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GROUPS OF GENERALIZED ISOTOPIES AND GENERALIZED G-SPACES

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Abstract. The group of generalized isotopies of topological space is studied. A relationship of this group with the group of homeomorphisms is established in case of locally compact and locally connected space. Notions of generalized G-spaces and there equivariant maps are introduced. It is proved that a new category of generalized G-spaces is a natural extension of the category of G-spaces.

1. Introduction

In [5] the first author has studied a group of invertible continuous binary operations of a topological space X. The concepts of binary G-spaces and binary equivariant maps have also been introduced and studied. The notions of binary G-space X and group of invertible binary operations of a topological space X are interdependent.

In this article natural generalizations of these objects are considered. We introduce the notion of generalized isotopy and study a group of generalized isotopies of a topological space X. The relationship between this group and the group of homeomorphisms is established. We also introduced the notions of generalized G-spaces and their equivariant maps related to generalized isotopies. So, we have a new category which is a natural extension of the category of G-spaces and equivariant maps.

Now let us recall some notations, notions and auxiliary results that we need in this article. Throughout this paper, all spaces are assumed to be Hausdorff. The category of topological spaces and continuous maps is denoted by Top.

By $C(X,Y)$ we denote the space of all continuous maps of X to Y endowed with the compact-open topology, that is, the topology generated by the subbase consisting of all sets of the form $W(K,U) = \{f : X \to Y;\ f(K) \subset U\}$, where K is a compact subset of X and U is an open subset of Y.

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If G is a topological group, then there is a natural group operation on $C(X,G)$: given any continuous maps $f,g \in C(X,G)$, their product $fg \in C(X,G)$ is defined by $(fg)(x) = f(x)g(x)$ for all $x \in X$. If G is a topological group, then so is $C(X,G)$ (see [7]).

The group of all self-homeomorphisms of X is denoted by $H(X)$. This group is not generally a topological group. However, if a space X is locally compact and locally connected, then $H(X)$ is a topological group (see [1]).

Let X be a topological space, and let G be a topological group G with identity element e. Suppose given a continuous map $\theta : G \times X \to X$ satisfying the conditions $\theta(g, \theta(h,x)) = \theta(gh, x)$ and $\theta(e,x) = x$ for $g,h \in G$ and $x \in X$. Then X is called a G-space, and the continuous map $\theta : G \times X \to X$ is called the action of the group G on the space X.

Let X and Y be G-spaces. A continuous map $f : X \to Y$ is said to be equivariant if $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$. All G-spaces and their equivariant maps form a category. This category is denoted by *G-Top*.

Details on these notions, as well as on all definitions, notions, and results used in this paper without reference, can be found in [2–4, 6].

2. The group of generalized isotopies

Let T be a topological space. For any topological space X let us denote the topological space of all continuous maps from $T \times X$ to X with the compact-open topology by $C(T \times X, X)$. We define an operation "$\circ$" on $C(T \times X, X)$ by

$$(f \circ g)(t,x) = f(t, g(t,x)) \tag{1}$$

for all $t \in T$ and $x \in X$.

Note that each map $f : T \times X \to X$ can be considered as a family $f = \{f_t\}$, $t \in T$, of continuous maps $f_t : X \to X$ defined by

$$f_t(x) = f(t,x) \tag{2}$$

for all $x \in X$. Now the formula (1) can be written as $(f \circ g)_t(x) = f_t(g_t(x)) = (f_t \circ g_t)(x)$, i.e. $(f \circ g)_t = f_t \circ g_t$, where $f_t \circ g_t$ is an ordinary composition of maps $f_t, g_t : X \to X$. So, $f \circ g$ is naturally called a composition of maps $f, g \in C(T \times X, X)$.

Proposition 2.1. *The space $C(T \times X, X)$ under the operation "$\circ$" is a semigroup with identity, i.e. a monoid.*

Proof. Let $f,g,h \in C(T \times X, X)$ be continuous maps. Let us check the semigroup axiom:

$$\begin{aligned}[f \circ (g \circ h)](t,x) &= f(t, (g \circ h)(t,x)) = f(t, g(t, h(t,x))) \\ &= (f \circ g)(t, h(t,x)) = [(f \circ g) \circ h](t,x).\end{aligned}$$

The continuous map $e : T \times X \to X$ defined by $e(t,x) = x$ for all $t \in T$ and $x \in X$ is the identity element of semigroup $C(T \times X, X)$, because $f \circ e = e \circ f = f$. Indeed, $(f \circ e)(t,x) = f(t, e(t,x)) = f(t,x)$ and $(e \circ f)(t,x) = e(t, f(t,x)) = f(t,x)$. □

Definition 2.2. A map $f = \{f_t\} \in C(T \times X, X)$ is said to be *invertible* if there exists a map $f^{-1} = \{(f^{-1})_t\} \in C(T \times X, X)$ such that

$$f \circ f^{-1} = f^{-1} \circ f = e. \tag{3}$$

In this case, f and f^{-1} are said to be *mutually inverse maps.*

Let us denote the set of all invertible maps of $C(T \times X, X)$ by $H(T \times X, X)$. So, $H(T \times X, X)$ is a group.

The following necessary condition of invertibility of continuous map $f : T \times X \to X$ holds.

Theorem 2.3 ([5, Theorem 3]). *If a map $f = \{f_t\} \in C(T \times X, X)$ is invertible, then for any $t \in T$ the map $f_t : X \to X$ is a homeomorphism and $f^{-1} = \{f_t^{-1}\}$.*

Proof. Suppose that a continuous map $f : T \times X \to X$ is invertible, i.e. there exists a map $f^{-1} = \{(f^{-1})_t\} \in C(T \times X, X)$ satisfying relations (3). Take any element $t \in T$. Let us prove that the continuous map $f_t : X \to X$ is a homeomorphism.

The map f_t is a injective. Indeed, suppose that $f_t(x) = f_t(x')$, i.e. $f(t, x) = f(t, x')$. Then $x = e(t, x) = (f^{-1} \circ f)(t, x) = f^{-1}(t, f(t, x)) = f^{-1}(t, f(t, x')) = (f^{-1} \circ f)(t, x') = e(t, x') = x'$. Since $(f_t \circ (f^{-1})_t)(x) = f_t((f^{-1})_t(x)) = f_t(f^{-1}(t, x)) = f(t, f^{-1}(t, x)) = (f \circ f^{-1})(t, x) = e(t, x) = x$, we have $f_t \circ (f^{-1})_t = 1_X$.

A similar argument proves that $(f^{-1})_t \circ f_t = 1_X$. Thus, the continuous map $(f^{-1})_t : X \to X$ is inverse to $f_t : X \to X$, i.e. a map $f_t : X \to X$ is a homeomorphism and $f_t^{-1} = (f^{-1})_t$. □

Note that if $T = [0, 1]$ then, by Theorem 2.3, an invertible continuous map $f : [0, 1] \times X \to X$ is an isotopy of a topological space X. So, we will call invertible maps $f : T \times X \to X$ *generalized isotopies* of topological space X over T and $H(T \times X, X)$ — *a group of generalized isotopies.*

The converse of Theorem 2.3 is not true, i.e. if for any $t \in T$ the map $f_t : X \to X$ is a homeomorphism then $f^{-1} = \{f_t^{-1}\}$ is not necessarily inverse to $f = \{f_t\}$.

Example 2.4. Let X be a subset of the plane consisting of the graph of function $y = \sin \dfrac{2\pi}{x}$, $x \in (0, 1]$, and point $O(0, 0)$ (see Figure 1):

$$X = \left\{(x, y) \in \mathbf{R}^2; \quad x \in (0, 1],\ y = \sin \frac{2\pi}{x}\right\} \cup \left\{(0, 0) \in \mathbf{R}^2\right\}$$

We will denote the intersection points of $y = \sin \dfrac{2\pi}{x}$, $x \in (0, 1]$, and x-axis by a_1, a_2, $\ldots$, a_n, $\ldots$ It is obvious that $a_{2n-1} = \left(\dfrac{1}{n}, 0\right)$, $a_{2n} = \left(\dfrac{2}{2n+1}, 0\right)$, $n = 1, 2, \ldots$

The points $b_1, b_2, \ldots, b_n, \ldots$ which represent the intersection of graph $y = \sin \dfrac{2\pi}{x}$, $x \in (0, 1]$, with line $y = 1$ have coordinates $b_n = \left(\dfrac{4}{4n+1}, 1\right)$, $n = 1, 2, \ldots$

Let $c_1, c_2, \ldots, c_n, \ldots$ be the intersection points of graph of the function $y = \sin \dfrac{2\pi}{x}$ with $y = -x$ and let ξ_n be the x-coordinate of point c_{2n-1}, $n = 1, 2, \ldots$

We denote the intersection points of $y = \sin\dfrac{2\pi}{x}$ with the line $y = \dfrac{1}{2}$ by d_1, d_2, $\dots$, d_n, $\dots$ and suppose that η_n is the x-coordinate of point d_{2n-1}, $n = 1, 2, \dots$, i.e., $d_{2n-1} = \left(\eta_n, \dfrac{1}{2}\right)$.

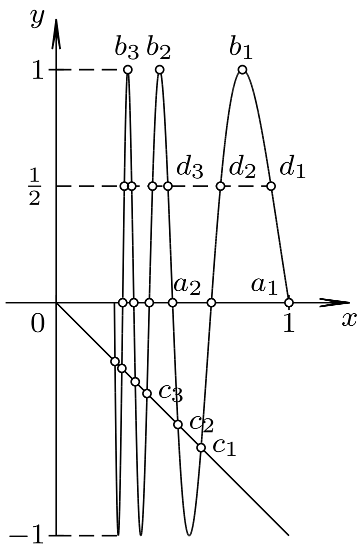

Figure 1

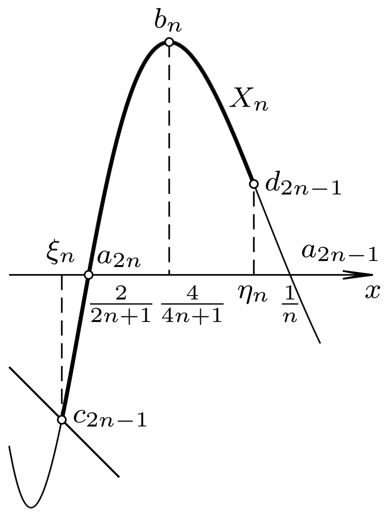

Figure 2

Now let us define the isotopy $f : [0,1] \times X \to X$, i.e. the family of homeomorphisms $f_t : X \to X$, $t \in [0,1]$, in the following way.

For $t = 0$ let $f_0 = 1_X$. Now we define the homeomorphism $f_t : X \to X$ on each interval $\left[\dfrac{1}{2n+1}, \dfrac{1}{2n-1}\right]$, $n = 1, 2, \dots$, as follows.

If $t \in \left[\dfrac{1}{2n}, \dfrac{1}{2n-1}\right]$, $n = 1, 2, \dots$, then

$$f_t(x,y) = \begin{cases} \left(p_t(x), \sin\dfrac{2\pi}{p_t(x)}\right) & x \in \left(\xi_n, \dfrac{4}{4n+1}\right], \\ \left(q_t(x), \sin\dfrac{2\pi}{q_t(x)}\right) & x \in \left(\dfrac{4}{4n+1}, \eta_n\right], \\ (x,y); & x \notin (\xi_n, \eta_n], \end{cases} \tag{4}$$

where

$$p_t(x) = \frac{\dfrac{4n(2n-1)}{(2n+1)(4n+1)}\left(t-\dfrac{1}{2n}\right)+\dfrac{2}{2n+1}-\xi_n}{\dfrac{4}{4n+1}-\xi_n}(x-\xi_n)+\xi_n.$$

$$q_t(x) = \frac{\eta_n-\left[\dfrac{4n(2n-1)}{(2n+1)(4n+1)}\left(t-\dfrac{1}{2n}\right)+\dfrac{2}{2n+1}\right]}{\eta_n-\dfrac{4}{4n+1}}(x-\eta_n)+\eta_n.$$

Let $t \in \left[\dfrac{1}{2n+1}, \dfrac{1}{2n}\right]$. Note that the interval $\left[\dfrac{1}{2n+1}, \dfrac{1}{2n}\right]$ is mapped linearly on the interval $\left[\dfrac{1}{2n}, \dfrac{1}{2n-1}\right]$ by the formula $t' = \dfrac{1}{2n} + \dfrac{2n+1}{2n-1}\left(\dfrac{1}{2n} - t\right)$. Moreover, $\dfrac{1}{2n+1} \to \dfrac{1}{2n-1}$ and $\dfrac{1}{2n} \to \dfrac{1}{2n}$. Now we define the homeomorphism $f_t(x,y)$ by putting $f_t(x,y) = f_{t'}(x,y)$, where $f_{t'}(x,y)$ is calculated by formula (4).

It is not difficult to see that the map $f_t : X \to X$ is an identity except on arc X_n which consists of the points of X with x-coordinate belonging to $(\xi_n, \eta_n]$ (see Figure 2). This arc is linearly mapped on itself by f_t such that $f_{\frac{1}{2n}}(b_n) = a_{2n}$.

One can easily verify that $p_t(x) = q_t(x) = x$ for $t = \dfrac{1}{2n-1}$. Therefore, $f_{\frac{1}{2n-1}}(x,y) = (x,y)$ for all $n = 1, 2, \dots$ and $(x,y) \in X$.

The map $f : [0,1] \times X \to X$, $f(t,x) = f_t(x)$ is obviously continuous on $X \backslash \{O\}$ and is fixed at O, i.e., $f(t,O) = O$ for all $t \in [0,1]$. It is readily seen that every circular neighborhood of O of radius $< \dfrac{1}{2}$ is mapped into itself by every f_t. Hence $f : [0,1] \times X \to X$ is continuous at the point O, i.e. is an isotopy of X.

Now let us prove that the map $f^{-1} : [0,1] \times X \to X$, $f^{-1} = \{f_t^{-1}\}$, is not continuous at the point O. Indeed, the sequence $a_{2n} = \dfrac{1}{2n}$, $n = 1, 2, \dots$, converges to O. However, the sequence $f^{-1}_{\frac{1}{2n}}(a_{2n}) = b_n$, $n = 1, 2, \dots$, has no limit in X.

However, the following statement is true.

Theorem 2.5 ([5, Theorem 4]). *Let T be any topological space, X be a locally compact and locally connected and $f : T \times X \to X$ be a continuous map. If the map $f_t : X \to X$, defined by* (2)*, is a homeomorphism for each $t \in X$, then the map f is a generalized isotopy.*

Proof. Consider the map f^{-1} given by $f^{-1}(t,x) = f_t^{-1}(x)$.

It is easy to show that f^{-1} is inverse to f. Indeed,

$$(f \circ f^{-1})(t,x) = f(t, f^{-1}(t,x)) = f(t, f_t^{-1}(x)) = f_t(f_t^{-1}(x)) = x,$$

i.e. $f \circ f^{-1} = e$. The relation $f^{-1} \circ f = e$ is proved in a similar way.

It remains to prove the continuity of map $f^{-1} : T \times X \to X$.

Let $(t_0, x_0) \in T \times X$ be any point. Denote $f^{-1}(t_0, x_0) = f_{t_0}^{-1}(x_0) = y_0$ and consider any open neighborhood $W \subset X$ of y_0 such that the closure $\overline{W}$ is compact. Since the map $f_{t_0}^{-1}$ is a homeomorphism, there exists a compact connected neighborhood K of x_0 for which

$$f_{t_0}^{-1}(K) \subset W. \tag{5}$$

Denote the interior of K by K°. It is evident that

$$f_{t_0}(y_0) = x_0 \in K^\circ. \tag{6}$$

Inclusion (5) implies

$$f_{t_0}(W^C \cap \overline{W}) \subset K^C, \tag{7}$$

where W^C and K^C are the complements of W and K, respectively.

Since $f : T \times X \to X$ is a continuous map, $\{y_0\}$ and $W^C \cap \overline{W}$ are compact subsets of X, and K° and K^C are open subsets, it follows from (6) and (7) that the point $t_0 \in T$ has an open neighborhood $U \subset T$ such that, for any $t \in U$, we have

$$f_t(y_0) \in K^\circ \tag{8}$$

and

$$f_t(W^C \cap \overline{W}) \subset K^C. \tag{9}$$

Inclusion (9) implies that $K \subset f_t(W \cup \overline{W}^C)$ for any $t \in U$. Therefore, $f_t^{-1}(K) \subset W \cup \overline{W}^C$. Since $f_t^{-1}(K)$ is connected and W and $\overline{W}^C$ are disjoint open sets, it follows from the last inclusion that $f_t^{-1}(K)$ is contained in one of the sets W and $\overline{W}^C$. However, by virtue of (8), we obviously have $f_t^{-1}(K) \subset W$. Hence

$$f_t^{-1}(K^\circ) \subset W \tag{10}$$

for all $t \in U$.

Thus, given any open neighborhood W of $y_0 = f_{t_0}^{-1}(x_0)$, we have found open neighborhoods U of t_0 and K° of x_0 for which (10) holds. This proves the continuity of map $f^{-1} : T \times X \to X$. □

Theorems 2.3 and 2.5 imply the following assertion.

Theorem 2.6 ([5, Theorem 5]). *Let T be any topological space, X be a locally compact and locally connected. A continuous map $f : T \times X \to X$ is a generalized isotopy if and only if the continuous map $f_t : X \to X$ defined by (2) is a homeomorphism for any $t \in X$.*

Proposition 2.7. *The group $H(X)$ of all self-homeomorphisms of a topological space X is isomorphic to a subgroup of the group $H(T \times X, X)$ of generalized isotopies.*

Proof. To each $f \in H(X)$ we assign the continuous map $\tilde{f} : T \times X \to X$ defined by $\tilde{f}(t, x) = f(x)$ for all $t \in T$, $x \in X$. Obviously, $\widetilde{f^{-1}} = \tilde{f}^{-1}$. Thus, $\tilde{f}$ is a generalized isotopy, i.e. $\tilde{f} \in H(T \times X, X)$. The correspondence $f \to \tilde{f}$ is the required isomorphism between the group $H(X)$ and a subgroup of $H(T \times X, X)$. □

Theorem 2.8 ([5, Theorem 6]). *Let T be any topological space and X be a locally compact and locally connected. The group $H(T \times X, X)$ is isomorphic (algebraically and topologically) to $C(T, H(X))$.*

Proof. Consider the map $p : C(T, H(X)) \to H(T \times X, X)$ defined by $p(f)(t, x) = f(t)(x)$ for $f \in C(T, H(X))$, $t \in T$ and $x \in X$. The map $f(t) : X \to X$ is a homeomorphism for each $t \in X$. Therefore, by virtue of Theorem 2.6, $p(f) : T \times X \to X$ is a generalized isotopy, i.e. $p(f) \in H(T \times X, X)$.

Let us prove that p is injective. Take $f, g \in C(T, H(X))$, $f \neq g$. There exists a point $t_0 \in T$ such that $f(t_0) \neq g(t_0)$. Since $f(t_0), g(t_0) \in H(X)$, it follows that $f(t_0)(x_0) \neq g(t_0)(x_0)$ for some $x_0 \in X$. Thus, $p(f)(t_0, x_0) \neq p(g)(t_0, x_0)$, and hence $p(f) \neq p(g)$.

The map p is also surjective. Indeed, let $\varphi \in H(T \times X, X)$ be any generalized isotopy. By virtue of Theorem 2.6, the map $\varphi_t : X \to X$ defined by $\varphi_t(x) = \varphi(t, x)$ is a homeomorphism for any $t \in T$. It is easy to see that the element $f \in C(T, H(X))$ determined by the condition $f(t) = \varphi_t$ is the preimage of φ. Indeed, we have $p(f)(t, x) = f(t)(x) = \varphi_t(x) = \varphi(t, x)$.

Thus, the map $p^{-1} : H(T \times X, X) \to C(T, H(X))$ defined by $p^{-1}(\varphi)(t)(x) = \varphi(t, x)$ for $\varphi \in H(T \times X, X)$ is inverse to $p : C(T, H(X)) \to H(T \times X, X)$.

The map p is a homomorphism, that is, $p(f \circ g) = p(f) \circ p(g)$. Indeed, for any $t \in T$, $x \in X$ we have

$$\begin{aligned} p(f \circ g)(t, x) &= (f \circ g)(t)(x) = (f(t) \circ g(t))(x) = f(t)(g(t)(x)) \\ &= f(t)(p(g)(t, x)) = p(f)(t, p(g)(t, x)) = (p(f) \circ p(g))(t, x). \end{aligned}$$

Let us prove the continuity of p. Take any element $W(K \times K', U)$ of the subbase of the compact-open topology on $H(T \times X, X)$, where $U \subset X$ open and $K \subset T$, $K' \subset X$ are compact subsets. Let us show that the preimage of $W(K \times K', U)$ is the set $W(K, W(K', U))$, which is an element of the subbase of the compact-open topology on $C(T, H(X))$. Indeed, for any $\varphi \in W(K \times K', U)$ and $f = p^{-1}(\varphi) \in C(T, H(X))$, we have

$$\begin{aligned} \varphi \in W(K \times K', U) &\iff \varphi(t, x) \in U \iff p(f)(t, x) \in U \\ &\iff f(t)(x) \in U \iff f \in W(K, W(K', U)), \end{aligned}$$

where $t \in K$ and $x \in K'$ are arbitrary elements.

The continuity of the inverse map $p^{-1} : H(T \times X, X) \to C(T, H(X))$ is proved in precisely the same way. □

Corollary 2.9. *If T is any topological space and X is locally compact and locally connected, then $H(T \times X, X)$ is a topological group.*

Proof. It is known that $H(X)$ is a topological group (see [1]). Therefore, $C(T, H(X))$ is a topological group as well (see [7]). According to Theorem 2.8, $H(T \times X, X)$ is a topological group. □

3. Generalized actions of groups and the category of generalized G-Spaces

The idea to introduce the notion of action of a group on a finite set is related to the permutation group of this set. The rule, by which the elements of the permutation group permute the elements of the set is called its group action.

The same applies when set X is infinite and we consider a symmetric group $S(X)$ of set X. Elements of $S(X)$, that are bijections of set X, also permute the elements of X. They satisfy the following conditions:

$$(gh)x = g(hx), \quad ex = x, \tag{11}$$

where $g, h \in S(X)$, and e — identity mapping of X.

These conditions are the basis of definition of abstract action of group G on X: *a map $\alpha : G \times X \to X$, $\alpha(g, x) = gx$ or $(g, x) \to gx$, is called an action of group G on X if it satisfies condition* (11) *for every $g, h \in G$ and identity element e of group G.*

Let's note that the map $g \to \alpha_g$, where $\alpha_g(x) = \alpha(g, x)$, is a homomorphism from G to the symmetric group $S(X)$.

When we consider a topological space X, we assume that all maps are continuous and so we get homomorphism from group G to the group $H(X)$ of all homeomorphisms of X.

Remark 3.1. Note that an action $\alpha : G \times X \to X$ of group G on X is a generalized isotopy. Indeed, the continuous map $\alpha^{-1} : G \times X \to X$ given by the formula $\alpha^{-1}(g, x) = \alpha(g^{-1}, x)$ is inverse to α in the sense of Definition 2.2: $(\alpha \circ \alpha^{-1})(g, x) = \alpha(g, \alpha^{-1}(g, x)) = \alpha(g, \alpha(g^{-1}, x)) = \alpha(gg^{-1}, x) = \alpha(e, x) = x$. Clearly, $\alpha^{-1} : G \times X \to X$ is the right action of group G on X. Hence, the set of all actions of group G on the topological space X is a subset of group $H(G \times X, X)$ of all generalized isotopies of X over G.

Now let us consider the group $H(T \times X, X)$ of generalized isotopies of topological space X over T. Since elements of the group $H(T \times X, X)$ satisfy condition (1), we can define the notion of generalized action of group G on X over T and generalized G-space in the following way.

Definition 3.2. A continuous map $\alpha : G \times T \times X \to X$ is called a *generalized action* of group G on X over T, if

$$\alpha(gh, t, x) = \alpha(g, t, \alpha(h, t, x)), \tag{12}$$

$$\alpha(e, t, x) = x, \tag{13}$$

where e is the identity element of G, $g, h \in G$, $t \in T$ and $x \in X$.

In the notation $\alpha(g, t, x) = g(t, x)$ relations (12) and (13) take the form $gh(t, x) = g(t, h(t, x))$, $e(t, x) = x$.

Definition 3.3. A topological space X with a fixed generalized action of group G on X over T, that is, a quadruple (G, X, T, α) is called a *generalized G-space* or a *G-space over T*.

Let α be a generalized action of a topological group G on space X over T. For each $g \in G$, we define a continuous map $\alpha_g : T \times X \to X$ as $\alpha_g(t, x) = \alpha(g, t, x)$.

The following proposition is valid.

Proposition 3.4. *The map $g \to \alpha_g$ is a continuous homomorphism from G to the group $H(T \times X, X)$ of generalized isotopies of topological space X over T.*

It is not difficult to note that for any $t \in T$ the continuous map $\alpha_t : G \times X \to X$ defined by $\alpha_t(g, x) = \alpha(g, t, x)$ is an action of group G on X. Hence, *a generalized action α of group G on space X over T induces the family $\{\alpha_t\}$, $t \in T$, of "ordinary" actions of G on X.*

Now, let (G, X, α) be a G-space. It is evident that one can define a generalized action $\tilde{\alpha}$ of G on X over T by formula

$$\tilde{\alpha}(g, t, x) = \alpha(g, x) \tag{14}$$

for all $t \in T$ and $x \in X$. This is a trivial generalized action induced by "ordinary" action of a group G on X.

The following example shows that there is a nontrivial generalized action induced by "ordinary" action of group G on X.

Example 3.5. Let (G, X, α) be a G-space. The continuous map $\tilde{\alpha} : G \times G \times X \to X$ defined by $\tilde{\alpha}(g, h, x) = \alpha(hgh^{-1}, x)$, or $g(h, x) = hgh^{-1}x$, is a generalized action of the topological group G on X over G.

Indeed, conditions (12) and (13) in Definition 3.2 are satisfied: we have $\tilde{\alpha}(g\tilde{g}, h, x) = hg\tilde{g}h^{-1}x = hgh^{-1}h\tilde{g}h^{-1}x = \tilde{\alpha}(g, h, h\tilde{g}h^{-1}x) = \tilde{\alpha}(g, h, \tilde{\alpha}(\tilde{g}, h, x))$ and $\tilde{\alpha}(e, h, x) = heh^{-1}x = ex = x$ for all $g, \tilde{g}, h \in G$ and $x \in X$.

Now, let us consider two generalized G-spaces (G, X, α) and (G, Y, β). One can define the notion of equivariant map between two generalized G-spaces in the following way.

Definition 3.6. A continuous map $f : X \to Y$ is called *an equivariant map* if $f(\alpha(g, t, x)) = \beta(g, t, f(x))$, or, equivalently, $f(g(t, x)) = g(t, f(x))$, for all $g \in G$, $t \in T$ and $x \in X$.

An equivariant map $f : X \to Y$ of generalized G-spaces (G, X, α) and (G, Y, β) which is also a homeomorphism is called an equivalence of generalized G-spaces.

The following assertion is valid.

Theorem 3.7. *Generalized G-spaces and equivariant maps form a category.*

We denote this category by G-Top_T. It is evident that if $T = \{*\}$ is a singleton then G-$Top_* = G$-Top.

Note that if X and Y are G-spaces, then any equivariant map $f : X \to Y$ is also equivariant as a map between generalized G spaces with trivial generalized actions defined by (14). Thus, the category G-Top can be regarded as a subcategory of the category G-Top_T. So, we have the following chain of natural extensions of categories: $Top \subset G$-$Top \subset G$-Top_T.

There is a natural covariant functor from the category $G\text{-}Top_T$ to the category G-Top. Indeed, if (G, T, X, α) is a generalized G-space then we can consider the G-space $(G, T \times X, \tilde{\alpha})$ on which group G acts as

$$\tilde{\alpha}(g, t, x) = (t, \alpha(g, t, x)) \tag{15}$$

for all $g \in G$, $t \in T$ and $x \in X$.

It remains to note that if X and Y are generalized G-spaces, then any equivariant map $f : X \to Y$ generates the equivariant map $\tilde{f} : T \times X \to T \times Y$ defined by

$$\tilde{f}(t, x) = (t, f(x)), \tag{16}$$

where $t \in T$ and $x \in X$.

Thus, the formulas (15) and (16) determine an isomorphism of $G\text{-}Top_T$ onto a subcategory of $G\text{-}Top$ whose objects are G-spaces of the form $T \times X$ with actions $\tilde{\alpha}$ over T while morphisms are equivariant maps (i.e. G-maps) $\tilde{f} : T \times X \to T \times Y$ over T. Clearly, "over T" means that the diagrams

$$\begin{tikzcd} G \times T \times X \arrow[rr, "\tilde{\alpha}"] \arrow[dr] & & T \times X \arrow[dl] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T \times X \arrow[rr, "\tilde{f}"] \arrow[dr] & & T \times Y \arrow[dl] \\ & T & \end{tikzcd}$$

commute, where the maps with the codomain T are natural projections. In other words, the established functor $G\text{-}Top_T \to G\text{-}Top$ is injective on objects and faithful.

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