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EXACT FORMULAE OF GENERAL SUM-CONNECTIVITY INDEX FOR SOME GRAPH OPERATIONS

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Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $a \in V(G)$ is denoted by $d_G(a)$. The general sum-connectivity index of G is defined as $\chi_\alpha(G) = \sum_{ab\in E(G)}(d_G(a) + d_G(b))^\alpha$, where α is a real number. In this paper, we compute exact formulae for general sum-connectivity index of several graph operations. These operations include tensor product, union of graphs, splices and links of graphs and Hajós construction of graphs. Moreover, we also compute exact formulae for general sum-connectivity index of some graph operations for positive integral values of α. These operations include cartesian product, strong product, composition, join, disjunction and symmetric difference of graphs.

1. Introduction and preliminary results

Let $G = (V(G), E(G))$ be a simple and connected graph. An edge with end vertices a and b is denoted by ab (or ba). The order and size of graph G are denoted by n_G and m_G, respectively. The degree of a vertex $a \in V(G)$, denoted by $d_G(a)$, is the number of vertices incident with a. A path P_n of length $n-1$ is a graph with vertex set $\{a_i \mid i = 1,\dots,n\}$ and edge set $\{a_ia_{i+1} \mid i = 1,\dots,n-1\}$. A cycle C_n of length n is a graph with vertex set $\{a_i \mid i = 1,\dots,n\}$ and edge set $\{a_ia_{i+1} \mid i = 1,\dots,n-1\} \cup \{a_na_1\}$. A complete graph of order n is denoted by K_n.

A topological index is a mathematical measure which correlate to the chemical structures of any simple finite graph. They play an important role in the study of QSAR/QSPR. There are numerous topological descriptors that have some applications in theoretical chemistry. Among these topological descriptors, the degree based topological indices are of great importance.

The first degree based topological indices that are defined by Gutman and Trinajstić [8] in 1972, are the first and second Zagreb indices. These indices are defined as follows: $M_1(G) = \sum\limits_{a\in V(G)} (d_G(a))^2$, $M_2(G) = \sum\limits_{ab\in E(G)} d_G(a)d_G(b)$. Here $M_1(G)$ and $M_2(G)$

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denote the first and second Zagreb index, respectively. Li and Zhao [13] introduced the *first general Zagreb index*:

$$M_\alpha(G) = \sum_{a\in V(G)} (d_G(a))^\alpha, \quad \alpha \in \mathbb{R}.$$

The general Randić index (product-connectivity index), introduced by Li and Gutman [12], is defined in the following way:

$$R_\alpha(G) = \sum_{ab\in E(G)} (d_G(a)d_G(b))^\alpha, \quad \alpha \in \mathbb{R}.$$

Then $R_{-1/2}(G)$ is called the Randić index which was defined by Randić [15] in 1975. The general sum-connectivity index is introduced by Zhou and Trinajstić [18] and is defined as:

$$\chi_\alpha(G) = \sum_{ab\in E(G)} (d_G(a) + d_G(b))^\alpha, \quad \alpha \in \mathbb{R}.$$

Then $\chi_{-1/2}(G)$ is the classical sum-connectivity index which was defined by Zhou and Trinajstić [17] in 2009. Another variant of the Randić index of G is the harmonic index, denoted by $H(G)$ and is defined as follows:

$$H(G) = \sum_{ab\in E(G)} \frac{2}{d_G(a) + d_G(b)} = 2\chi_{-1}(G).$$

Recently Ashrafi et al. [1] computed the exact formulae for Zagreb coindices of some graph operations. Ashrafi et al. [2] calculated some topological indices of splices and links of graphs. In [16], Yarahmadi computed some topological indices of tensor product of graphs. For a detailed study on topological indices of graph operations, we refer the reader to [3, 4, 6, 7, 9–11, 14].

This paper is organized as follows. In Section 2, we present some known graph operations. In Section 3, we give exact formulae of the general sum-connectivity index for several graph operations. These graph operations include tensor product, union of graphs, splices and links of graphs and Hajós construction of graphs for real values of α. In Section 4, we compute exact formulae of the general sum-connectivity index of some graph operations including cartesian product, strong product, composition, join, disjunction and symmetric difference of graphs for positive integral values of α.

2. Graph operations

Let G and H be two simple connected graphs whose vertex sets are disjoint. The set of real numbers and the set of positive integers are denoted by $\mathbb{R}$ and $\mathbb{Z}^+$, respectively.

The *tensor product* of G and H, denoted by $G \otimes H$, is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and $(a, b)(c, d) \in E(G \otimes H)$ whenever $ac \in E(G)$ and $bd \in E(H)$. The order and the size of $G \otimes H$ are $n_G n_H$ and $2m_G m_H$, respectively. The degree of a vertex $(a, b) \in V(G \otimes H)$ is given by

$$d_{G\otimes H}(a, b) = d_G(a)d_H(b). \tag{1}$$

The *union* of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The order and size of $G \cup H$ are $n_G + n_H$ and $m_G + m_H$, respectively. For $a \in V(G \cup H)$, either $a \in V(G)$ or $a \in V(H)$ but not both. The degree of a in $G \cup H$ is given by

$$d_{G \cup H}(a) = \begin{cases} d_G(a) & \text{if } a \in V(G), \\ d_H(a) & \text{if } a \in V(H). \end{cases}$$

Splices of graphs (also known as coalescences of graphs) were introduced by Doslić [5] in 2005. A splice of G and H for given vertices $a \in V(G)$ and $b \in V(H)$, denoted by $(G \bullet H)(a, b)$, is defined by identifying the vertices a and b in the union of G and H. The order and the size of $(G \bullet H)(a, b)$ are $n_G + n_H - 1$ and $m_G + m_H$, respectively. Let x be the vertex obtained by identifying $a \in V(G)$ and $b \in V(H)$. Then $V((G \bullet H)(a, b)) = \{x\} \cup \big(V(G) \cup V(H)\big)$ and $E((G \bullet H)(a, b)) = E(G) \cup E(H)$, respectively. The degree of a vertex $c \in V((G \bullet H)(a, b))$ is given by

$$d_{(G \bullet H)(a,b)}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(a) + d_H(b) & \text{if } c = x. \end{cases} \tag{2}$$

The motivation for considering these graphs comes from chemistry, where splices of cycles serves as models of spirane molecules and models of complex molecules are built from simpler building blocks by iterating and/or combining the splice and link operations.

Links of graphs were introduced by Doslić [5] in 2005. The link of G and H for given vertices $a \in V(G)$ and $b \in V(H)$, denoted by $(G \sim H)(a, b)$, is obtained by joining the vertices a and b by an edge in the union of G and H. The order and the size of $(G \sim H)(a, b)$ are $n_G + n_H$ and $m_G + m_H + 1$, respectively. The set of vertices and edges are $V((G \sim H)(a, b)) = V(G) \cup V(H)$ and $E((G \sim H)(a, b)) = (E(G) \cup V(H)) \cup \{ab\}$, respectively. The degree of a vertex $c \in V((G \sim H)(a, b))$ is given by

$$d_{(G \sim H)(a,b)}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(c) + 1 & \text{if } c = a, \\ d_H(c) + 1 & \text{if } c = b. \end{cases} \tag{3}$$

Let $a\acute{a}$ be an edge in G and $b\acute{b}$ be an edge in H. Then the *Hajós construction* of graphs, denoted by $(G \triangle H)(a\acute{a}, b\acute{b})$, is obtained by identifying a and b, deleting the edges $a\acute{a}$ and $b\acute{b}$, and adding an edge $\acute{a}\acute{b}$. The order and size of $(G \triangle H)(a\acute{a}, b\acute{b})$ are $n_G + n_H - 1$ and $m_G + m_H - 1$, respectively. Let x be the vertex obtained by identifying $a \in V(G)$ and $b \in V(H)$. Then $V((G \triangle H)(a\acute{a}, b\acute{b})) = \{x\} \cup (V(G) \setminus \{a\}) \cup (V(H) \setminus \{b\})$ and $E((G \triangle H)(a\acute{a}, b\acute{b})) = (E(G) \setminus \{a\acute{a}\}) \cup (V(H) \setminus \{b\acute{b}\}) \cup \{\acute{a}\acute{b}\}$. The degree of a vertex $c \in V((G \triangle H)(a\acute{a}, b\acute{b}))$ is given by

$$d_{(G \triangle H)(a\acute{a}, b\acute{b})}(c) = \begin{cases} d_G(c) & \text{if } c \in V(G), c \neq a, \\ d_H(c) & \text{if } c \in V(H), c \neq b, \\ d_G(a) + d_H(b) - 2 & \text{if } c = x. \end{cases} \tag{4}$$

The *cartesian product* of G and H, denoted by $G \square H$, is the graph with vertex set

$V(G\Box H)=V(G)\times V(H)$ and $(a,b)(c,d)\in E(G\Box H)$ whenever $[a=c$ and $bd\in E(H)]$ or $[ac\in E(G)$ and $b=d]$. The order and size of $G\Box H$ are n_Gn_H and $m_Gn_H+m_Hn_G$, respectively. The degree of a vertex $(a,b)\in V(G\Box H)$ is given by

$$d_{G\Box H}(a,b)=d_G(a)+d_H(b). \tag{5}$$

The *strong product* of G and H, denoted by $G\boxtimes H$, is the graph with vertex set $V(G\boxtimes H)=V(G)\times V(H)$ and $(a,b)(c,d)\in E(G\boxtimes H)$ whenever $[a=c$ and $bd\in E(H)]$ or $[ac\in E(G)$ and $b=d]$ or $[ac\in E(G)$ and $bd\in E(H)]$. The order and size of $G\boxtimes H$ are n_Gn_H and $n_Gm_H+n_Hm_G+2m_Gm_H$, respectively. The degree of a vertex $(a,b)\in V(G\boxtimes H)$ is given by

$$d_{G\boxtimes H}(a,b)=d_G(a)+d_H(b)+d_G(a)d_H(b). \tag{6}$$

The *composition* (lexicographic product) of G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H])=V(G)\times V(H)$ and $(a,b)(c,d)\in E(G[H])$ whenever $[a$ is adjacent to c in $G]$ or $[a=c$ and b is adjacent to d in $H]$. The order and size of $G[H]$ are n_Gn_H and $m_Gn_H^2+n_Gm_H$, respectively. The degree of a vertex $(a,b)\in V(G[H])$ is given by

$$d_{G[H]}(a,b)=n_Hd_G(a)+d_H(b). \tag{7}$$

The *join* of G and H, denoted by $G+H$, is the graph union $G\cup H$ together with all the edges joining $V(G)$ and $V(H)$. The order and size of $G+H$ are n_G+n_H and $m_G+m_H+n_Gn_H$, respectively. The degree of a vertex a in $G+H$ is given by

$$d_{G+H}(a)=\begin{cases} d_G(a)+n_H & \text{if } a\in V(G),\\ d_H(a)+n_G & \text{if } a\in V(H).\end{cases} \tag{8}$$

The *disjunction* of G and H, denoted by $G\vee H$, is the graph with vertex set $V(G\vee H)=V(G)\times V(H)$ and $(a,b)(c,d)\in E(G\vee H)$ whenever $ac\in E(G)$ or $bd\in E(H)$. The order and size of $G\vee H$ are n_Gn_H and $m_Gn_H^2+m_Hn_G^2-2m_Gm_H$, respectively. The degree of a vertex (a,b) in $G\vee H$ is given by

$$d_{G\vee H}(a,b)=n_Hd_G(a)+n_Gd_H(b)-d_G(a)d_H(b). \tag{9}$$

The *symmetric difference* of G and H, denoted by $G\oplus H$, is the graph with vertex set $V(G\oplus H)=V(G)\times V(H)$ and $(a,b)(c,d)\in E(G\oplus H)$ whenever $ac\in E(G)$ or $bd\in E(H)$ but not both. The order and size of $G\oplus H$ are n_Gn_H and $m_Gn_H^2+m_Hn_G^2-4m_Gm_H$, respectively. The degree of a vertex (a,b) in $G\oplus H$ is given by

$$d_{G\oplus H}(a,b)=n_Hd_G(a)+n_Gd_H(b)-2d_G(a)d_H(b).$$

3. Formulae of general sum-connectivity index when $\alpha\in\mathbb{R}$

In this section, we derive exact formulae of general sum-connectivity index for some graph operations defined in Section 2. In the following theorem, we compute the general sum-connectivity index of $G\otimes H$.

Theorem 3.1. *Let G and H be two graphs such that either G or H is regular. Then* *the general sum-connectivity index of $G \otimes H$ is given by the formula:*

$$\chi_\alpha(G \otimes H) = \frac{1}{2^{\alpha-1}}\chi_\alpha(G)\chi_\alpha(H).$$

Proof. By equation (1) and the definition of general sum-connectivity index, we have

$$\chi_\alpha(G \otimes H) = 2\sum_{ac\in E(G)}\sum_{bd\in E(H)}\left(d_G(a)d_H(b) + d_G(c)d_H(d)\right)^\alpha. \tag{10}$$

Without loss of generality, assume that G is a regular graph.

For each $(a,b)(c,d) \in E(G \otimes H)$, we have $d_G(a) = d_G(b)$. Thus

$$\begin{aligned} d_G(a)d_H(b)+d_G(c)d_H(d) &= \frac{1}{2}(d_G(a)d_H(b) + d_G(a)d_H(b) + d_G(c)d_H(d) + d_G(c)d_H(d)) \\ &= \frac{1}{2}(d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d)) \qquad (11) \\ &= \frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d)). \end{aligned}$$

Then (10) together with (11) gives

$$\begin{aligned} \chi_\alpha(G \otimes H) &= 2\sum_{ac\in E(G)}\sum_{bd\in E(H)}(d_G(a)d_H(b) + d_G(c)d_H(d))^\alpha \\ &= 2\sum_{ac\in E(G)}\sum_{bd\in E(H)}\left(\frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d))\right)^\alpha \\ &= \left(\frac{2}{2^\alpha}\right)\sum_{ac\in E(G)}(d_G(a) + d_G(c))^\alpha\sum_{bd\in E(H)}\left(d_H(b) + d_H(d)\right)^\alpha = \frac{1}{2^{\alpha-1}}\chi_\alpha(G)\chi_\alpha(H). \qquad \square \end{aligned}$$

Example 3.2. Applying Theorem 3.1, the general sum-connectivity index for tensor product of some graphs is given below:

1) $\chi_\alpha(C_n \otimes C_m) = 2^{3\alpha+1}mn$, 2) $\chi_\alpha(K_n \otimes K_m) = \frac{1}{2^{\alpha+1}}mn[(n+1)(m+1)]^\alpha$.

In the following theorem, we give the general sum-connectivity index of union of finite number of graphs. The proof is omitted since it can be easily derived.

Theorem 3.3. *Let $G_1, G_2, \ldots, G_n$ be vertex-disjoint graphs. Then the general sum-connectivity index of $\bigcup\limits_{i=1}^{k} G_i$ is given by the following formula:*

$$\chi_\alpha\Big(\bigcup_{i=1}^{k} G_i\Big) = \chi_\alpha(G_1) + \chi_\alpha(G_2) + \ldots + \chi_\alpha(G_n).$$

Example 3.4. Using Theorem 3.3, the general sum-connectivity index for union of some graphs is given below:

1) $\chi_\alpha\left(\bigcup\limits_{i=1}^{k} C_{n_i}\right) = 4^\alpha\sum\limits_{i=1}^{k} n_i$, 2) $\chi_\alpha\left(\bigcup\limits_{i=1}^{k} P_{n_i}\right) = 2k(3^\alpha + 4^\alpha) + 4^\alpha\sum\limits_{i=1}^{k} n_i$.

In the following theorem, we compute the general sum-connectivity index of splices of two graphs G and H for given vertices a and b.

THEOREM 3.5. *Let G and H be two graphs. Then the general sum-connectivity index of $(G \bullet H)(a,b)$ is given by the formula:*

$$\chi_\alpha\left((G \bullet H)(a,b)\right) = \chi_\alpha(G) + \sum_{d \in N_G(a)} \left[(d_G(a) + d_H(b) + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha\right]$$
$$+ \chi_\alpha(H) + \sum_{d \in N_H(b)} \left[(d_H(b) + d_G(a) + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha\right].$$

Proof. By equation (2) and the definition of general sum-connectivity index, we have

$$\begin{aligned}
\chi_\alpha\left((G \bullet H)(a,b)\right) &= \sum_{\substack{cd \in E(G)\\ c,d \neq a}} (d_G(c) + d_G(d))^\alpha + \sum_{\substack{cd \in E(H)\\ c,d \neq b}} (d_H(c) + d_H(d))^\alpha \\
&+ \sum_{\substack{cd \in E(G), c=a\\ d \in V(G)}} (d_G(c) + d_G(d) + d_H(b))^\alpha + \sum_{\substack{cd \in E(H), c=b\\ d \in V(H)}} (d_G(a) + d_H(c) + d_H(d))^\alpha \\
&= \chi_\alpha(G) - \sum_{d \in N_G(a)} (d_G(a) + d_G(d))^\alpha + \chi_\alpha(H) - \sum_{d \in N_H(b)} (d_H(b) + d_H(d))^\alpha \\
&+ \sum_{d \in N_G(a)} (d_G(a) + d_H(b) + d_G(d))^\alpha + \sum_{d \in N_H(b)} (d_H(b) + d_G(a) + d_H(d))^\alpha \\
&= \chi_\alpha(G) + \sum_{d \in N_G(a)} \left[(d_G(a) + d_H(b) + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha\right] \\
&+ \chi_\alpha(H) + \sum_{d \in N_H(b)} \left[(d_H(b) + d_G(a) + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha\right]. \qquad \square
\end{aligned}$$

EXAMPLE 3.6. Applying Theorem 3.5, the general sum-connectivity index for splices of K_n and C_m is given below:

$$\chi_\alpha((K_n \bullet C_m)(a,b)) = 2^{\alpha-1} n(n-1)^\alpha + 4^\alpha(m-2) + 2^\alpha(n-1)(n^\alpha + (n-1)^\alpha) + 2(n+3)^\alpha.$$

In the following theorem, we compute the general sum-connectivity index of links of two graphs G and H for given vertices a and b.

THEOREM 3.7. *Let G and H be two graphs. Then the general sum-connectivity index of $(G \sim H)(a,b)$ is given by the formula:*

$$\chi_\alpha\left((G \sim H)(a,b)\right) = \chi_\alpha(G) + \chi_\alpha(H) + \sum_{d \in N_G(a)} \left[(d_G(a) + 1 + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha\right]$$
$$+ (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d \in N_H(b)} \left[(d_H(b) + 1 + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha\right].$$

Proof. Equation (3) and the definition of general sum-connectivity index give

$$\chi_\alpha\left((G \sim H)(a,b)\right) = \sum_{\substack{cd \in E(G)\\ c,d \neq a}} (d_G(c) + d_G(d))^\alpha + \sum_{\substack{cd \in E(H)\\ c,d \neq b}} (d_H(c) + d_H(d))^\alpha$$

$$+ (d_G(a) + d_H(b) + 2)^\alpha + \sum_{\substack{cd\in E(G),c=a\\ d\in V(G)}} (d_G(c) + 1 + d_G(d))^\alpha + \sum_{\substack{cd\in E(H),c=b\\ d\in V(H)}} (d_H(c) + 1 + d_H(d))^\alpha$$
$$= \chi_\alpha(G) - \sum_{d\in N_G(a)} (d_G(a) + d_G(d))^\alpha + \chi_\alpha(H) - \sum_{d\in N_H(b)} (d_H(b) + d_H(d))^\alpha$$
$$+ (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d\in N_G(a)} (d_G(a) + 1 + d_G(d))^\alpha + \sum_{d\in N_H(b)} (d_H(b) + 1 + d_H(d))^\alpha$$
$$= \chi_\alpha(G) + \chi_\alpha(H) + \sum_{d\in N_G(a)} [(d_G(a) + 1 + d_G(d))^\alpha - (d_G(a) + d_G(d))^\alpha]$$
$$+ (d_G(a) + d_H(b) + 2)^\alpha + \sum_{d\in N_H(b)} [(d_H(b) + 1 + d_H(d))^\alpha - (d_H(b) + d_H(d))^\alpha]. \qquad \square$$

Example 3.8. Applying Theorem 3.7, the general sum-connectivity index of $(P_n \sim C_m)(a,b)$ is given by $\chi_\alpha((P_n \sim C_m)(a,b)) = 2^{\alpha+1}(2 \times 3^\alpha m + m(n-3)4^\alpha)$.

Theorem 3.9. *Let G and H be two graphs. Then the general sum-connectivity index of $(G \triangle H)(a\acute{a}, b\acute{b})$ is given by the formula:*

$$\chi_\alpha((G \triangle H)(a\acute{a}, b\acute{b})) = \chi_\alpha(G) + \chi_\alpha(H) + (d_G(\acute{a}) + d_H(\acute{b}))^\alpha$$
$$+ \sum_{\acute{a}\neq d\in N(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha - \sum_{d\in N_G(a)} (d_G(a) + d_G(d))^\alpha$$
$$+ \sum_{\acute{b}\neq d\in N(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha - \sum_{d\in N_H(b)} (d_H(b) + d_H(d))^\alpha.$$

Proof. By equation (4) and the definition of general sum-connectivity index, we get

$$\chi_\alpha((G \triangle H)(a\acute{a}, b\acute{b})) = \sum_{\substack{cd\in E(G)\\ c,d\neq a}} (d_G(c) + d_G(d))^\alpha + \sum_{\substack{cd\in E(H)\\ c,d\neq b}} (d_H(c) + d_H(d))^\alpha + (d_G(\acute{a}) + d_H(\acute{b}))^\alpha$$
$$+ \sum_{\substack{cd\in E(G),c=a\\ \acute{a}\neq d\in V(G)}} (d_G(c) + d_H(b) - 2 + d_G(d))^\alpha + \sum_{\substack{cd\in E(H),c=b\\ \acute{b}\neq d\in V(H)}} (d_H(c) + d_G(a) - 2 + d_H(d))^\alpha$$
$$= \chi_\alpha(G) - (d_G(a) + d_G(\acute{a}))^\alpha - \sum_{\acute{a}\neq d\in N_G(a)} (d_G(a) + d_G(d))^\alpha + \chi_\alpha(H) - (d_H(b) + d_H(\acute{b}))^\alpha$$
$$- \sum_{\acute{b}\neq d\in N_H(b)} (d_H(b) + d_H(d))^\alpha + \sum_{\acute{a}\neq d\in N_G(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha$$
$$+ \sum_{\acute{b}\neq d\in N_H(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha + (d_G(\acute{a}) + d_H(\acute{b}))^\alpha$$
$$= \chi_\alpha(G) + \chi_\alpha(H) + (d_G(\acute{a}) + d_H(\acute{b}))^\alpha + \sum_{\acute{a}\neq d\in N_G(a)} (d_G(a) + d_H(b) - 2 + d_G(d))^\alpha$$
$$- \sum_{d\in N(a)} (d_G(a) + d_G(d))^\alpha + \sum_{\acute{b}\neq d\in N_H(b)} (d_H(b) + d_G(a) - 2 + d_H(d))^\alpha - \sum_{d\in N(b)} (d_H(b) + d_H(d))^\alpha.$$

$\square$

EXAMPLE 3.10. Applying Theorem 3.9, the general sum-connectivity index of $(C_n \sim K_m)(aa', bb')$ is given by:

$$\chi_\alpha((C_n \sim K_m)(aa', bb')) = (n-2)4^\alpha + 2(m+1)^\alpha + 2^{\alpha-1}(m-1)^\alpha(m^2-m-2).$$

4. Formulae of general sum-connectivity index when $\alpha \in \mathbb{Z}$

In this section, we derive exact formulae of general sum-connectivity index for some graph operations defined in Section 2 when $\alpha \in \mathbb{Z}$.

THEOREM 4.1. *Let G and H be two graphs and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G\Box H$ is given by the formula:*

$$\chi_\alpha(G\Box H) = \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} (M_n(G)\chi_{\alpha-n}(H) + M_n(H)\chi_{\alpha-n}(G)).$$

Proof. By equation (5) and the definition of general sum-connectivity index, we have

$$\chi_\alpha(G\Box H) = \sum_{u\in V(G)} \sum_{bd\in E(H)} (2d_G(u) + d_H(b) + d_H(d))^\alpha + \sum_{ac\in E(G)} \sum_{v\in V(H)} (2d_H(v) + d_G(a) + d_G(c))^\alpha. \quad (12)$$

Now, for $u, a, c \in V(G)$ and $v, b, d \in V(H)$, using binomial expansion, we obtain

$$(2d_G(u) + (d_H(b) + d_H(d)))^\alpha = \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} d_G(u)^n (d_H(b) + d_H(d))^{\alpha-n}, \quad (13)$$

$$(2d_H(v) + (d_G(a) + d_G(c)))^\alpha = \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} d_H(v)^n (d_G(a) + d_G(c))^{\alpha-n}. \quad (14)$$

Using equations (13) and (14) in equation (12), we get

$$\begin{aligned}\chi_\alpha(G\Box H) &= \sum_{u\in V(G)} \sum_{bd\in E(H)} \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} d_G(u)^n (d_H(b) + d_H(d))^{\alpha-n} \\ &\quad + \sum_{ac\in E(G)} \sum_{v\in V(H)} \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} d_H(v)^n (d_G(a) + d_G(c))^{\alpha-n} \\ &= \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} \sum_{u\in V(G)} d_G(u)^n \sum_{bd\in E(H)} (d_H(b) + d_H(d))^{\alpha-n} \\ &\quad + \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} \sum_{v\in V(H)} d_H(v)^n \sum_{ac\in E(G)} (d_G(a) + d_G(c))^{\alpha-n} \\ &= \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} M_n(G)\chi_{\alpha-n}(H) + \sum_{n=0}^{\alpha} 2^n \binom{\alpha}{\alpha-n} M_n(H)\chi_{\alpha-n}(G).\end{aligned}$$

□

Example 4.2. Using Theorem 4.1, the general sum-connectivity index of cartesian product of C_l and C_m is $\chi_\alpha(C_l \Box C_m) = 2^{3\alpha+1}lm$.

In the following theorem, we compute the general sum-connectivity index of $G \boxtimes H$.

Theorem 4.3. *Let G and H be two graphs such that either G or H is regular and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G \boxtimes H$ is given by the formula:*

$$\chi_\alpha(G \boxtimes H) = \sum_{n=0}^{\alpha} \frac{1}{2^{n-1}} \binom{\alpha}{\alpha-n} \chi_n(G) \sum_{\beta=0}^{n} 2^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha-n+\beta}(H) \tag{15}$$

$$+ \sum_{n=0}^{\alpha} \binom{\alpha}{\alpha-n} (M_n(G) \sum_{\beta=0}^{n} 2^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha+\beta-n}(H) + M_n(H) \sum_{\beta=0}^{n} 2^{n-\beta} \binom{n}{n-\beta} \chi_{\alpha+\beta-n}(G)).$$

Proof. By equation (6) and the definition of general sum-connectivity index, we get

$$\chi_\alpha(G \boxtimes H) = \sum_{u \in V(G)} \sum_{bd \in E(H)} (2d_G(u) + (d_H(b) + d_H(d)) + d_G(u)(d_H(b) + d_H(d)))^\alpha$$

$$+ \sum_{ac \in E(G)} \sum_{v \in V(H)} (2d_H(v) + (d_G(a) + d_G(c)) + d_H(v)(d_G(a) + d_G(c)))^\alpha$$

$$+ 2 \sum_{ac \in E(G)} \sum_{bd \in E(H)} ((d_G(a) + d_G(c)) + (d_H(b) + d_H(d)) + (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha$$

Now, for $u \in V(G)$ and $b, d \in V(H)$, using binomial expansion, we obtain

$$(2d_G(u) + (d_H(b) + d_H(d)) + d_G(u)(d_H(b) + d_H(d)))^\alpha$$
$$= \sum_{n=0}^{\alpha} \binom{\alpha}{\alpha-n} d_G(u)^n \sum_{\beta=0}^{n} 2^{n-\beta} \binom{n}{n-\beta} (d_H(b) + d_H(d))^{\alpha-n+\beta}. \tag{16}$$

Similarly, for $v \in V(H)$ and $a, c \in V(G)$, we obtain

$$(2d_H(v) + (d_G(a) + d_G(c)) + d_H(v)(d_G(a) + d_G(c)))^\alpha$$
$$= \sum_{n=0}^{\alpha} \binom{\alpha}{\alpha-n} d_H(v)^n \sum_{\beta=0}^{n} 2^{n-\beta} \binom{n}{n-\beta} (d_G(a) + d_G(c))^{\alpha-n+\beta}. \tag{17}$$

Without loss of generality, assume that G is a regular graph. Then for $(a, b)(c, d) \in E(G \boxtimes H)$, we obtain

$$\begin{aligned} d_G(a)d_H(b) + d_G(c)d_H(d) &= \frac{1}{2}(d_G(a)d_H(b) + d_G(a)d_H(b) + d_G(c)d_H(d) + d_G(c)d_H(d)) \\ &= \frac{1}{2}(d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d)) \\ &= \frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d)). \end{aligned}$$

Using binomial expansion, we get

$$(d_G(a) + d_G(c)) + (d_H(b) + d_H(d)) + (d_G(a)d_H(b) + d_G(c)d_H(d))^\alpha$$

$$=\sum_{n=0}^{\alpha}\frac{1}{2^{n-1}}\binom{\alpha}{\alpha-n}(d_G(a)+d_G(c))^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}(d_H(b)+d_H(d))^{\alpha-n+\beta}. \tag{18}$$

Using equations (16)–(18) in equation (3), we get

$$\begin{aligned}
\chi_\alpha(G\boxtimes H) &= \sum_{u\in V(G)}\sum_{bd\in E(H)}\sum_{n=0}^{\alpha}\binom{\alpha}{\alpha-n}d_G(u)^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}(d_H(b)+d_H(d))^{\alpha-n+\beta}\\
&+\sum_{ac\in E(G)}\sum_{v\in V(H)}\sum_{n=0}^{\alpha}\binom{\alpha}{\alpha-n}d_H(v)^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}(d_G(a)+d_G(c))^{\alpha-n+\beta}\\
&+\sum_{ac\in E(G)}\sum_{bd\in E(H)}\sum_{n=0}^{\alpha}\frac{1}{2^{n-1}}\binom{\alpha}{\alpha-n}(d_G(a)+d_G(c))^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}(d_H(b)+d_H(d))^{\alpha-n+\beta}\\
&=\sum_{n=0}^{\alpha}\binom{\alpha}{\alpha-n}\sum_{u\in V(G)}d_G(u)^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}\sum_{bd\in E(H)}(d_H(b)+d_H(d))^{\alpha-n+\beta}\\
&+\sum_{n=0}^{\alpha}\binom{\alpha}{\alpha-n}\sum_{v\in V(H)}d_H(v)^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}\sum_{ac\in E(G)}(d_G(a)+d_G(c))^{\alpha-n+\beta}\\
&+\sum_{n=0}^{\alpha}\frac{1}{2^{n-1}}\binom{\alpha}{\alpha-n}\sum_{ac\in E(G)}(d_G(a)+d_G(c))^n\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}\sum_{bd\in E(H)}(d_H(b)+d_H(d))^{\alpha-n+\beta}.
\end{aligned}$$

Thus, (15) holds. □

EXAMPLE 4.4. Using Theorem 4.3, the general sum-connectivity index of strong product of C_l and P_m is given by:

$$\begin{aligned}
\chi_\alpha(C_l\boxtimes P_m) &= \sum_{n=0}^{\alpha}\binom{\alpha}{\alpha-n}\sum_{\beta=0}^{n}2^{n-\beta}\binom{n}{n-\beta}\Big[2\times 3^{\alpha+\beta}\left(\frac{1}{3^m}+\frac{2^{n+2}l}{3^n}\right)\\
&+4^{\alpha+\beta}(m-2)\left(\frac{1}{4^m}+\frac{l}{2^{n+1}}\right)+4^\beta\left(\frac{1}{2^{2n-1}}+m-2\right)\Big].
\end{aligned}$$

In the following theorem, we give the general sum-connectivity index of composition of two graphs.

THEOREM 4.5. *Let G and H be two graphs such that either G or H is regular and $\alpha\in\mathbb{Z}$. Then the general sum-connectivity index of $G[H]$ is given by the formula:*

$$\begin{aligned}
\chi_\alpha(G[H]) &= \sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}2^k n_H^k M_k(G)\chi_{\alpha-k}(H)\\
&+\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}\chi_{\alpha-k}(G)\sum_{\beta=0}^{k}\binom{k}{k-a\beta}M_\beta(H)M_{k-\beta}(H).
\end{aligned}\tag{19}$$

Proof. Equation (7) and the definition of general sum-connectivity index give

$$\chi_\alpha(G[H]) = \sum_{w\in V(G)}\sum_{uv\in E(H)}(2n_Hd_G(w)+(d_H(u)+d_H(v)))^\alpha$$

$$+\sum_{ac\in E(G)}\sum_{u\in V(H)}\sum_{v\in V(H)}(n_H(d_G(a)+d_G(c))+(d_H(u)+d_H(v)))^{\alpha}. \tag{20}$$

Now, for $w \in V(G)$ and $u, v \in V(H)$, by using binomial expansion, we get

$$(2n_Hd_G(w)+(d_H(u)+d_H(v)))^{\alpha} = \sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}2^kn_H^kd_G(w)^k(d_H(u)+d_H(v))^{\alpha-k}. \tag{21}$$

Similarly, for $a, c \in V(G)$ and $u, v \in V(H)$, we obtain

$$(n_H(d_G(a)+d_G(c))+(d_H(u)+d_H(v)))^{\alpha} = \sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}(d_G(a)+d_G(c))^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}d_H(u)^{\beta}d_H(v)^{k-\beta}. \tag{22}$$

Equation (20) along with equations (21) and (22) gives

$$\chi_\alpha(G[H]) = \sum_{w\in V(G)}\sum_{uv\in E(H)}\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}2^kn_H^kd_G(w)^k(d_H(u)+d_H(v))^{\alpha-k}$$
$$+\sum_{ac\in E(G)}\sum_{u\in V(H)}\sum_{v\in V(H)}\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}(d_G(a)+d_G(c))^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}d_H(u)^{a}d_H(v)^{k-\beta}$$
$$=\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}2^kn_H^k\sum_{w\in V(G)}d_G(w)^k\sum_{uv\in E(H)}(d_H(u)+d_H(v))^{\alpha-k}$$
$$+\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}\sum_{ac\in E(G)}(d_G(a)+d_G(c))^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}\sum_{u\in V(H)}d_H(u)^{\beta}\sum_{v\in V(H)}d_H(v)^{k-\beta}.$$

Thus the required result is given by (19). $\square$

Example 4.6. Using Theorem 4.5, we obtain the general sum-connectivity index of the fence graph $P_n[P_2]$ below:

$$\chi_\alpha(P_n[P_2]) = \sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}2^{\alpha-k}(2\times 3^{\alpha-k}+(n-2)4^{\alpha-k})\sum_{\beta=0}^{k}4\binom{k}{k-\alpha\beta}\left[1+2(4^{k-\beta}+4^{\beta})+4^{k+1}\right].$$

In the following theorem, we compute the general sum-connectivity index of join of finite number of graphs for $\alpha \in \mathbb{Z}$.

Theorem 4.7. *Let $\alpha \in \mathbb{Z}$ and $G_1, G_2, \ldots, G_n$ be vertex-disjoint graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$, $1 \le i \le n$, $G = G_1 + G_2 + \ldots + G_n$ and $V = V(G)$. Then the general sum-connectivity index of join of graphs is given by formula:*

$$\chi_\alpha(G) = \sum_{i=1}^{n}\sum_{k=0}^{\alpha}2^k\binom{\alpha}{\alpha-k}(|V|-|V_i|)^k\chi_{\alpha-k}(G_i) \tag{23}$$

$$+\frac{1}{2}\sum_{i\neq j,i,j=1}^{n}\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}(2|V|-|V_i|-|V_j|)^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}M_{\beta}(G_i)M_{k-\beta}(G_j).$$

Proof. By equation (8) and the definition of general sum-connectivity index, we obtain

$$\chi_\alpha(G)=\sum_{i=1}^{n}\sum_{uv\in E_i}((d_{G_i}(u)+d_{G_i}(v))+2(|V|-|V_i|))^{\alpha}$$
$$+\frac{1}{2}\sum_{i\neq j,i,j=1}^{n}\sum_{u\in V_i}\sum_{v\in V_j}((d_{G_i}(u)+d_{G_j}(v))+(2|V|-|V_i|-|V_j|))^{\alpha} \tag{24}$$

Now, for $u,v\in V_i$, $1\leq i\leq n$, we use binomial expansion to obtain

$$((d_{G_i}(u)+d_{G_i}(v))+2(|V|-|V_i|))^{\alpha}=$$
$$\sum_{k=0}^{\alpha}2^k\binom{\alpha}{\alpha-k}(|V|-|V_i|)^k(d_{G_i}(u)+d_{G_i}(v))^{\alpha-k}. \tag{25}$$

Similarly, for $u\in V(G_i)$ and $v\in V(G_j)$, $1\leq i,j\leq n$, we obtain

$$((d_{G_i}(u)+d_{G_j}(v))+(2|V|-|V_i|-|V_j|))^{\alpha}=$$
$$\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}(2|V|-|V_i|-|V_j|)^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}d_{G_i}(u)^{\beta}d_{G_j}(v)^{k-\beta}. \tag{26}$$

Using equations (25) and (26) in equation (24), we get

$$\chi_\alpha(G)=\sum_{i=1}^{n}\sum_{uv\in E_i}\sum_{k=0}^{\alpha}2^k\binom{\alpha}{\alpha-k}(|V|-|V_i|)^k(d_{G_i}(u)+d_{G_i}(v))^{\alpha-k}$$
$$+\frac{1}{2}\sum_{i\neq j,i,j=1}^{n}\sum_{u\in V_i}\sum_{v\in V_j}\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}(2|V|-|V_i|-|V_j|)^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}d_{G_i}(u)^{\beta}d_{G_j}(v)^{k-\beta}$$
$$=\sum_{i=1}^{n}\sum_{k=0}^{\alpha}2^k\binom{\alpha}{\alpha-k}(|V|-|V_i|)^k\sum_{uv\in E_i}(d_{G_i}(u)+d_{G_i}(v))^{\alpha-k}$$
$$+\frac{1}{2}\sum_{i\neq j,i,j=1}^{n}\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-k}(2|V|-|V_i|-|V_j|)^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}\sum_{u\in V_i}d_{G_i}(u)^{\beta}\sum_{v\in V_j}d_{G_j}(v)^{k-\beta}.$$

After simplification, we get exactly (23). □

Example 4.8. Using Theorem 4.7, the general sum-connectivity index of join of P_n and P_m is given below:

$$\chi_\alpha(P_n+P_m)=\sum_{k=0}^{\alpha}\binom{\alpha}{\alpha-n}\Big[2\times 3^{\alpha}(m^k+n^k)+4^{\alpha}(m^k(n-2)+n^k(m-2))$$
$$+\frac{1}{2}(n+m)^{\alpha-k}\sum_{\beta=0}^{k}\binom{k}{k-\beta}(4+(m-2)2^{2k-2\beta+1}+2(n-2)4^{\beta}+(n-2)(m-2)4^k)\Big].$$

In the following theorem, we compute the general sum-connectivity index of $G \vee H$.

THEOREM 4.9. *Let G and H be two graphs such that either G or H is regular and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G \vee H$ is given by the formula:*

$$\chi_\alpha(G \vee H) =$$
$$\sum_{k=0}^{\alpha} \frac{1}{2^k} \binom{\alpha}{\alpha-k} n_G^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_n(G) M_{k-n}(G) \sum_{n=0}^{k} n_H^{n-1} (-1)^n 2^{k-n} \binom{k}{k-n} \chi_{\alpha+n-k}(H)$$
$$+ \sum_{k=0}^{\alpha} \frac{1}{2^k} \binom{\alpha}{\alpha-k} n_H^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_n(H) M_{k-n}(H) \sum_{n=0}^{k} n_H^{k-n} (-1)^n 2^{k-n} \binom{k}{k-n} \chi_{\alpha+n-k}(G)$$
$$-4 \sum_{k=0}^{\alpha} \frac{1}{2^k} \binom{\alpha}{\alpha-k} n_G^{\alpha-k} \chi_k(G) \sum_{n=0}^{k} (-1)^n 2^{k-n} n_H^{k-n} \binom{k}{k-n} \chi_{\alpha-k+n}(H).$$

Proof. Equation (9) and the definition of general sum-connectivity index imply

$$\chi_\alpha(G \vee H) = \tag{27}$$
$$\sum_{a\in V(G)} \sum_{c\in V(G)} \sum_{bd\in E(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha$$
$$+ \sum_{ac\in E(G)} \sum_{b\in V(H)} \sum_{d\in V(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha$$
$$- 4 \sum_{ac\in E(G)} \sum_{bd\in E(H)} (n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha.$$

Now, for $a, c \in V(G)$ and $b, d \in V(H)$, we obtain the following by using binomial expansion:

$$(n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha$$
$$= \sum_{k=0}^{\alpha} \frac{1}{2^k} \binom{\alpha}{\alpha-k} n_G^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} d_G(a)^n d_G(c)^{k-n}$$
$$\sum_{n=0}^{k} n_H^{n-1} (-1)^n 2^{k-n} \binom{k}{k-n} (d_H(b) + d_H(d))^{\alpha+n-k}. \tag{28}$$

Similarly, for $a, c \in V(G)$ and $b, d \in V(H)$, using binomial expansion, we obtain

$$(n_G(d_H(b) + d_H(d)) + n_H(d_G(a) + d_G(c)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha$$
$$= \sum_{k=0}^{\alpha} \frac{1}{2^k} \binom{\alpha}{\alpha-k} n_H^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} d_H(b)^n d_H(d)^{k-n}$$
$$\sum_{n=0}^{k} n_H^{k-n} (-1)^n 2^{k-n} \binom{k}{k-n} (d_G(a) + d_G(c))^{\alpha+n-k} \tag{29}$$

Without loss of generality, assume that G is a regular graph. Then $(a,b)(c,d) \in E(G \vee H)$, we obtain

$$d_G(a)d_H(b) + d_G(c)d_H(d) = \frac{1}{2}(d_G(a)d_H(b) + d_G(a)d_H(b) + d_G(c)d_H(d) + d_G(c)d_H(d))$$
$$= \frac{1}{2}(d_G(a)d_H(b) + d_G(c)d_H(b) + d_G(a)d_H(d) + d_G(c)d_H(d)) = \frac{1}{2}(d_G(a) + d_G(c))(d_H(b) + d_H(d)).$$

Using binomial expansion, we obtain

$$(n_H(d_G(a) + d_G(c)) + n_G(d_H(b) + d_H(d)) - (d_G(a)d_H(b) + d_G(c)d_H(d)))^\alpha \tag{30}$$
$$= \sum_{k=0}^{\alpha} \frac{1}{2^k}\binom{\alpha}{\alpha-k} n_G^{\alpha-k}(d_G(a) + d_G(c))^k \sum_{n=0}^{k}(-1)^n 2^{k-n} n_H^{k-n}\binom{k}{k-n}(d_H(b) + d_H(d))^{\alpha-k+n}.$$

Using equations (28)–(30) in equation (27), we get

$$\chi_\alpha(G \vee H) = \sum_{a\in V(G)}\sum_{c\in V(G)}\sum_{bd\in E(H)}\sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_G^{\alpha-k}$$
$$\sum_{n=0}^{k}\binom{k}{k-n}d_G(a)^n d_G(c)^{k-n}\sum_{n=0}^{k} n_H^{n-1}(-1)^n 2^{k-n}\binom{k}{k-n}(d_H(b) + d_H(d))^{\alpha+n-k}$$
$$+ \sum_{ac\in E(G)}\sum_{b\in V(H)}\sum_{d\in V(H)}\sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}$$
$$\sum_{n=0}^{k}\binom{k}{k-n}d_H(b)^n d_H(d)^{k-n}\sum_{n=0}^{k} n_H^{k-n}(-1)^n 2^{k-n}\binom{k}{k-n}(d_G(a) + d_G(c))^{\alpha+n-k}$$
$$-4\sum_{ac\in E(G)}\sum_{bd\in E(H)}\sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_G^{\alpha-k}\chi_k(G)\sum_{n=0}^{k}(-1)^n 2^{k-n} n_H^{k-n}\binom{k}{k-n}\chi_{\alpha-k+n}(H)$$
$$= \sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_G^{\alpha-k}\sum_{n=0}^{k}\binom{k}{k-n}\sum_{a\in V(G)} d_G(a)^n \sum_{c\in V(G)} d_G(c)^{k-n}$$
$$\sum_{n=0}^{k} n_H^{n-1}(-1)^n 2^{k-n}\binom{k}{k-n}\sum_{bd\in E(H)}(d_H(b) + d_H(d))^{\alpha+n-k}$$
$$+ \sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_H^{\alpha-k}\sum_{n=0}^{k}\binom{k}{k-n}\sum_{b\in V(H)} d_H(b)^n \sum_{d\in V(H)} d_H(d)^{k-n}$$
$$\sum_{n=0}^{k} n_H^{k-n}(-1)^n 2^{k-n}\binom{k}{k-n}\sum_{ac\in E(G)}(d_G(a) + d_G(c))^{\alpha+n-k}$$
$$-4\sum_{k=0}^{\alpha}\frac{1}{2^k}\binom{\alpha}{\alpha-k}n_G^{\alpha-k}\sum_{ac\in E(G)}(d_G(a) + d_G(c))^k\sum_{n=0}^{k}(-1)^n 2^{k-n} n_H^{k-n}\binom{k}{k-n}$$
$$\sum_{bd\in E(H)}(d_H(b) + d_H(d))^{\alpha-k+n}.$$

Thus the statement of the theorem is true. □

Example 4.10. Using Theorem 4.9, the general sum-connectivity index of $C_l \vee C_m$ is given below:

$$\chi_\alpha(C_l \vee C_m) = \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} \Big[l^{\alpha-k+1} \sum_{n=0}^{k} \binom{k}{k-n} \sum_{n=0}^{k} \binom{k}{k-n} m^{n+1} (-1)^n 2^{n+2\alpha}$$
$$+ \sum_{n=0}^{k} \binom{k}{k-n} m^{\alpha-k+2} \sum_{n=0}^{k} \binom{k}{k-n} (-1)^n 2^{n+2\alpha} - 4ml^{\alpha-k+1} \sum_{n=0}^{k} \binom{k}{k-n} (-1)^n 2^{n+2\alpha} \Big].$$

In the following theorem, we give the general sum-connectivity index of symmetric difference of two graphs for $\alpha \in \mathbb{Z}^+$. The proof is similar to the proof of Theorem 4.9, hence omitted.

Theorem 4.11. *Let G and H be two graphs such that either G or H is regular and $\alpha \in \mathbb{Z}$. Then the general sum-connectivity index of $G \oplus H$ is given by the formula:*

$$\chi_\alpha(G \oplus H) =$$
$$\sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_G^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_n(G) M_{k-n}(G) \sum_{n=0}^{k} n_H^{k-n} (-1)^n \binom{k}{k-n} \chi_{\alpha+n-k}(H)$$
$$+ \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_H^{\alpha-k} \sum_{n=0}^{k} \binom{k}{k-n} M_n(H) M_{k-n}(H) \sum_{n=0}^{k} n_G^{k-n} (-1)^n \binom{k}{k-n} \chi_{\alpha+n-k}(G)$$
$$-2 \sum_{k=0}^{\alpha} \binom{\alpha}{\alpha-k} n_G^{\alpha-k} \chi_k(G) \sum_{n=0}^{k} (-1)^n n_H^{k-n} \binom{k}{k-n} \chi_{\alpha-k+n}(H).$$

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