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GENERALIZED WINTGEN INEQUALITY FOR BI-SLANT SUBMANIFOLDS IN LOCALLY CONFORMAL KAEHLER SPACE FORMS

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Abstract. In 1999, De Smet et al. conjectured the generalized Wintgen inequality for submanifolds in real space forms. This conjecture is also known as the DDVV conjecture and it was proved by Ge and Tang. Recently, Mihai established such inequality for Lagrangian submanifold in complex space forms. In this paper, we obtain the generalized Wintgen inequality for bi-slant submanifolds in locally conformal Kaehler space forms. Further, we discuss the particular cases of this inequality i.e. for semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, invariant submanifolds and anti-invariant submanifolds in the same ambient space.

1. Introduction

The locally conformally Kaehler manifolds are those complex manifolds which have the property that on their universal cover there exists a Kaehler metric upon which the deck transformations act by homotheties (see the next section for definition). I. Vaisman [11] introduced the notion of locally conformal manifolds. In the last three decades, locally conformal Kaehler manifolds have been studied intensively by many geometers due to its rich geometric importance [6, 12].

On the other hand, Wintgen inequality is a sharp geometric inequality for surfaces in 4-dimensional Euclidean space involving Gauss curvature (intrinsic invariant), normal curvature and square mean curvature (extrinsic invariants).

P. Wintgen [13], proved that the Gauss curvature $\mathcal{K}$, the normal curvature $\mathcal{K}^{\perp}$ and the squared mean curvature $\|\mathcal{H}\|^2$ for any surface $\mathcal{N}^2$ in E^4 satisfy the following inequality: $\|\mathcal{H}\|^2 \geq \mathcal{K} + |\mathcal{K}^{\perp}|$. The equality holds if and only if the ellipse of curvature $\mathcal{N}^2$ in E^4 is a circle.

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Further, it was extended by I. V. Gaudalupe et al. [2] for arbitrary codimension m in real space forms $\overline{\mathcal{N}}^{m+2}(c)$ as $\|\mathcal{H}\|^2 + c \geq \mathcal{K} + |\mathcal{K}^{\perp}|$. They also discussed the equality case of the inequality.

In 2014, I. Mihai [9] obtained the DDVV inequality for Lagrangian submanifolds in complex space forms and investigated some of its applications. In 2017, M. N. Boyom et al. [10] studied generalized Wintegen type inequality for Lagrangian submanifolds in holomorphic statistical space forms and provided some of its applications.

In the present article, we obtain generalized Wintgen inequalities for bi-slant submanifolds in locally conformal Kaehler space forms. We also investigate such inequality for different slant cases.

2. Submanifolds in locally conformal Kaehler space form

A Hermitian manifold $(\overline{\mathcal{N}}, J, g)$ equipped with complex structure J and Hermitian metric g, is called locally Kaehler manifold if each point $p \in \overline{\mathcal{N}}$ has an open neighbourhood U with a differentiable map $\phi : U \to \mathbb{R}$ such that the local metric $g = e^{-2\phi} g_{|U}$ is a Kaehler metric on U. The fundamental 2-form ψ of $\overline{\mathcal{N}}$ is defined by $\psi(X, Y) = g(JX, Y)$, for any tangent vector fields $X, Y \in T\overline{\mathcal{N}}$ (see [1]).

Proposition 2.1. [5] *A Hermitian manifold $\overline{\mathcal{N}}$ is a locally conformal Kaehler manifold if and only if there exists a global 1-form ω, satisfying*

$$g(\overline{\nabla}_Z JX, Y) = \omega(JX)g(Y, Z) - \omega(X)g(JY, Z) - \omega(JY)g(X, Z) - \omega(Y)g(JX, Z),$$

for all $X, Y, Z \in T\overline{\mathcal{N}}$.

The 1-form ω is called the Lee form and its dual vector field is said to be the Lee vector field. On a locally conformal Kaehler manifold, a symmetric $(0, 2)$-tensor $\overline{P}$ is defined as

$$\overline{P}(X, Y) = -(\overline{\nabla}_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\|\omega\|^2 g(X, Y),$$

where $\|\omega\|$ is the length of the Lee form ω with respect to g. The tensor field $\overline{P}$ is said to be hybrid if $\overline{P}(JX, Y) + \overline{P}(X, JY) = 0$, for $X, Y \in T\overline{\mathcal{N}}$.

The locally conformal Kaehler manifold with constant holomorphic sectional curvature c is called locally conformal Kaehler space form and is denoted by $\overline{\mathcal{N}}(c)$. In the rest part of the paper we assume that $\overline{P}$ is hybrid in a locally conformal Kaehler space form.

The curvature tensor $\overline{R}$ for locally conformal Kaehler space forms is given as [5, 8]

$$\begin{aligned}
\overline{R}(X, Y, Z, W) = {}& \frac{c}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
& + \frac{c}{4}\{g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) - 2g(JX, Y)g(JZ, W)\} \\
& - \frac{3}{4}\{g(Y, Z)\overline{P}(X, W) - g(X, Z)\overline{P}(Y, W) + g(X, W)\overline{P}(Y, Z) - g(Y, W)\overline{P}(X, Z)\}
\end{aligned}$$

$$-\frac{1}{4}\{g(JY,Z)\overline{P}(JX,W)-g(JX,Z)\overline{P}(JY,W)+g(JX,W)\overline{P}(JY,Z)$$
$$-g(JY,W)\overline{P}(JX,Z)-2g(JZ,W)\overline{P}(JX,Y)-2g(JX,Y)\overline{P}(JZ,W)\}, \quad (1)$$

for all $X,Y,Z,W \in T\overline{\mathcal{N}}$.

Let $\mathcal{N}$ be a submanifold of an almost Hermitian manifold $\overline{\mathcal{N}}$ with induced metric g; if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle $T\mathcal{N}$ and the normal bundle $T^\perp\mathcal{N}$ of $\mathcal{N}$, respectively, then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\overline{\nabla}_X N = -S_N X + \nabla^\perp_X N,$$

for vector fields $X, Y \in T\mathcal{N}$ and $N \in T^\perp\mathcal{N}$, where h, S_N and $\nabla^\perp$ are the second fundamental form, the shape operator and the normal connection, respectively.

The second fundamental form and the shape operator are related by the equation $g(h(X,Y),N) = g(S_N X, Y)$, for vector fields $X, Y \in T\mathcal{N}$ and $N \in T^\perp\mathcal{N}$.

The equation of Gauss is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \quad (2)$$

for $X,Y,Z,W \in T\mathcal{N}$, where $\overline{R}$ and R represent the curvature tensor of $\overline{\mathcal{N}}(c)$ and $\mathcal{N}$ respectively.

For any tangent vector field $X \in T\mathcal{N}$, we can write $JX = PX + QX$, where P and Q are the tangential and normal components of JX respectively. If $P = 0$, the submanifold is said to be an anti-invariant submanifold and if $Q = 0$, the submanifold is said to be an invariant submanifold.

The squared norm of P at $p \in \mathcal{N}$ is given as

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j), \quad (3)$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space $T\mathcal{N}$ of $\mathcal{N}$.

Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathcal{N}$. The mean curvature vector field is given by

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

3. Generalized Wintgen inequality

We denote by $\mathcal{K}$ and $R^\perp$ the sectional curvature function and the normal curvature tensor on $\mathcal{N}$, respectively. Then the normalized scalar curvature ρ is given by [9]

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathcal{K}(e_i \wedge e_j), \quad (4)$$

where τ is the scalar curvature.

In terms of the components of the second fundamental form, we can express the scalar normal curvature $\mathcal{K}_N$ of $\mathcal{N}$ by the formula [9]

$$\mathcal{K}_N = \sum_{1\leq r<s\leq 2m-n+1} \sum_{1\leq i<j\leq n} \Big(\sum_{k=1}^{n} h^r_{jk} h^s_{ik} - h^r_{jk} h^s_{ik}\Big)^2. \tag{5}$$

A submanifold $\mathcal{N}$ of a locally conformal Kaehler manifold $\overline{\mathcal{N}}$ is said to be a C-totally real submanifold if J maps each tangent space of $\mathcal{N}$ into the normal space, i.e. $J(T\mathcal{N}) \subset T^{\perp}\mathcal{N}$. In particular, if $n = 2m$, then $\mathcal{N}$ is called a Lagrangian submanifold.

A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a slant submanifold if for any $p \in \mathcal{N}$ and a non zero vector $X \in T_p\mathcal{N}$, the angle between JX and $T_p\mathcal{N}$ is constant, i.e., the angle does not depend on the choice of $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of $\mathcal{N}$ in $\overline{\mathcal{N}}$.

A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2, such that:

(i) $T\mathcal{N}$ admits the orthogonal direct decomposition i.e $T\mathcal{N} = D_1 + D_2$.

(ii) For i=1,2, D_i is the slant distribution with slant angle θ_i.

In fact, semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds can be obtained from bi-slant submanifolds in particular. We can see the cases in the following table:

Table 1: Definition

S.N.	$\overline{\mathcal{N}}$	$\mathcal{N}$	D_1	D_2	θ_1	θ_2
(1)	$\overline{\mathcal{N}}$	bi-slant	slant	slant	slant angle	slant angle
(2)	$\overline{\mathcal{N}}$	semi-slant	invariant	slant	0	slant angle
(3)	$\overline{\mathcal{N}}$	hemi-slant	slant	anti-invariant	slant angle	$\frac{\pi}{2}$
(4)	$\overline{\mathcal{N}}$	CR	invariant	anti-invariant	0	$\frac{\pi}{2}$
(5)	$\overline{\mathcal{N}}$	slant	either $D_1 = 0$ or $D_2 = 0$		either $\theta_1 = \theta_2 = \theta$ or $\theta_1 = \theta_2 \neq \theta$	

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively, and when $0 < \theta < \frac{\pi}{2}$, then the slant submanifold is called a proper slant submanifold.

If $\mathcal{N}$ is a bi-slant submanifold in generalized locally conformal Kaehler space form $\overline{\mathcal{N}}(c)$, then one can easily see that

$$\|P\|^2 = 2(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2), \tag{6}$$

where $\dim D_1 = 2d_1$ and $\dim D_2 = 2d_2$.

The normalized scalar normal curvature is given by [9] $\rho_N = \frac{2}{n(n-1)}\sqrt{\mathcal{K}_N}$.

Now, we shall state and prove the generalized Wintgen inequality for bi-slant submanifolds in locally conformal Kaehler space forms.

THEOREM 3.1. *Let $\mathcal{N}$ be a bi-slant submanifold of locally conformal Kaehler space forms $\overline{\mathcal{N}}(c)$. Then*

$$\rho_N \leq \|\mathcal{H}\|^2 - (\rho - c) - \frac{3}{n-1}\operatorname{trace}\overline{P} + \frac{c}{2n(n-1)}(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - \frac{3}{2n(n-1)}\sum_{1\leq i<j\leq n} g(Je_i,e_j)\overline{P}(Je_i,e_j). \tag{7}$$

Proof. We choose $\{e_1,\dots,e_n\}$ and $\{e_{n+1},\dots,e_{2m}\}$ as orthonormal frame and orthonormal normal frame on $\mathcal{N}$ respectively. Putting $X=W=e_i$, $Y=Z=e_j$, $i\neq j$, from (1), we have

$$\begin{aligned}\overline{R}(e_i,e_j,e_j,e_i) &= \frac{c}{4}\{g(e_j,e_j)g(e_i,e_i) - g(e_i,e_j)g(e_j,e_i)\}\\
&+ \frac{c}{4}\{g(e_i,Je_i)g(e_j,Je_j) - g(Je_i,e_j)g(Je_j,e_i) - 2g(Je_i,e_j)g(Je_j,e_i)\}\\
&- \frac{3}{4}\{g(e_j,e_j)\overline{P}(e_i,e_i) - g(e_i,e_j)\overline{P}(e_j,e_i) + g(e_i,e_i)\overline{P}(e_j,e_j) - g(e_j,e_i)\overline{P}(e_i,e_j)\}\\
&+ \frac{1}{4}\{g(Je_j,e_j)\overline{P}(Je_i,e_i) - g(Je_i,e_j)\overline{P}(Je_j,e_i) + g(Je_i,e_i)\overline{P}(Je_j,e_j)\\
&- g(Je_j,e_i)\overline{P}(Je_i,e_j) - 2g(Je_j,e_i)\overline{P}(Je_i,e_j) - 2g(Je_i,e_j)\overline{P}(Je_j,e_i).\end{aligned} \tag{8}$$

Combining equations (2) and (8), we obtain

$$\begin{aligned}R(e_i,e_j,e_j,e_i) &= \frac{c}{4}\{g(e_j,e_j)g(e_i,e_i) - g(e_i,e_j)g(e_j,e_i)\}\\
&+ \frac{c}{4}\{g(e_i,Je_i)g(e_j,Je_j) - g(Je_i,e_j)g(Je_j,e_i) - 2g(Je_i,e_j)g(Je_j,e_i)\}\\
&- \frac{3}{4}\{g(e_j,e_j)\overline{P}(e_i,e_i) - g(e_i,e_j)\overline{P}(e_j,e_i) + g(e_i,e_i)\overline{P}(e_j,e_j) - g(e_j,e_i)\overline{P}(e_i,e_j)\}\\
&+ \frac{1}{4}\{g(Je_j,e_j)\overline{P}(Je_i,e_i) - g(Je_i,e_j)\overline{P}(Je_j,e_i) + g(Je_i,e_i)\overline{P}(Je_j,e_j)\\
&- g(Je_j,e_i)\overline{P}(Je_i,e_j) - 2g(Je_j,e_i)\overline{P}(Je_i,e_j) - 2g(Je_i,e_j)\overline{P}(Je_j,e_i)\\
&+ g(h(e_i,e_i),h(e_j,e_j)) - g(h(e_i,e_j),h(e_i,e_j)).\end{aligned} \tag{9}$$

By taking summation for $1\leq i<j\leq n$ and using (3) in (9), we derive

$$\sum_{1\leq i<j\leq n} R(e_i,e_j,e_j,e_i) = \frac{n(n-1)}{8}c + \frac{1}{8}c\|P\|^2 + \frac{3n}{4}\operatorname{trace}\overline{P} - \frac{3}{4}\sum_{1\leq i<j\leq n} g(Je_i,e_j)\overline{P}(Je_i,e_j) + \sum_{r=n+1}^{2m-n}\sum_{1\leq i<j\leq n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{10}$$

Also, we know that

$$\tau = \sum_{1\leq i<j\leq n} R(e_i,e_j,e_j,e_i). \tag{11}$$

Now, using equations (6) and (11) in (10), we obtain

$$\begin{aligned}\tau =&\frac{n(n-1)}{8}c+\frac{1}{4}c(d_1\cos^2\theta_1+d_2\cos^2\theta_2)+\frac{3n}{4}\operatorname{trace}\overline{P}\\&-\frac{3}{4}\sum_{1\le i<j\le n}g(Je_i,e_j)\overline{P}(Je_i,e_j)+\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}[h^r_{ii}h^r_{jj}-(h^r_{ij})^2].\end{aligned}\tag{12}$$

On the other hand, we have

$$\begin{aligned}n^2\|\mathcal{H}\|^2 &= \sum_{r=n+1}^{2m-n}\big(\sum_{i=1}^{n}h^r_{ii}\big)^2=\frac{1}{n-1}\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}(h^r_{ii}-h^r_{jj})^2\\&+\frac{2n}{n-1}\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}h^r_{ii}h^r_{jj}.\end{aligned}\tag{13}$$

Further, from [7] we have

$$\begin{aligned}&\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}(h^r_{ii}-h^r_{jj})^2+2n\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}(h^r_{ij})^2\ge\\&\qquad 2n\Big[\sum_{n+1\le r<s\le 2m-n}\sum_{1\le i<j\le n}\big(\sum_{k=1}^{n}(h^r_{jk}h^s_{ik}-h^r_{ik}h^s_{jk})\big)^2\Big]^{\frac{1}{2}}.\end{aligned}\tag{14}$$

Now, combining (5), (13) and (14), we find

$$n^2\|\mathcal{H}\|^2-n^2\rho_N\ge\frac{2n}{n-1}\sum_{r=n+1}^{2m-n}\sum_{1\le i<j\le n}[h^r_{ii}h^r_{jj}-(h^r_{ij})^2].\tag{15}$$

Taking into account (4), (12) and (15), we find (7). □

Remark 3.2. Using Table 1 and Theorem 3.1 one can derive the corresponding inequalities for a semi-slant submanifold, a hemi-slant submanifold, a CR-submanifold, and a slant submanifold.

Remark 3.3. Since an invariant and anti-invariant submanifolds are slant submanifolds with $\theta=0$ and $\theta=\frac{\pi}{2}$, it follows that the inequalities

$$\begin{aligned}\rho_N \le&\|\mathcal{H}\|^2-(\rho-c)-\frac{3}{n-1}\operatorname{trace}\overline{P}+\frac{c}{4(n-1)}\\&-\frac{3}{2n(n-1)}\sum_{1\le i<j\le n}g(Je_i,e_j)\overline{P}(Je_i,e_j).\end{aligned}$$

and

$$\rho_N \le\|\mathcal{H}\|^2-(\rho-c)-\frac{3}{n-1}\operatorname{trace}\overline{P}.$$

are valid for them, respectively.

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