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NON-NORMAL p-BICIRCULANTS, p A PRIME

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Abstract. A graph Γ is called a semi-Cayley graph over a group G, if there exists a semiregular subgroup R_G of $\mathrm{Aut}(\Gamma)$ isomorphic to G with two orbits (of equal size). We say that Γ is normal if R_G is a normal subgroup of $\mathrm{Aut}(\Gamma)$. Semi-Cayley graphs over cyclic groups are called bicirculants. In this paper, we determine all non-normal bicirculants over a group of prime order.

1. Introduction and result

For a graph Γ, we let $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Aut}(\Gamma)$ and Γ^c denote the vertex set, the edge set, the full automorphism group and the complement of Γ, respectively. We say that Γ is vertex-transitive, primitive or imprimitive when $\mathrm{Aut}(\Gamma)$ acts transitively, primitively or imprimitively on $V(\Gamma)$, respectively. Our notation and terminology are standard. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [3] and [5], respectively. Throughout the paper all graphs are finite and simple. Also, for a group G we denote $G \setminus \{1_G\}$ by G^* and we use the multiplicative notation for cyclic groups.

Let G be a finite group and $S = S^{-1} \subseteq G^*$. The Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S has vertex set G and edge set $\{(g, sg) \mid g \in G, s \in S\}$. It is well-known that the right regular representation $R(G)$ of G is a regular subgroup of $\mathrm{Aut}(\Gamma)$. If $R(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$, then Γ is called a normal Cayley graph over G [13]. The study of normality of Cayley graphs, which plays an important role in the investigation of various symmetry properties of graphs, was started by Xu in [13] and it is still an active topic in algebraic graph theory. We encourage the reader to consult [4] for a survey up to 2008.

By a theorem of Sabidussi [12], a graph Γ is a Cayley graph of a group G if and only if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. In analogy to the Sabidussi's Theorem, a graph Γ is called a *semi-Cayley* graph over a group G if there exists a semi-regular subgroup R_G of $\mathrm{Aut}(\Gamma)$ isomorphic to G with two orbits

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(of equal size) [11]. Semi-Cayley graphs are called by some authors *bi-Cayley* graphs, see for example [14]. Recently, some authors studied the structure of automorphism group of semi-Cayley graphs [1, 14]. In analogy to the concept of normality of Cayley graphs, Arezoomand and Taeri defined normal semi-Cayley graphs. A semi-Cayley graph Γ over a group G is called *normal* if R_G is a normal subgroup of $\mathrm{Aut}(\Gamma)$ [1]. It is clear that Γ is a normal semi-Cayley graph over a group G if and only if its complement, Γ^c, is a normal semi-Cayley graph over G. An important subclass of semi-Cayley graphs are *bicirculants*, which are semi-Cayley graphs over cyclic groups. For an equivalent definition of bicirculants see [9]. Recently, the study of bicirculants have been the object of many papers, see for example [6]– [10]). In [9], the symmetry structure of bicirculants over a group of prime order p is determined. In this paper, our aim is to classify non-normal bicirculants over a group of prime order p.

Resmini and Jungnickel [11] determined the structure of semi-Cayley graphs: A graph Γ is a semi-Cayley graph over a group G if there exist subsets $R = R^{-1} \subseteq G^*$, $L = L^{-1} \subseteq G^*$ and S of G such that $\Gamma \cong \mathrm{SC}(G; R, L, S)$ where $\mathrm{SC}(G; R, L, S)$ is a graph with vertex set $G \times \{1, 2\}$ and edge set $E_R \cup E_L \cup E_S$, where

$$\{\{(x,1),(y,1)\} \mid yx^{-1} \in R\} \qquad \text{(right edges)},$$
$$\{\{(x,2),(y,2)\} \mid yx^{-1} \in L\} \qquad \text{(left edges)},$$
$$\{\{(x,1),(y,2)\} \mid yx^{-1} \in S\} \qquad \text{(spoke edges)}.$$

Let $g \in G$ and ρ_g be a permutation of the vertex set of $\mathrm{SC}(G; R, L, S)$ such that $(x, i)^{\rho_g} = (xg, i)$ for all $x \in G$ and $i = 1, 2$. Then $R_G = \{\rho_g \mid g \in G\}$ is a semi-regular subgroup of $\mathrm{Aut}(\mathrm{SC}(G; R, L, S))$ isomorphic to G with two orbits $G \times \{1\}$ and $G \times \{2\}$. Hence, we may denote a semi-Cayley graph over a group G by $\mathrm{SC}(G; R, L, S)$ for some suitable subsets R, L and S of G. We denote the subgraph of $\Gamma = \mathrm{SC}(G; R, L, S)$ induced by all the edges of Γ having one end-vertex in $G \times \{1\}$ and the other in $G \times \{2\}$ (in other words when $R = L = \emptyset$) with $\mathrm{BCay}(G, S)$. Note that in $\mathrm{BCay}(G, S)$ maybe $S \neq S^{-1}$. But if S is inverse-closed then $\mathrm{BCay}(G, S) \cong \mathrm{Cay}(G, S) \otimes K_2$, where $\otimes$ denotes the tensor product of graphs [2, Lemma 3.2]. Note that in [9], a bicirculant $\mathrm{SC}(G; R, L, S)$, $G \times \{1\}$, $G \times \{2\}$ and $\mathrm{BCay}(G, S)$ are denoted by $[R, L, S]$, U, W and $[U, W]$, respectively.

Using the classification of p-bicirculants, p a prime, given in [9], we classify all non-normal bicirculants over a group of prime order p:

Theorem 1.1. *Let Γ be a non-normal bicirculant over a group $G = \langle x \rangle$ of prime order p. Then Γ is one of the following graphs.*

(a) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, G^*, G) \cong K_4$, $p = 2$.

(b) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, G^*, \{1_G\})$, $p = 2$.

(c) Γ *or* $\Gamma^c = \mathrm{BCay}(G, \{1_G\})$, $p = 2$.

(d) Γ *or* $\Gamma^c \cong \Gamma_1 + \Gamma_2$, *where* Γ_i *are two non-isomorphic Cayley graphs of order* p, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ *and* $p > 2$.

(e) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong pK_2$*, in which case* $\mathrm{Aut}(\Gamma) \cong S_p$ *and* $p > 3$.

(f) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong B(PG(n, q))$ *where* $p = \frac{q^n-1}{q-1}$*, in which* $\mathrm{Aut}(\Gamma) = P\Sigma L(n, q)$ *and* $p > 3$.

(g) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong B(H(11))$*, in which case* $\mathrm{Aut}(\Gamma) \cong PSL(2, 11)$ *and* $S = \{x, x^3, x^4, x^5, x^9\}$, $p = 11$.

(h) Γ *or* $\Gamma^c \cong 2pK_1$, pK_2 *or* $2X$*, where* X *is connected Cayley graph of order* p *and* $p > 2$.

(i) Γ *or* $\Gamma^c \cong P$*, where* P *is the Petersen graph,* $p = 5$.

(j) Γ *or* $\Gamma^c \cong Y[2K_1]$*, where* Y *is a Cayley graph of order* p *and* $p > 2$.

(k) Γ *or* $\Gamma^c \cong B(PG(n, q))$ *or* $C(PG(n, q))$ *where* $p = \frac{q^n-1}{q-1}$*, in which* $\mathrm{Aut}(\Gamma) = P\Gamma L(n, q)$ *and* $p > 3$.

(l) Γ *or* $\Gamma^c \cong B(H(11))$ *or* $C(H(11))$*, in which* $\mathrm{Aut}(\Gamma) = PGL(2, 11)$ *and* $p = 11$*, where the incidence graph of the projective space* $PG(n, q)$ *and the Hadamard design* $H(11)$ *on 11 points are denoted by* $B(PG(n, q))$ *and* $B(H(11))$ *and their non-incidence graphs are denoted by* $C(PG(n, q))$ *and* $C(H(11))$*, respectively.*

2. Preliminaries

In this section we recall some preliminaries and results which are used in the proof of Theorem 1.1. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ and X be the set of all maps $\psi : V(\Gamma) \to V(\Gamma)$, where $(x, 1)^\psi = (x^\sigma, 1)$ and $(x, 2)^\psi = (gx^\sigma, 2)$, for some $g \in G$ and $\sigma \in \mathrm{Aut}(G)$ such that $R^\sigma = R$, $L^\sigma = g^{-1}Lg$, and $S^\sigma = g^{-1}S$. Also, let Y be the set of all maps $\varphi : V(\Gamma) \to V(\Gamma)$, where $(x, 1)^\varphi = (x^\theta, 2)$ and $(x, 2)^\varphi = (hx^\theta, 1)$, for some $h \in G$ and $\theta \in \mathrm{Aut}(G)$ such that $R^\theta = L$, $L^\theta = h^{-1}Rh$ and $S^\theta = h^{-1}S^{-1}$ with the convention that if one of the pair sets R, L is empty and the other is non-empty or $S = \emptyset$, we put $Y = \emptyset$. Also if in the above equalities, one of the subsets is empty, then we omit the equality including it. The structure of normalizer of R_G in $\mathrm{Aut}(\Gamma)$ is determined in [1] as follows:

THEOREM 2.1. *([1, Theorem 1]) Let* $\Gamma = \mathrm{SC}(G; R, L, S)$ *be a semi-Cayley graph over a group* G*, and* X, Y *be the sets defined above. Then* $N_{\mathrm{Aut}(\Gamma)}(R_G) = ZR_G$*, where* $Z = X \cup Y$*. Furthermore,* $R_G \cap Z = \{1_G\}$.

PROPOSITION 2.2. *([1, Proposition 2]) Let* $\Gamma = \mathrm{SC}(G; R, L, S)$ *be a semi-Cayley graph over* G*. Then*

(1) $R_G \trianglelefteq \mathrm{Aut}(\Gamma)$ *if and only if* $\mathrm{Aut}(\Gamma) = ZR_G$,

(2) if $R_G \trianglelefteq \mathrm{Aut}(\Gamma)$*, then* $\mathrm{Aut}(\Gamma)_{(1,1)} = X$ *and the converse holds if* $\mathrm{Aut}(\Gamma)$ *is not transitive on* $V(\Gamma)$*.*

COROLLARY 2.3. *([1, Corollary 3.2]) Let* Γ *be a normal semi-Cayley graph over a group* G *such that* $\mathrm{Aut}(G)$ *is solvable. Then* $\mathrm{Aut}(\Gamma)$ *is solvable. In particular, the automorphism group of every normal semi-Cayley graph over a cyclic group is solvable.*

The symmetry structure of bicirculants over a group of prime order is fully given in [9]. We collect its result as follows. Note that in the following theorem the lexicographic product and the disjoint union of graphs Γ_1 and Γ_2 are denoted by $\Gamma_1[\Gamma_2]$ and $\Gamma_1 + \Gamma_2$, respectively.

THEOREM 2.4. *([9, Theorem 2.1, Theorem 2.2]) Let* Γ *be a bicirculant over a group* $G = \langle x \rangle$ *of prime order* p*. Then one of the following occurs.*

(1) Γ *or* $\Gamma^c = \mathrm{SC}(G; R, L, \emptyset) \cong \mathrm{Cay}(G, R) + \mathrm{Cay}(G, L)$*, where* $\mathrm{Cay}(G, R)$ *and* $\mathrm{Cay}(G, L)$ *are two non-isomorphic Cayley graphs of order* p *and* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Cay}(G, R)) \times \mathrm{Aut}(\mathrm{Cay}(G, L))$*.*

(2) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong pK_2$*, in which case* $\mathrm{Aut}(\Gamma) \cong S_p$*.*

(3) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong B(PG(n, q))$*, where* $p = \frac{q^n-1}{q-1}$*, in which case* $\mathrm{Aut}(\Gamma) = P\Sigma L(n, q)$*.*

(4) Γ *or* $\Gamma^c = \mathrm{SC}(G; G^*, \emptyset, S)$ *and* $\mathrm{BCay}(G, S) \cong B(H(11))$*, in which case* $\mathrm{Aut}(\Gamma) \cong PSL(2, 11)$ *and* $S = \{x, x^3, x^4, x^5, x^9\}$*,* $p = 11$*.*

(5) There exists $\sigma \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = R_G \rtimes \langle \sigma \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_d$*, where* d *divides* $p-1$ *(for more details about the map* σ *and the structure of* Γ*, see [9, Theorem 2.1(iii)]).*

(6) Γ *or* $\Gamma^c \cong 2pK_1$*,* pK_2 *or* $2X$*, where* X *is a connected Cayley graph of order* p*.*

(7) Γ *or* $\Gamma^c \cong P$*, where* P *is the Petersen graph.*

(8) Γ *or* $\Gamma^c \cong Y[2K_1]$*, where* Y *is a Cayley graph.*

(9) Γ *or* $\Gamma^c \cong B(PG(n, q))$ *or* $C(PG(n, q))$*, where* $p = \frac{q^n-1}{q-1}$*, in which* $\mathrm{Aut}(\Gamma) = P\Gamma L(n, q)$*.*

(10) Γ *or* $\Gamma^c \cong B(H(11))$ *or* $C(H(11))$*, in which case* $\mathrm{Aut}(\Gamma) = PGL(2, 11)$*.*

(11) There exist $\alpha, \sigma \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = \langle \alpha \rangle \rtimes \langle \sigma \rangle \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_d$*, where* d *is a divisor of* $p-1$ *and* $\rho_x = \alpha^{p-1}$*, where* $R_G = \langle \rho_x \rangle$ *(for more details about the maps* α *and* σ *and the structure of* Γ*, see [9, Theorem 2.2(v)]).*

(12) There exists $\omega \in \mathrm{Aut}(\Gamma)$ *such that* $\mathrm{Aut}(\Gamma) = R_G \rtimes \langle \omega \rangle$ *(for more details about the map* ω *and the structure of* Γ*, see [9, Theorem 2.2(vi)]).*

Remark 2.5. In Theorem 2.4, all graphs other than (1)–(5) are vertex-transitive. Also in all cases other than (1) and (6), Γ and Γ^c are both connected. Moreover, in the cases (8)–(12), Γ is imprimitive and in case (8), Γ has only 2-blocks and in the cases (9)–(12), Γ has at least one p-block (see the proofs of Theorems 2.1 and 2.2 of [9] for more details).

3. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a bicirculant over a group $G = \langle x \rangle$ of prime order p. We denote the vertex set and the automorphism group of Γ by V and A, respectively. Also we assume that X is the set defined in Theorem 2.1.

Proof. Suppose that Γ is non-normal. Then Γ is one of the twelve graphs given in Theorem 2.4. In the cases (5) and (12), Γ is normal. Also, in the case (11), $\langle \alpha \rangle$ is a normal subgroup of A and R_G is a characteristic subgroup of $\langle \alpha \rangle$, which means that $R_G \trianglelefteq A$, i.e. Γ is normal. So Γ is one of the graphs (1)–(4) or (6)–(10).

First we assume that $p = 2$ and $G = \langle x \rangle \cong \mathbb{Z}_2$. Then Γ has 4 vertices and $R, L \in \{\emptyset, \{x\}\}$, and $S \in \{\emptyset, \{1\}, \{x\}, G\}$. By considering all possibilities of R, L and S, since Γ is non-normal, we have one of the following cases:

(a) Γ or $\Gamma^c = \mathrm{SC}(G, G^*, G^*) \cong K_4$,

(b) Γ or $\Gamma^c = \mathrm{SC}(G, G^*, G^*, \{1_G\})$,

(c) Γ or $\Gamma^c = \mathrm{SC}(G, \emptyset, \emptyset, \{1_G\})$.

Now suppose that $p > 2$. First, let Γ be a graph of type (1), i.e. $\Gamma = \mathrm{SC}(G; R, L, \emptyset) = \Gamma_1 + \Gamma_2$, where $\Gamma_1 \cong \mathrm{Cay}(G, R)$ and $\Gamma_2 \cong \mathrm{Cay}(G, L)$. We claim that Γ is non-normal. Let $B = \mathrm{Aut}(\Gamma_1)$ and $C = \mathrm{Aut}(\Gamma_2)$. Then $A = B \times C$. Without loss of generality, we may assume that $V(\Gamma_1) = G \times \{1\}$ and $V(\Gamma_2) = G \times \{2\}$. By [5, Exercise 14.13], $B \ncong \mathbb{Z}_p$. Hence $B_{(1,1)} \neq 1_B$. Choose an element $\varphi \in B_{(1,1)} \setminus \{1_B\}$. Then $(\varphi, 1_C) \in A_{(1,1)}$. Suppose, contrary to our claim, that Γ is normal. Then by Proposition 2.2, there exist $\sigma \in \mathrm{Aut}(G)$ and $g \in G$ such that for all $x \in G$, $(x, 1)^\varphi = (x^\sigma, 1)$ and $(x, 2)^{1_C} = (gx^\sigma, 2)$. The second equation implies that $g = 1_G$ and $\sigma = 1_{\mathrm{Aut}(G)}$. Hence $\varphi = 1_B$, a contradiction.

Now let Γ be a graph of type (6) or (7). Then Γ is primitive, by Remark 2.5, and so by [3, Theorem 1.6A(v)], Γ is non-normal. If Γ is of type (8), then by Remark 2.5 and [3, Theorem 1.6A(i)], Γ is non-normal. In the cases (4) and (10), since $\mathrm{PSL}(2, 11)$ and $\mathrm{PGL}(2, 11)$ are not solvable, by Corollary 2.3, Γ is non-normal.

Finally, we examine the remaining graphs Γ of types (2), (3) and (9). First note that if $\mathrm{Aut}(\Gamma) \cong S_3$, then Γ is normal. Hence $\mathrm{Aut}(\Gamma) \ncong S_3$. In the cases (3) and (9), $p = \frac{q^n-1}{q-1}$ is a prime. If $p = 3$, then $n = q = 2$ and $\mathrm{P\Gamma L}(n, q) \cong \mathrm{P\Sigma L}(n, q) \cong S_3$, contradicting the non-normality of Γ. Hence $p > 3$. Since S_p has no normal subgroup of order p, the graph (2) is non-normal. In the cases (3) and (9), $p = \frac{q^n-1}{q-1}$ is a prime and the assumption $p > 3$ implies that $(n,q) \neq (2,2)$. Since $\frac{q^n-1}{q-1}$ is a prime, we conclude that $(n,q) \neq (2,2),(2,3)$. Since $\mathrm{PG}(n,q)$ and $\mathrm{PSL}(n,q)$ are solvable only when $(n,q) \in \{(2,2),(2,3)\}$, and $\mathrm{PGL}(n,q)$ and $\mathrm{PSL}(n,q)$ are isomorphic to a normal subgroup of $\mathrm{P\Gamma L}(n,q)$ and $\mathrm{P\Sigma L}(n,q)$, respectively, we conclude that $\mathrm{P\Gamma L}(n,q)$ and $\mathrm{P\Sigma L}$ are not solvable and so the graphs of type (3) and (9) are non-normal, by Corollary 2.3. We have showed that in the case $p > 3$, the graphs (2), (3) and (9) are non-normal, which completes the proof. □

References

[1] M. Arezoomand, B. Taeri, *Normality of 2-Cayley digraphs*, Discrete Math., **338** (2015), 41–47.

[2] M. Arezoomand, B. Taeri, *A classification of finite groups with integral bi-Cayley graphs*, Trans. Comb., **4(4)** (2015), 55–61.

[3] J. D. Dixon, B. Mortimer, *Permutation Groups*, New York, Springer-Verlag, 1996.

[4] Y. Q. Feng, Z. P. Lu, M. Y. Xu, *Automorphism groups of Cayley digraphs*, Applications of Group Theory to Combinatorics, CRC Press/Balkema 2008, 13–25.

[5] F. Harary, *Graph Theory*, Addison-Welsey Publishing Company, 1969.

[6] H. Koike, I. Kovács, *Isomorphic tetravalent cyclic Haar graphs*, Ars Math. Contemp., **7** (2014), 215–235.

[7] I. Kovács, B. Kuzman, A. Malnič, *On non-normal arc transitive 4-valent dihedrants*, Acta Math. Sin. Eng. Ser. **26** (2010), 1485–1498.

[8] I. Kovács, B. Kuzman, A. Malnič, S. Wilson, *Characterization of edge-transitive 4-valent bicirculants*, J. Graph Theory, **69** (2012), 441–463.

[9] A. Malnič, D. Marušič, P. Šparl, B. Frelih, *Symmetry structure of bicirculants*, Discrete Math., **307** (2007), 409–414.

[10] T. Pisanski, *A classification of cubic bicirculants*, Discrete Math., **307** (2007), 567–578.

[11] M. J. de Resmini, D. Jungnickel, *Strongly regular semi-Cayley graphs*, J. Algebraic Combin., **1** (1992), 217–228.

[12] G. Sabidussi, *Vertex-transitive graphs*, Monatsh. Math., **68** (1964), 426–438.

[13] M. Y. Xu, *Automorphism groups and isomorphisms of Cayley digraphs*, Discrete Math., **182** (1998), 309–319.

[14] J. X. Zhou, Y. Q. Feng, *The automorphisms of bi-Cayley graphs*, J. Combin. Theory Ser. B, **116** (2016), 504–532.

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