in $H_*(\Omega X)$ induced by loop multiplication is called Pontryagin homology ring. In this paper we consider the rational Pontryagin homology ring of the based loop space ΩM of a simply connected four-manifold M. We first show, using only the techniques of the rational homotopy theory, that the rational homotopy type for ΩM is classified by the second Betti number $b_2(M)$. We further compute the rational Pontryagin homology ring for ΩM with the small second Betti numbers $b_2(M)$. When $b_2(M) = 0$ the following well-known result holds:

LEMMA 1.1. *For $b_2(M) = 0$, the rational Pontryagin homology ring for ΩM is given by*

$$H_*(\Omega M, \mathbb{Q}) \cong T(u),\ \deg u = 3.$$

For $b_2(M) = 1$, the rational Pontryagin homology ring for ΩM is given by

$$H_*(\Omega M, \mathbb{Q}) \cong \wedge(u_1) \otimes \mathbb{Q}[u_2],\ \deg u_1 = 1,\ \deg u_2 = 4.$$

In the case $b_2(M) = 2$ or $b_2(M) = 3$ we prove the following:

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Proposition 1.2. *If $b_2(M) = 2$ the rational Pontryagin homology ring for ΩM is given by*

$$H_*(\Omega M, \mathbb{Q}) \cong T(u_1, u_2)/\langle u_1^2 = -u_2^2\rangle, \ \deg u_1 = \deg u_2 = 1.$$

Theorem 1.3. *If $b_2(M) = 3$ the rational Pontryagin homology ring for ΩM up to degree 5 is given by*

$$H_*(\Omega M, \mathbb{Q}) \cong T(u_1, u_2, u_3)/\langle u_1^2 = -u_2^2 - u_3^2\rangle,$$

where $\deg u_1 = \deg u_2 = \deg u_3 = 1$.

2. The method of the proof

The simply connected four-manifolds are known to be formal in the sense of the rational homotopy theory [5], implying that their rational homotopy is completely determined by their rational cohomology structure. In particular, the minimal models of these manifolds, as defined in rational homotopy theory, can be calculated from their rational cohomology algebras. The rational cohomology algebras of the simply connected four-manifolds with a nontrivial second Betti number, are known to be determined by their intersection forms over $\mathbb{Q}$. These forms can be diagonalized with ± 1 as the diagonal elements, implying (see [9])

$$H^*(M, \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^2 = \ldots = x_k^2 = -x_{k+1}^2 = \ldots = -x_n^2\rangle,$$

where $n = b_2(M)$ is the rank of M and $1 \leq k \leq n$. The common notation is $k = b_2^+(M)$, $n-k = b_2^-(M)$, where $\sigma = b_2^+(M) - b_2^-(M)$ is known to be the signature for M. It is the well known result of Pontryagin-Wall [4] that the homotopy type of a simply connected four-manifold is classified by its integer intersection form. Using techniques of the rational homotopy theory, it is proved in [9] that the rational homotopy type of M is determined by its rank and signature, implying that a simply connected four manifold is rationally homotopy equivalent to connected sum of b_2^+ copies of $\mathbb{C}P^2$ and σ copies of $\overline{\mathbb{C}P^2}$. Further, in [10] it is proved that the rational homotopy groups for M are determined by the second Betti number $b_2(M)$.

Here we are interested in the description of the Pontryagin homology ring for ΩM. We first prove by Theorem 3.1 that the rational homotopy type for ΩM is classified by $b_2(M)$. Therefore, in describing the rational Pontryagin homology ring for ΩM one can ignore the signature. In particular, it implies when considering four-manifolds M with $b_2(M) = 2$ or $b_2(M) = 3$ one can assume their cohomology algebra to be of the form given by Lemma 3.3 or Lemma 3.4 respectively. Then the formality condition implies that the minimal models for these manifolds are given by Lemma 3.3 and Lema 3.4 and consequently their homotopy Lie algebras by Lemma 4.4 and Proposition 4.5. The Milnor-Moore theorem implies that the Lemma 4.8 and Proposition 4.9 prove the Proposition 1.2 and Theorem 1.3 on the structure of the rational homology ring for ΩM in these cases.

Remark 2.1. Quite recently the stronger result was proven [1] that the homotopy type of ΩM is classified by $b_2(M)$ for a simply connected four-manifold M, i.e. if M

and N are simply connected four-manifolds then it holds: ΩM is homotopy equivalent to ΩN if and only if $b_2(M) = b_2(N)$. The proof relies on hard homotopy techniques and constructions.

Remark 2.2. Note that a simply connected four-manifold M is known to be rationally elliptic if and only if $b_2(M) \leq 2$, where the rational ellipticity means that $\dim(\pi_*(M) \otimes \mathbb{Q}) < \infty$. For $b_2(M) \geq 3$, M is rationally hyperbolic meaning that its rationally homotopy groups grow exponentially. It implies that the number of generators in the minimal model for M as well as in its homotopy Lie algebra is infinite and moreover it grows exponentially. It in particular means that the procedure of the construction of the minimal model in Lemma 3.4 will never end. Nevertheless, by Theorem 1.3 we proved that in the Pontryagin homology ring for ΩM there are no generators in degrees $2, 3, 4$ and 5.

In the next sections we recall necessary definitions and background from rationally homotopy theory and prove the statements that lead to the proof of Proposition 1.2 and Theorem 1.3. For the the detailed account on rational homotopy theory see [3] and for original sources [7, 8].

3. Rational homotopy theory of differential graded algebras

Let $\mathcal{A} = (A, d_A)$ be a commutative graded differential algebra over the real numbers. A differential graded algebra (μ, d) is called *minimal model* for $\mathcal{A}$ if
(i) there exists differential graded algebra morphism $h\colon (\mu, d) \longrightarrow \mathcal{A}$ inducing an isomorphism in their cohomology algebras (such h is called quasi-isomorphism);
(ii) (μ, d) is a free algebra in the sense that $\mu = \wedge V$ is an exterior algebra over graded vector space V;
(iii) differential d is indecomposable meaning that for a fixed set $V = \{P_\alpha, \alpha \in I\}$ of free generators of μ for any $P_\alpha \in V$, $d(P_\alpha)$ is a polynomial in generators P_β with no linear part.
If in (iii) we omit the condition "with no linear part“, (μ, d) is just called a Sullivan model for (A, d).

Two algebras are said to be *weakly equivalent* if there exists quasi-isomorphism between them. This is equivalent to say that these algebras have isomorphic minimal models. The algebra (A, d_A) is said to be *formal* if it is weakly equivalent to the algebra $(H^*(A), 0)$.

The minimal model of a smooth connected manifold M is by definition the minimal model of its de Rham algebra of differential forms $\Omega_{DR}(M)$. In the case when M is simply connected manifold its minimal model completely classifies its rational homotopy type. The manifold M is said to be formal (in the sense of Sullivan) if $\Omega_{DR}(M)$ is a formal algebra. Note also that the rational homotopy groups for M are determined by the graded vector space V, that is $\dim(\pi_k(M) \otimes \mathbb{Q}) = \dim V_k, \;\; k \geq 2$, where V_k is the subspace in V consisting of the elements of degree k.

Using rational model techniques we prove:

Theorem 3.1. *The rational homotopy type of the based loop space ΩM for a simply connected four-manifold M is classified by the second Betti number $b_2(M)$.*

Proof. The proof directly follows from the construction of the minimal model of the path space fibration [3] and the result [10] that the generators of the minimal model for M are determined by $b_2(M)$. For the sake of clearness we present it in detail. Consider the path space fibration $\Omega M \to PM \to M$, where PM is the path space for M. Then the Sullivan model PM has the form $(\wedge V_M \otimes \wedge V, d)$, where $(\wedge V_M, d)$ is the minimal Sullivan model for M. Moreover, $(\wedge V_M \otimes \wedge V, d)$ factors to give the minimal Sullivan model for $(\wedge V, \bar{d})$ for the space ΩM. It follows that $d_0 : V_M \to V$, where d_0 is the linear part of the differential d. Moreover, the Sullivan model $(\wedge V_M \otimes \wedge V, d)$ is contractible since it can be represented as the product of the minimal model for PM and contractible algebra, and the minimal model for PM is trivial as it is contractible. It implies that $H^*(V_M \oplus V, d_0) = 0$, which means that $d_0 : V_M \to V$ is an isomorphism. Therefore $(\wedge V_M, \bar{d})$ is a minimal Sullivan model for ΩM. Since ΩM is an H - space we have that $\bar{d} = 0$. Thus, the rational homotopy type for ΩM is classified by the isomorphism type of the graded vector space V_M, that is by the rational homotopy groups for M. These are, by the result of [10] determined by $b_2(M)$. □

3.1 Construction of minimal model

Let (A, d) be a differential graded algebra which is connected $H^0(A, d) = \mathbb{R}$ and simply connected $H^1(A, d) = 0$. We recall the general procedure for computing the minimal model for a simply-connected commutative differential graded algebra (A, d). It begins with the choice of μ_2 and $m_2 : (\mu_2, 0) \to (A, d)$ such that $m_2^{(2)} : \mu_2 \to H^2(A, d)$ is an isomorphism. We put $\mu_2 = H^2(A)$ and $m_2 : (\mu_2, 0) \to (A, d)$, $m_2 = id$, then $m_2^{(2)} : H^2(\mu_2, 0) \to H^2(A, d)$ is an isomorphism.

In the inductive step, supposing that μ_k and $m_k : (\mu_k, d_k) \to (A, d)$ are constructed such that $m_k^{(k)} : H^k(\mu_k, d_k) \to H^k(A, d)$ is an isomorphism, we extend μ_k and m_k to μ_{k+1} and m_{k+1} so that $m_{k+1}^{(k+1)} : H^{k+1}(\mu_{k+1}, d_{k+1}) \to H^{k+1}(A, d)$ is an isomorphism. We define μ_{k+1} i m_{k+1}in the following way:

$$\mu_{k+1} = \mu_k \otimes \mathfrak{L}(u_i, v_i),$$

where $\mathfrak{L}(u_i, v_i)$ denotes the vector space spanned by the elements u_i, v_i corresponding to elements y_i, z_i respectively. The latter are given by:

$$H^{k+1}(A) = \operatorname{Im} m_k^{k+1} \oplus \mathfrak{L}(y_i), \qquad \operatorname{Ker} m_k^{k+2} = \mathfrak{L}(z_j).$$

Then we have that $m_k(z_j) = dw_j$, for some $w_j \in A$ and the homomorphism m_{k+1} is defined by: $m_{k+1}(u_i) = y_i, m_{k+1}(v_j) = w_j$ and $du_i = 0, dv_j = z_j$.

3.2 Minimal models for some algebras

Following the general construction of the minimal models described above, we provide the description of the minimal model for some algebras. For the sake of clearness we start with the results and their proofs for the following simple examples.

LEMMA 3.2. *The minimal model for the algebra $\mathcal{A} = (A, d = 0)$ is given by*

$$\mu = \mathbb{R}[x] \otimes \wedge(z),\ dz{=}x^2,\ \textit{where}\ \deg x{=}4, \deg z{=}7\ \textit{for}\ A = \{\langle x\rangle\,|x^2{=}0, \deg x{=}4\},$$

$$\mu = \mathbb{R}[x] \otimes \wedge(z),\ dz{=}x^3,\ \textit{where}\ \deg x{=}2, \deg z{=}5\ \textit{for}\ A = \{\langle x\rangle\,|x^3{=}0, \deg x{=}2\}.$$

Proof. We provide the proof for the second statement, the first one goes in an analogous simple way.

The nontrivial cohomology groups of the algebra (A, d) are $H^2(A) = \mathfrak{L}(x)$ and $H^4(A) = \mathfrak{L}(x^2)$. We define that $\mu_2 \cong H^2(A, d)$ and $m_2 = id$ Since the third and fifth cohomology groups for both μ_2 and A are trivial, and their fourth cohomology groups are isomorphic, we obtain that $\mu_2 = \mu_3 = \mu_4$. We further have that $H^6(\mu_4, d) = \operatorname{Ker} d_6/\operatorname{Im} d_5 = \mathfrak{L}(x^3)$ and $H^6(A, d) = 0$, so $m_4^6 : H^6(\mu_4, d) \to H^6(A, d)$ is surjective and $\operatorname{Ker} m_4^6 = \mathfrak{L}(x^3)$. In order to make this mapping injective, we introduce the new generator z of degree 5, with the differential $dz = x^3$ and define the commutative differential graded algebra $\mu_5 = \mu_2 \otimes \mathfrak{L}(z)$, $\deg z = 5$, $dz = x^3$. Now, $H^6(\mu_5, d) = 0$, so the mapping $m_4^6 : H^6(\mu_5, d) \to H^6(A, d)$ is an isomorphism. Since $H^i(\mu_5, d) = 0$, $i \geq 9$ and $H^i(A) = 0$, $i \geq 7$, we complete the construction of the minimal model. □

LEMMA 3.3. *The minimal model for the algebra $\mathcal{A} = (A, d = 0)$, $A = \{\langle x_1, x_2\rangle \mid x_1^2 = x_2^2,\ x_1x_2 = 0,\ \deg x_1 = \deg x_2 = 2\}$, is given by*

$$\mu = \mathbb{R}[x_1, x_2] \otimes \wedge(z_1, z_2),\ \ dz_1 = x_1^2 - x_2^2,\ dz_2 = x_1x_2,$$

where $\deg x_1 = \deg x_2 = 2,\ \deg z_1 = \deg z_2 = 3.$

Proof. The nontrivial cohomology groups for (A, d) are $H^2(A) = \mathfrak{L}(x_1, x_2)$ and $H^4(A) = \mathfrak{L}(x_1^2)$. The first non-trivial cohomology group is $H^2(A)$, so we define $\mu_2 \cong H^2(A, d)$, $m_2 = id$. Since the third cohomology groups for μ_2 and A are both trivial, we consider $H^4(\mu_2, d) = \mathfrak{L}(x_1^2, x_2^2, x_1x_2)$ and $H^4(A, d) = \mathfrak{L}(x_1^2)$, so $m_2^4 : H^4(\mu_2, d) \to H^4(A, d)$ is surjective and $\operatorname{Ker} m_2^4 = \mathfrak{L}(x_1^2{-}x_2^2, x_1x_2)$. In order to make this mapping injective, we introduce the new generators z_1, z_2 of degree 3, with the differentials $dz_1 = x_1^2{-}x_2^2, dz_2 = x_1x_2$ and define the new commutative differential graded algebra $\mu_3 = \mu_2 \otimes \mathfrak{L}(z_1, z_2)$. By simple calculations we arrive at the conclusion that $\mu_3 = \mu_4 = \mu_5$. Continuing the process we conclude that $\mu_k = \mu_3$ for $k \geq 6$. In this way we obtain that μ_3 is the minimal Sullivan algebra that we are looking for. □

Note that the algebras in the previous lemmas are the polynomial quotients by the Borel ideals, that is of the form $\mathbb{Q}[x_1, \ldots, x_n]/\langle P_1, \ldots, P_k\rangle$, where the polynomials $P_1, \ldots, P_k$ are without relations in $\mathbb{Q}[x_1, \ldots, x_n]$. It is proved in [2] that the minimal model for such algebra is given by

$$\mathbb{Q}[x_1, \ldots, x_n] \otimes \wedge(y_1, \ldots, y_k),\ \ dx_i = 0,\ dy_i = P_k,$$

which confirms the results of our application of the general method. The algebra that is considered in lemma that follows is the polynomial algebra quietened by the ideal which is not a Borel ideal any more and one has to apply the general approach for minimal model construction.

Lemma 3.4. *The generators and their differentials up to degree* 6 *for the minimal model of the algebra* $\mathcal{A} = (A, d = 0)$ *for* $A = \{\langle x_1, x_2, x_3\rangle \,|x_1^2 = x_2^2 = x_3^2, x_1^3 = x_2^3 = x_3^3 = 0, x_1x_2 = x_1x_3 = x_2x_3 = 0, \deg x_1 = \deg x_2 = \deg x_3 = 2\}$ *are*

$$x_1, x_2, x_3, z_1, \ldots, z_5, y_1, \ldots, y_5, w_1, \ldots, w_{10}, q_1, \ldots, q_{24},$$

where their differentials are given by the formulas (1)*,* (2)*,* (3) *and* (4) *and* $\deg x_i = 2$*,* $\deg z_i = 3$*,* $\deg y_i = 4$*,* $\deg w_i = 5$ *and* $\deg q_i = 6$*.*

Proof. The nontrivial cohomology groups for the algebra $\mathcal{A}$ are $H^2(\mathcal{A}) = \mathfrak{L}(x_1, x_2, x_3)$ and $H^4(\mathcal{A}) = \mathfrak{L}(x_1^2)$. We define $\mu_2 \cong H^2(A, d)$ and $m_2 = id$. The map $m_2^3 : H^3(\mu_2, d) \to H^3(A, d)$ is an isomorphism, while for the homomorphism $m_2^4 : H^4(\mu_2, d) \to H^4(A, d)$ we have that $\operatorname{Ker} m_2^4 = \mathfrak{L}(x_1x_2, x_1x_3, x_2x_3, x_1^2 - x_2^2, x_1^2 - x_3^2)$. We want the homomorphism $m_2^4 : H^4(\mu_2, d) \to H^4(A, d)$ to be an isomorphism, so we introduce the new generators z_1, z_2, z_3, z_4, z_5 such that

$$dz_1 = x_1x_2,\ dz_2 = x_1x_3,\ dz_3 = x_2x_3,\ dz_4 = x_1^2 - x_2^2,\ dz_5 = x_1^2 - x_3^2. \qquad (1)$$

Then $\deg z_i = 3$, $1 \leq i \leq 5$ and we define $\mu_3 = \mu_2 \otimes \mathfrak{L}(z_1, z_2, z_3, z_4, z_5)$ Now we consider the homomorphism $m_3^4 : H^4(\mu_3, d) \to H^4(A, d)$. Since $H^4(\mu_3, d) = \mathfrak{L}(x^2)$ and $H^4(A, d) = \mathfrak{L}(x^2)$ we see that $m_3^4 = id$.

In the next step we consider the mapping $m_3^5 : H^5(\mu_3, d) \to H^5(A, d)$ and need to determine $H^5(\mu_3, d)$. We first find the cycles c_{ij} of degree 5. The general form of the fifth-degree element in μ_3 is $c_{ij} = \sum\limits_{i=1}^{3}\sum\limits_{j=1}^{5} \alpha_{ij}x_iz_j$ and

$$\begin{aligned}
d_5(c_{ij}) =& \sum_{i=1}^{3}\sum_{i=1}^{5} \alpha_{ij}x_i dz_j \\
=& x_1(\alpha_{11}x_1x_2 + \alpha_{12}x_1x_3 + \alpha_{13}x_2x_3 + \alpha_{14}(x_1^2 - x_2^2) + \alpha_{15}(x_1^2 - x_3^2)) \\
&+ x_2(\alpha_{21}x_1x_2 + \alpha_{22}x_1x_3 + \alpha_{23}x_2x_3 + \alpha_{24}(x_1^2 - x_2^2) + \alpha_{25}(x_1^2 - x_3^2)) \\
&+ x_3(\alpha_{31}x_1x_2 + \alpha_{32}x_1x_3 + \alpha_{33}x_2x_3 + \alpha_{34}(x_1^2 - x_2^2) + \alpha_{35}(x_1^2 - x_3^2)) \\
=& x_1^3(\alpha_{14} + \alpha_{15}) + x_1^2x_2(\alpha_{11} + \alpha_{24} + \alpha_{25}) + x_1^2x_3(\alpha_{12} + \alpha_{34} + \alpha_{35}) \\
&+ x_1x_2^2(-\alpha_{14} + \alpha_{21}) + x_1x_3^2(-\alpha_{15} + \alpha_{32}) + x_1x_2x_3(\alpha_{13} + \alpha_{22} + \alpha_{31}) \\
&+ x_2^3(-\alpha_{24}) + x_2^2x_3(\alpha_{23} - \alpha_{34}) + x_2x_3^2(-\alpha_{25} + \alpha_{33}) + x_3^3(-\alpha_{35}).
\end{aligned}$$

Then $d_5(c_{ij}) = 0$ iff the coefficients α_{ij} satisfy the following system:

$$\begin{array}{ccc}
\alpha_{14} + \alpha_{15} = 0 & \alpha_{11} + \alpha_{24} + \alpha_{25} = 0 & \alpha_{12} + \alpha_{34} + \alpha_{35} = 0 \\
-\alpha_{14} + \alpha_{21} = 0 & -\alpha_{15} + \alpha_{32} = 0 & \alpha_{13} + \alpha_{22} + \alpha_{31} = 0 \\
\alpha_{24} = \alpha_{35} = 0 & \alpha_{23} - \alpha_{34} = 0 & -\alpha_{25} + \alpha_{33} = 0.
\end{array}$$

The system has five free variables, we put them to be

$$\alpha_{14} = a,\ \alpha_{34} = b,\ \alpha_{33} = c,\ \alpha_{13} = d,\ \alpha_{22} = e.$$

Then we have

$$\begin{array}{ll}
\alpha_{21} = -\alpha_{32} = -\alpha_{15} = a & \alpha_{23} = -\alpha_{12} = b \\
\alpha_{25} = -\alpha_{11} = c & \alpha_{31} = -\alpha_{13} - \alpha_{22} = -d - e.
\end{array}$$

It follows that $\operatorname{Ker} d_5$ is given by

$$\mathfrak{L}(x_1z_4 - x_1z_5 + x_2z_1 - x_3z_2, -x_1z_2 + x_2z_3 + x_3z_4, \\ - x_1z_1 + x_2z_5 + x_3z_3, x_1z_3 - x_3z_1, x_2z_2 - x_3z_1).$$

Elements of degree 4 u μ_3 have the form $x_ix_j, i, j = 1, 2, 3$ so $d_4(x_ix_j) = 0$, implying $\operatorname{Im} d_4 = 0$. Therefore, $H^5(\mu_3, d) = \operatorname{Ker} d_5$ and since we have that $H^5(A, d) = 0$, we conclude that $\operatorname{Ker} m_3^5 = H^5(\mu_3, d)$.

In order to achieve the isomorphism between the fifth cohomology groups for $\mathcal{A}$ and the algebra μ_3 we introduce the new generators y_1, y_2, y_3, y_4, y_5, such that

$$dy_1 = x_1z_4 - x_1z_5 + x_2z_1 - x_3z_2, \quad dy_2 = -x_1z_2 + x_2z_3 + x_3z_4, \tag{2}$$
$$dy_3 = -x_1z_1 + x_2z_5 + x_3z_3, \quad dy_4 = x_1z_3 - x_3z_1, \quad dy_5 = x_2z_2 - x_3z_1.$$

Therefore, $\mu_4 = \mu_3 \otimes \mathfrak{L}(y_1, y_2, y_3, y_4, y_5)$, where obviously $\deg y_1 = \deg y_2 = \deg y_3 = \deg y_4 = \deg y_5 = 4$.

Now $H^5(\mu_4, d) = 0$, so $m_4^5 : H^5(\mu_4, d) \to H^5(A, d)$ is an isomorphism. We further consider the homomorphism $m_4^6 : H^6(\mu_4, d) \to H^6(A, d)$, where, by the short calculation, we obtain that $H^6(A, d) = 0$. We are now calculating the sixth cohomology group for (μ_4, d). The general form of the elements of the sixth degree in μ_4 is

$$c_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{5} \alpha_{ij}x_iy_j + \sum_{i,j=1}^{5} \beta_{ij}z_iz_j + \sum_{i,j,k=1}^{3} \gamma_{ij}x_ix_jx_k.$$

Similarly, as in the previous step, we obtain that $H^6(\mu_4, d)$ is given by

$$\mathfrak{L}(x_1y_1-x_3y_2+x_2y_3+z_4z_5, z_1z_2-x_1y_5, x_2y_1-z_1z_4+x_1y_3-x_3y_4+x_3y_5, \\ z_1z_3-x_2y_4, x_3y_1+z_2z_5-x_1y_2+x_2y_4, x_3y_3-z_3z_5-x_1y_4, x_2y_5-z_2z_4+x_1y_2-x_2y_4, \\ z_2z_3-x_3y_4+x_3y_5, x_2y_2-z_3z_4-x_1y_4+x_1y_5, z_1z_5-x_1y_3+x_3y_4),$$

and $\operatorname{Ker} m_4^6 = H^6(\mu_4, d)$.

In order to have the isomorphism between the sixth cohomology groups of $\mathcal{A}$ and μ_4 we introduce the new generators $w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}$, such that

$$dw_1 = x_1y_1 - x_3y_2 + x_2y_3 + z_4z_5, \quad dw_2 = x_2y_1 - z_1z_4 + x_1y_3 - x_3y_4 + x_3y_5, \tag{3}$$
$$dw_3 = x_3y_1 + z_2z_5 - x_1y_2 + x_2y_4, \quad dw_4 = x_2y_5 - z_2z_4 + x_1y_2 - x_2y_4,$$
$$dw_5 = x_2y_2 - z_3z_4 - x_1y_4 + x_1y_5, \quad dw_6 = x_3y_3 - z_3z_5 - x_1y_4,$$
$$dw_7 = z_1z_2 - x_1y_5, \quad dw_8 = z_1z_3 - x_2y_4,$$
$$dw_9 = z_2z_3 - x_3y_4 + x_3y_5, \quad dw_{10} = z_1z_5 - x_1y_3 + x_3y_4.$$

Therefore, $\mu_5 = \mu_4 \otimes \mathfrak{L}(w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10})$, where obviously $\deg w_1 = \deg w_2 = \cdots = \deg w_{10} = 5$. Now, $H^6(\mu_5, d) = 0$, as well as $H^6(A, d)$, so m_5^6 is isomorphism.

Now we are considering $m_5^7 : H^7(\mu_5, d) \to H^7(A, d)$. We have that $H^7(A, d) = 0$, and we compute that $H^7(\mu_5, d)$ is given by

$$\mathfrak{L}(y_3z_1+x_2w_{10}+x_3w_8, y_2z_2+y_5z_3+x_3w_8-x_3w_4, y_4z_3+x_3w_8, y_1z_4-x_1w_1+x_2w_2-x_3w_4, \\ y_1z_5-x_1w_1-x_2w_{10}+x_3w_3, y_1z_1-x_1w_2-x_1w_{10}-x_3w_7, y_5z_2+x_3w_7, y_2z_3+x_1w_9-x_3w_5, \\ y_3z_5+x_1w_{10}+x_3w_6, y_1z_2-x_1w_4-x_1w_3-x_2w_7, y_3z_3+x_1w_8-x_2w_6, y_2z_4-x_1w_4+x_2w_5,$$

$y_5z_1{+}x_2w_7, y_4z_1{+}x_1w_8, y_5z_5{-}y_5z_4{-}x_2w_4{-}x_2w_3{+}x_3w_{10}{+}x_3w_2,$
$y_1z_3{+}y_5z_4{-}x_1w_5{+}x_1w_6{-}x_2w_8{+}x_2w_4{+}x_3w_9{-}x_3w_2, y_4z_4{-}y_5z_4{+}x_1w_5{-}x_2w_4,$
$-y_5z_3{+}x_2w_9{-}x_3w_8, y_2z_5{+}x_1w_3{+}x_2w_6{-}x_3w_1, y_3z_4{-}x_1w_2{+}x_2w_1{+}x_3w_5,$
$y_4z_2{+}x_1w_9{+}x_3w_7, y_2z_1{-}y_5z_4{-}x_1w_7{+}x_2w_8{-}x_2w_4,$
$y_3z_2{+}y_5z_4{+}x_1w_7{+}x_2w_4{+}x_2w_3{+}x_3w_9{-}x_3w_2, y_4z_5{+}x_1w_6{+}x_3w_{10}).$

Thus, $\mathrm{Ker}\, m_5^7 = H^7(\mu_5, d)$ and to have the isomorphism between the seventh cohomology groups of $\mathcal{A}$ and μ_5 we introduce the new generators $q_1, q_2, q_3, \ldots, q_{24}$, such that

$$\begin{aligned}
dq_1 &= y_3z_1{+}x_2w_{10}{+}x_3w_8, & dq_2 &= y_2z_2{+}y_5z_3{+}x_3w_8{-}x_3w_4,\\
dq_3 &= y_4z_3{+}x_3w_8, & dq_4 &= y_1z_4{-}x_1w_1{+}x_2w_2{-}x_3w_4,\\
dq_5 &= y_1z_5{-}x_1w_1{-}x_2w_{10}{+}x_3w_3, & dq_6 &= y_1z_1{-}x_1w_2{-}x_1w_{10}{-}x_3w_7,\\
dq_7 &= y_5z_2{+}x_3w_7, & dq_8 &= y_2z_3{+}x_1w_9{-}x_3w_5,\\
dq_9 &= y_3z_5{+}x_1w_{10}{+}x_3w_6, & dq_{10} &= y_1z_2{-}x_1w_4{-}x_1w_3{-}x_2w_7,\\
dq_{11} &= y_3z_3{+}x_1w_8{-}x_2w_6, & dq_{12} &= y_2z_4{-}x_1w_4{+}x_2w_5,\\
dq_{13} &= y_5z_1{+}x_2w_7, & dq_{14} &= y_4z_1{+}x_1w_8,\\
dq_{15} &= y_5z_5{-}y_5z_4{-}x_2w_4{-}x_2w_3{+}x_3w_{10}{+}x_3w_2,\\
dq_{16} &= y_1z_3{+}y_5z_4{-}x_1w_5{+}x_1w_6{-}x_2w_8{+}x_2w_4{+}x_3w_9{-}x_3w_2,\\
dq_{17} &= y_4z_4{-}y_5z_4{+}x_1w_5{-}x_2w_4, & dq_{18} &= -y_5z_3{+}x_2w_9{-}x_3w_8,\\
dq_{19} &= y_2z_5{+}x_1w_3{+}x_2w_6{-}x_3w_1, & dq_{20} &= y_3z_4{-}x_1w_2{+}x_2w_1{+}x_3w_5,\\
dq_{21} &= y_4z_2{+}x_1w_9{+}x_3w_7, & dq_{22} &= y_2z_1{-}y_5z_4{-}x_1w_7{+}x_2w_8{-}x_2w_4,\\
dq_{23} &= y_3z_2{+}y_5z_4{+}x_1w_7{+}x_2w_4{+}x_2w_3{+}x_3w_9{-}x_3w_2,\\
dq_{24} &= y_4z_5{+}x_1w_6{+}x_3w_{10}.
\end{aligned} \tag{4}$$

Therefore,$\mu_6 = \mu_5 \otimes \mathfrak{L}(q_1, q_2, q_3, \ldots, q_{24})$, where $\deg q_1 = \deg q_2 = \cdots = \deg q_{24} = 6$. Now, $H^7(\mu_6, d) = 0$, as well as $H^7(A, d)$ so m_6^7 isomorphism. □

4. Homotopy Lie algebra of a minimal Sullivan algebra and universal enveloping algebra

DEFINITION 4.1. : A Lie algebra L on $\mathbb{Q}$ is a graded vector space together with linear mapping $[-,-] : L_p \otimes L_q \to L_{p+q}$ such that:

$$[a, b] = -(-1)^{\deg a \cdot \deg b}[b, a], \quad [a, [b, c]] = [[a, b], c] + (-1)^{\deg a \cdot \deg b}[b, [a, c]].$$

There is a standard procedure which to any minimal Sullivan algebra assigns a Lie algebra. More precisely, let $(\wedge V, d)$ be a minimal Sullivan algebra. Define a graded vector space L by requiring that $sL = \mathrm{Hom}(V, k)$, where the suspension sL is defined by $(sL)_k = L_{k-1}$. Thus, a pairing $\langle , \rangle : V \times sL \to k$ is defined by $\langle v; sx\rangle = (-1)^{\deg v} sx(v)$. Extend this to (k+1)-linear maps $\wedge^k V \times sL \cdots \times sL \to k$ by setting

$$\langle v_1 \wedge \cdots \wedge v_k; sx_k, \ldots, sx_1\rangle = \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \cdots \langle v_{\sigma(k)}; sx_k\rangle,$$

where as usual S_k is the permutation group on k symbols and

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \epsilon_\sigma v_1 \wedge \cdots \wedge v_k.$$

Definition 4.2. A pair of dual bases for V and for L consists of a basis (v_i) for V and a basis (x_j) for L such that

$$\langle v_i; sx_j\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

We observe now that L inherits a Lie bracket [,] from d_1. Indeed, a bilinear map $[,] : L \times L \to L$ is uniquely determined by the formula

$$\langle v; s[x, y]\rangle = (-1)^{\deg y+1} < d_1 v; sx, sy >, x, y \in L, v \in V. \tag{5}$$

Here d_1 is quadratic part in the differential d for the minimal model $(\wedge V, d)$ and it is defined by $d - d_1 \in \wedge^{k\geq 3} V$. Thus, if $d_1 v = v_1 \wedge v_2$ for some $v_1, v_2 \in V$ the expression $\langle d_1 v; sx, sy\rangle$ is defined by

$$\langle v_1 \wedge v_2; sx, sy\rangle = \langle v_1; sx\rangle\langle v_2; sy\rangle - \langle v_2; sx\rangle\langle v_1; sy\rangle.$$

The relation $v \wedge w = (-1)^{\deg x \deg y} w \wedge v$ leads at once to $[x, y] = -(-1)^{\deg x \deg y}[y, x]$ and an easy computation gives

$$\langle d_1^2 v, sx, sy, sz\rangle = (-1)^{\deg y}\langle v; s[x, [y, z]] - s[[x, y], z] - (-1)^{\deg x \deg y} s[y, [x, z]]\rangle.$$

Thus, the Jacobi identity is equivalent to the relation $d_1^2 = 0$. The Lie algebra L is called the *homotopy Lie algebra of the Sullivan algebra* $\mu = (\wedge V, d)$.

On the other hand in the category of topological spaces and continuous maps, it is defined the Samelson products $[f, g] \colon S^{p+q} \longrightarrow \Omega M$ of maps $f \colon S^p \longrightarrow \Omega M$ and $g \colon S^q \longrightarrow \Omega M$ by the composite

$$S^p \wedge S^q \xrightarrow{f \wedge g} \Omega M \wedge \Omega M \xrightarrow{c} \Omega M,$$

where c is given by the multiplicative commutator, that is, $c(x, y) = x \cdot y \cdot x^{-1} \cdot y^{-1}$. The graded Lie algebra $L_M = (\pi_*(\Omega M) \otimes \mathbb{Q}; [\,,\,])$ for which the commutator $[\,,\,]$ is given by the Samelson product is called the *rational homotopy Lie algebra of* M. There is an isomorphism between the rational homotopy Lie algebra L_M and the homotopy Lie algebra L of μ ([3]).

We describe the homotopy Lie algebras of the minimal Sullivan algebras that we constructed in the previous chapter.

Lemma 4.3. *The homotopy Lie algebra for the algebra $\mathcal{A}$ from Lemma 3.2 is given by*
1. (for the first case) $L = \mathfrak{L}(v_1^{'}, v_2^{'})$, $[v_1^{'}, v_1^{'}] = 2v_2^{'}$, *while the other Lie brackets are trivial and* $\deg v_1^{'} = 3, \deg v_2^{'} = 6$.
2. (for the second case) $L = \mathfrak{L}(v_1^{'}, v_2^{'}), \deg v_1^{'} = 1, \deg v_2^{'} = 4$, *where all Lie brackets are trivial.*

Proof. We prove the first statement. the second one goes in an analogous way. The minimal Sullivan algebra for $\mathcal{A}$ is by Lemma 3.2 given by

$$\wedge V = \mathbb{Q}[x] \otimes \wedge(z), \ \ \deg x = 4, dx = 0, \deg z = 7, dz = x^2.$$

For the sake of easy calculation, we will denote elements x and z by a_1 and a_2. We first define the vector space $\mathrm{Hom}(V, k) = \mathfrak{L}(v_1, v_2)$, where $v_1(a_1) = 1, v_1(a_2) = 0$ and $v_2(a_1) = 0,\ v_2(a_2) = 1$. Then we have $L = \mathfrak{L}(v_1^{'}, v_2^{'}), \deg v_1^{'} = 3,\ \deg v_2^{'} = 6$. The elements $v_1^{'}, v_2^{'}$ are the elements v_1, v_2 displaced in the graduation by -1. The Lie brackets are determined by (5):

$$\langle a_1, s[v_1^{'}, v_1^{'}]\rangle = \langle d_1 a_1; sv_1^{'}, sv_1^{'}\rangle = 0,$$

$$\langle a_2, s[v_1^{'}, v_1^{'}]\rangle = \langle a_1^2; sv_1^{'}, sv_1^{'}\rangle = \langle a_1; sv_1^{'}\rangle\langle a_1, sv_1^{'}\rangle + \langle a_1; sv_1^{'}\rangle\langle a_1, sv_1^{'}\rangle = 2.$$

So, we can conclude that $[v_1^{'}, v_1^{'}] = 2v_2^{'}$. For the bracket $[v_2^{'}, v_2^{'}]$ we have that $\langle a_1, s[v_2^{'}, v_2^{'}]\rangle = 0$ and $\langle a_2, s[v_2^{'}, v_2^{'}]\rangle = -\langle a_1^2; sv_2^{'}, sv_2^{'}\rangle = 0$.

Thus, $[v_2^{'}, v_2^{'}] = 0$. For the Lie bracket $[v_1^{'}, v_2^{'}]$ we have that $< a_1, s[v_1^{'}, v_2^{'}] >= 0$ and $< a_2, s[v_1^{'}, v_2^{'}] >= 0$. So, $[v_1^{'}, v_2^{'}] = 0$. □

LEMMA 4.4. *The homotopy Lie algebra of the algebra $\mathcal{A}$ from Lemma 3.3 is given by*

$$L = \mathfrak{L}(v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}),\ [v_1^{'}, v_1^{'}] = 2v_3^{'},\ [v_1^{'}, v_2^{'}] = v_4^{'},\ [v_2^{'}, v_2^{'}] = -2v_3^{'},$$

while all others Lie brackets are equal to zero and $\deg v_1^{'} = 1$, $\deg v_2^{'} = 1$, $\deg v_3^{'} = 2$, $\deg v_3^{'} = 2$.

Proof. The minimal minimal Sullivan algebra for $\mathcal{A}$ is, according to Lemma 3.3 given by $\mu = \mathbb{Q}(a_1, a_2,) \otimes \wedge(a_3, a_4)$, $da_3 = a_1^2 - a_2^2$, $da_4 = a_1 a_2$, where $\deg a_1 = \deg a_2 = 2$, $da_1 = da_2 = 0$, $\deg a_3 = \deg a_4 = 4$. We consider the vector space $\mathrm{Hom}(V, k) = \mathfrak{L}(v_1, v_2, v_3, v_4)$, where $v_i(a_j) = 1$ iff $i = j$, $i.j = 1, \ldots, 4$ and define that $L = \mathfrak{L}(v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'})$, where the graduation is given by $\deg v_1^{'} = \deg v_2^{'} = 1$, $\deg v_3^{'} = \deg v_4^{'} = 2$.

The Lie brackets on L are determined by the (5). From that formulas we can conclude that the Lie bracket $[v_i^{'}, v_j^{'}] = 0$ iff in there is no generator in the minimal model whose differential d_1 contains element of the form $a_i a_j$. When we look at expressions that represent differentials of the generators of the minimal model, we see that they do not contain the elements of the form $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_3, a_3a_4, a_4a_4$, so

$$[v_1^{'}, v_3^{'}] = [v_1^{'}, v_4^{'}] = [v_2^{'}, v_3^{'}] = [v_2^{'}, v_4^{'}] = [v_3^{'}, v_3^{'}] = [v_3^{'}, v_4^{'}] = [v_4^{'}, v_4^{'}] = 0.$$

It remains to determine the Lie bracket $[v_1^{'}, v_1^{'}], [v_1^{'}, v_2^{'}]$, $[v_2^{'}, v_2^{'}]$. We have that $d_1 a_3 = a_1^2 - a_2^2$, implying

$$\langle a_3, s[v_1^{'}, v_1^{'}]\rangle = 2, \qquad \langle a_i, s[v_1^{'}, v_1^{'}]\rangle = 0,\ i \neq 3$$

$$\langle a_3, s[v_2^{'}, v_2^{'}]\rangle = -2, \qquad \langle a_i, s[v_2^{'}, v_2^{'}]\rangle = 0,\ i \neq 3.$$

We have that $d_1 a_4 = a_1 a_2$, so

$$\langle a_4, s[v_1^{'}, v_2^{'}]\rangle = 1,\ \langle a_i, s[v_1^{'}, v_2^{'}]\rangle = 0,\ i \neq 4.$$

It follows that $[v_1^{'}, v_1^{'}] = 2v_3^{'},\ [v_1^{'}, v_2^{'}] = v_4^{'},\ [v_2^{'}, v_2^{'}] = -2v_3^{'}$. □

Proposition 4.5. *The homotopy Lie algebra up to degree* 5 *of the algebra* $\mathcal{A}$ *from Lemma 3.4 is given by*

$$L = \mathfrak{L}(v_1^{'}, v_2^{'}, \dots, v_{47}^{'}), \tag{6}$$

where $\deg v_i^{'} = 1$, $1 \leq i \leq 3$, $\deg v_i^{'} = 2$, $4 \leq i \leq 8$, $\deg v_i^{'} = 3$, $9 \leq i \leq 13$, $\deg v_i^{'} = 4$, $14 \leq i \leq 23$ *and* $\deg v_i^{'} = 5$, $24 \leq i \leq 47$. *The non trival Lie brackets on* L *up to degree* 5 *are given by the table below.*

Remark 4.6. Note that, since we described in Lemma 3.4 the generators of the minimal model up to degree 6, we are not able to calculate all the Lie brackets for elements of L. More precisely, we can calculate the Lie brackets up to degree 5. For example, $[v_{10}^{'}, v_{25}^{'}]$ should be an element of degree 7 and can not be described in terms of the elements in L. Nevertheless, we can calculate the Lie brackets for a lot of elements in L.

Proof. The minimal model for $\mathcal{A}$ is given by Lemma 3.4. Like in the previous examples, for the sake of simplicity, we will denote elements $x_1, x_2, x_3, z_1, z_2, \dots, q_{24}$ by $a_1, a_2, a_3, a_4, a_5, \dots, a_{47}$ and correspondingly write their differentials. We denote by v_i, $1 \leq i \leq 47$ the duals for a_i that is $v_i(a_j) = 1$, for $i = j$ and $v_i(a_j) = 0, i \neq j$.

We define that $L = \mathfrak{L}(v_1^{'}, v_2^{'}, \dots, v_{47}^{'})$, where the the elements $v_i^{'}$ are the elements v_i displaced in the graduation by -1.

There are some general rules that we can observe and use for calculating a Lie bracket. If we want to calculate the Lie bracket $[v_k^{'}, v_l^{'}]$ we are looking for element $a_n, n \in \{1, 2, \dots, 47\}$ which contains element $a_k a_l$ in his differential d_1. If there is no such an element the Lie bracket is equal to zero. Moreover, for $k \neq l$ and there is one element a_n which has element $a_k a_l$ in his differential d_1, then the Lie bracket $[v_k^{'}, v_l^{'}]$ is equal to $v_n^{'}$ or to $-v_n^{'}$. When $v_k^{'}$ is odd degree then $[v_k^{'}, v_l^{'}] = v_n^{'}$ and when $v_k^{'}$ is even degree then $[v_k^{'}, v_l^{'}] = -v_n^{'}$. If in the differential we have an element $-a_k a_l$ the situation is opposite. If there is more then one element whose differential d_1 contains $a_k a_l$, for example a_p, a_s, a_q then $[v_k^{'}, v_l^{'}]$ is equal to the sum of $v_p^{'}, v_s^{'}, v_q^{'}$ with appropriate sign for any of the elements $v_p^{'}, v_s^{'}, v_q^{'}$.

For $k = l$ and $v_k^{'}$ has even degree, $[v_k^{'}, v_k^{'}]$ is equal to zero. If $v_k^{'}$ has odd degree then $[v_k^{'}, v_k^{'}]$ is equal to $2v_n^{'}$, where a_n is the only element in the minimal model whose differential d_1 contains the element $a_k^{'} a_k^{'}$. If in the differential d_1 of an element a_n appears $-a_k^{'} a_k^{'}$ then $[v_k^{'}, v_k^{'}] = -2v_n^{'}$. If there is more then one element whose differential d_1 contains the element $a_k^{'} a_k^{'}$, then the rule is same like in case when $k \neq l$.

Using these rules, we list the Lie brackets in L which gives the elements up to degree 5.

$[v_1^{'}, v_1^{'}] = 2v_7^{'} + 2v_8^{'}$	$[v_1^{'}, v_2^{'}] = v_4^{'}$	$[v_1^{'}, v_3^{'}] = v_5^{'}$
$[v_1^{'}, v_4^{'}] = -v_{11}^{'}$	$[v_1^{'}, v_5^{'}] = -v_{10}^{'}$	$[v_1^{'}, v_6^{'}] = v_{12}^{'}$
$[v_1^{'}, v_7^{'}] = v_9^{'}$	$[v_1^{'}, v_8^{'}] = -v_9^{'}$	$[v_1^{'}, v_9^{'}] = v_{14}^{'}$
$[v_1^{'}, v_{10}^{'}] = -v_{16}^{'} + v_{17}^{'}$	$[v_1^{'}, v_{11}^{'}] = v_{15}^{'} - v_{23}^{'}$	$[v_1^{'}, v_{12}^{'}] = -v_{18}^{'} - v_{19}^{'}$

$[v_1', v_{13}'] = v_{18}' - v_{20}'$	$[v_1', v_{14}'] = -v_{27}' - v_{28}'$	$[v_1', v_{15}'] = -v_{29}' - v_{43}'$
$[v_1', v_{16}'] = -v_{33}' + v_{42}'$	$[v_1', v_{17}'] = -v_{33}' - v_{35}'$	$[v_1', v_{18}'] = -v_{39}' + v_{40}'$
$[v_1', v_{19}'] = v_{39}' + v_{47}'$	$[v_1', v_{20}'] = -v_{45}' + v_{46}'$	$[v_1', v_{21}'] = v_{34}' + v_{37}'$
$[v_1', v_{22}'] = v_{31}' + v_{44}'$	$[v_1', v_{23}'] = -v_{29}' + v_{32}'$	
$[v_2', v_2'] = -2v_7'$	$[v_2', v_3'] = v_6'$	$[v_2', v_4'] = v_9'$
$[v_2', v_5'] = v_{13}'$	$[v_2', v_6'] = v_{10}'$	$[v_2', v_7'] = 0$
$[v_2', v_8'] = v_{11}'$	$[v_2', v_9'] = v_{15}'$	$[v_2', v_{10}'] = v_{18}'$
$[v_2', v_{11}'] = v_{14}'$	$[v_2', v_{12}'] = v_{16}' - v_{17}' - v_{21}'$	$[v_1', v_{13}'] = v_{17}'$
$[v_2', v_{14}'] = v_{43}'$	$[v_1', v_{15}'] = v_{27}'$	$[v_2', v_{16}'] = -v_{38}' + v_{46}'$
$[v_2', v_{18}'] = v_{35}'$	$[v_2', v_{19}'] = -v_{34}' + v_{42}'$	$[v_2', v_{23}'] = v_{24}' - v_{28}'$
$[v_2', v_{20}'] = v_{31}' - v_{33}'$	$[v_2', v_{21}'] = -v_{39}' + v_{45}'$	$[v_2', v_{22}'] = v_{41}'$
$[v_2', v_{17}'] = -v_{38}' + v_{39}' - v_{40}' - v_{45}' + v_{46}'$		
$[v_3', v_3'] = -2v_8'$	$[v_3', v_4'] = -v_{12}' - v_{13}'$	$[v_3', v_5'] = -v_9'$
$[v_3', v_6'] = v_{11}'$	$[v_3', v_7'] = v_{10}'$	$[v_3', v_8'] = 0$
$[v_3', v_9'] = v_{16}'$	$[v_3', v_{10}'] = -v_{14}'$	$[v_3', v_{11}'] = v_{19}'$
$[v_3', v_{12}'] = -v_{15}' - v_{22}' + v_{23}'$	$[v_3', v_{13}'] = v_{15}' + v_{22}'$	$[v_3', v_{14}'] = -v_{42}'$
$[v_3', v_{15}'] = v_{38}' - v_{39}' - v_{46}'$	$[v_3', v_{16}'] = v_{28}'$	$[v_3', v_{17}'] = -v_{25}' - v_{27}'$
$[v_3', v_{18}'] = -v_{31}' + v_{43}'$	$[v_3', v_{19}'] = v_{32}'$	$[v_3', v_{20}'] = -v_{29}' + v_{30}' + v_{44}'$
$[v_3', v_{21}'] = v_{24}' + v_{25}' + v_{26}' - v_{41}'$	$[v_3', v_{22}'] = v_{39}' + v_{46}'$	$[v_3', v_{23}'] = v_{38}' + v_{47}'$
$[v_4', v_4'] = 0$	$[v_4', v_5'] = -v_{20}'$	
$[v_4', v_6'] = -v_{21}'$	$[v_4', v_7'] = v_{15}'$	$[v_4', v_8'] = -v_{23}'$
$[v_4', v_9'] = -v_{29}'$	$[v_4', v_{10}'] = -v_{45}'$	$[v_4', v_{11}'] = -v_{24}'$
$[v_4', v_{12}'] = -v_{37}'$	$[v_4', v_{13}'] = -v_{36}'$	
$[v_5', v_5'] = 0$	$[v_5', v_6'] = -v_{22}'$	$[v_5', v_7'] = v_{17}'$
$[v_5', v_8'] = -v_{16}'$	$[v_5', v_9'] = -v_{33}'$	$[v_5', v_{10}'] = -v_{25}'$
$[v_5', v_{11}'] = -v_{46}'$	$[v_5', v_{12}'] = -v_{44}'$	$[v_5', v_{13}'] = -v_{30}'$
$[v_6', v_6'] = 0$	$[v_6', v_7'] = v_{18}'$	$[v_6', v_8'] = v_{19}'$
$[v_6', v_9'] = -v_{39}'$	$[v_6', v_{10}'] = -v_{31}'$	$[v_6', v_{11}'] = -v_{34}'$
$[v_6', v_{12}'] = -v_{26}'$	$[v_6', v_{13}'] = -v_{25}' + v_{41}'$	$[v_7', v_7'] = 0$
$[v_7', v_8'] = -v_{14}'$	$[v_7', v_9'] = -v_{27}'$	$[v_7', v_{10}'] = -v_{35}'$

$[v_7', v_{11}'] = -v_{43}'$ $\quad [v_7', v_{12}'] = -v_{40}'$

$[v_7', v_{13}'] = v_{38}' - v_{39}' + v_{40}' + v_{45}' - v_{46}'$

$[v_8', v_8'] = 0$ $\quad [v_8', v_9'] = -v_{28}'$ $\quad [v_8', v_{10}'] = -v_{42}'$

$[v_8', v_{11}'] = -v_{32}'$ $\quad [v_8', v_{12}'] = -v_{47}'$ $\quad [v_8', v_{13}'] = -v_{38}'$. □

4.1 Universal enveloping algebra of a homotopy Lie algebra

The universal enveloping algebra for a Lie algebra L is defined by

$$\mathrm{UL} \cong T(L)/\left\langle x \otimes y - (-1)^{\deg x \deg y} y \otimes x - [x, y]\right\rangle.$$

Milnor and Moore [see Appendix in [6]] showed that for a path connected homotopy associative H-space with unit E, there is an isomorphism of Hopf algebras $U(\pi_*(E) \otimes \mathbb{Q}) \cong H_*(E; \mathbb{Q})$. As loop multiplication is homotopy associative with unit, applying the Milnor and Moore theorem to our case, it follows that $H_*(\Omega M; \mathbb{Q}) \cong UL$, where UL is the universal enveloping algebra for L. For a more detailed account on this construction see for example [3], Chapters 12 and 16.

We compute the universal enveloping algebras for the Lie algebras we previously calculated.

Lemma 4.7. *The universal enveloping algebras for the Lie algebras given by Lemma 4.3 are given by* $T(v_1')$ *and* $T(v_1', v_2')/\langle v_1' \otimes v_1' = 0,\ v_1' \otimes v_2' = v_2' \otimes v_1'\rangle$ *respectively.*

Proof. We determine the universal enveloping algebra UL in the second case. We find that the generating elements for the ideal I are $2v_1' \otimes v_1'$, $v_1' \otimes v_2' - v_2' \otimes v_1'$. So, in UL we have two relations $v_1' \otimes v_1' = 0$ and $v_1' \otimes v_2' = v_2' \otimes v_1'$. Therefore, the universal enveloping algebra which we are looking for is:

$$\mathrm{UL} \cong T(v_1', v_2')/\langle v_1' \otimes v_1' = 0,\ v_1' \otimes v_2' = v_2' \otimes v_1'.\rangle$$ □

Lemma 4.8. *The universal enveloping algebra for the Lie algebra given by Lemma 4.4 is* $T(v_1', v_2')/\langle v_1' \otimes v_1' = -v_2' \otimes v_2'\rangle$.

Proof. Like in previous examples, the generating elements in I are:

$$2v_1' \otimes v_1' - 2v_3',\ v_1' \otimes v_2' + v_2' \otimes v_1' - v_4',\ 2v_2' \otimes v_2' + 2v_3',\ v_1' \otimes v_3' - v_3' \otimes v_1',$$
$$v_1' \otimes v_4' - v_4' \otimes v_1',\ v_2' \otimes v_3' - v_3' \otimes v_2',\ v_2' \otimes v_4' - v_4' \otimes v_2',\ v_3' \otimes v_4' - v_4' \otimes v_3'.$$

So, in UL we have three relations $v_1' \otimes v_1' = v_3'$, $v_1' \otimes v_2' + v_2' \otimes v_1' = v_4'$ and $v_2' \otimes v_2' = -v_3'$. All other relations are consequences of these three relations. The first and the third relation give us relation $v_1' \otimes v_1' = -v_2' \otimes v_2'$. Therefore, the universal enveloping algebra is given by: $\mathrm{UL} \cong T(v_1', v_2')/\langle v_1' \otimes v_1' = -v_2' \otimes v_2'\rangle$. □

Proposition 4.9. *The universal enveloping algebra up to degree* 5 *for the Lie algebra given by Proposition 4.5 is given by* $UL \cong T(v_1', v_2', v_3')/\langle v_1' \otimes v_1' = -v_2' \otimes v_2' - v_3' \otimes v_3'\rangle$.

Proof. From the previous examples we conclude that, if some Lie bracket $[v_k', v_l']$ is equal to some element $v_n', n \geq k, l$ then in the universal enveloping algebra we have the relation between these three elements, that is we can express element v_n' using elements $v_k^{'}, v_l^{'}$. For example, $v_1^{'} \otimes v_2^{'} - (-1) v_2^{'} \otimes v_1^{'} - [v_1^{'}, v_2^{'}] = v_1^{'} \otimes v_2^{'} + v_2^{'} \otimes v_1^{'} - v_4^{'}$, so in UL we have $v_4^{'} = v_1^{'} \otimes v_2^{'} + v_2^{'} \otimes v_1^{'}$. When we look at the list of Lie brackets given in the proof of Proposition 4.5 we can see that every element $v_k^{'}, k \geq 4$ can be obtained by the Lie bracket of the generators of the lower degree. So, we can conclude that every element $v_k^{'}, k \geq 4$ in UL we can express as a linear combination of the products of the elements $v_1^{'}, v_2^{'}$ and $v_3^{'}$. Therefore, in UL we have just elements $v_1^{'}, v_2^{'}, v_3^{'}$ and the linear combination of their tensor products. It remains just to see if there is some relation between $v_1^{'}, v_2^{'}, v_3^{'}$ in UL. From the list of the Lie brackets we have that $[v_1^{'}, v_1^{'}] = -[v_2^{'}, v_2^{'}] - [v_3^{'}, v_3^{'}]$. In UL we have that $2v_1^{'} \otimes v_1^{'} = [v_1^{'}, v_1^{'}], 2v_2^{'} \otimes v_2^{'} = [v_2^{'}, v_2^{'}], 2v_3^{'} \otimes v_3^{'} = [v_3^{'}, v_3^{'}]$, so in UL we have the relation $v_1^{'} \otimes v_1^{'} = -v_2^{'} \otimes v_2^{'} - v_3^{'} \otimes v_3^{'}$. Therefore, the universal enveloping algebra which we are looking for is:

$$\mathrm{UL} \cong T(v_1^{'}, v_2^{'}, v_3^{'}) / \langle v_1^{'} \otimes v_1^{'} = -v_2^{'} \otimes v_2^{'} - v_3^{'} \otimes v_3^{'} \rangle \qquad \square$$

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In this way we completed the proofs of Proposition 1.2 and Theorem 1.3.

References

[1] P. Beben, S. Theriault, *The loop space homotopy type of simply-connected four-manifolds and their generalizations*, Adv. Math. **262** (2014), 213–238.

[2] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8(179)** (1976), 1–94.

[3] Y. Félix, S. Halperin, J.-C. Thomas, *Rational Homotopy Theory*, Springer Verlag 2000.

[4] R. Mandelbaum, *Four-dimensional topology: an introduction*, Bull. Amer. Math. Society 2, (1980), 1–159.

[5] T. Miller, J. Neisendorfer, *Formal and coformal spaces*, Illinois J. Math. **22** (1978), 565–580.

[6] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

[7] D. Quillen, *Rational homotopy theory*, Annals of Math. **90** (1969), 205–295.

[8] D. Sullivan, *Infinitesimal computations in topology*, Publ. I. H. E. S **47** (1977), 269–331.

[9] S. Terzić, *On rational homotopy of four-manifolds*, Contemporary Geometry and Related Topics, Proceedings of the Workshop, Edited by: N. Bokan, M. Djorić, Z. Rakić, A. T. Fomenko and J. Wess, World Sci. Publ., River Edge, NJ, (2004), 375–388.

[10] S. Terzić, *The rational homology ring of the based loop space of the gauge groups and the spaces of connections on a four-manifold*, (in Russian) Fund. i Prikladnaya Matemat. **21(6)** (2016), 205–215.

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Faculty of Science, University of Montenegro, Dzordza Vasingtona bb, 81000 Podgorica, Montenegro

E-mail: gaga.borovic@gmail.com

Faculty of Science, University of Montenegro, Dzordza Vasingtona bb, 81000 Podgorica, Montenegro

E-mail: sterzic@ucg.ac.me