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INITIAL–BOUNDARY VALUE PROBLEMS FOR FUSS-WINKLER-ZIMMERMANN AND SWIFT–HOHENBERG NONLINEAR EQUATIONS OF 4TH ORDER

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Abstract. This paper presents results of the investigation of bifurcations of stationary solutions of the Swift–Hohenberg equation and dynamic descent to the points of minimal values of the functional of energy for this equation, obtained with the use of the modification of the Lyapunov–Schmidt variation method and some methods from the theory of singularities of smooth functions. Nonstationary case is investigated by the construction of paths of descent along the trajectories of the infinite-dimensional SH dynamical system from arbitrary initial states to points of the minimum energy.

1. Introduction

The main works in bifurcation theory were initially obtained by H. Poincare, A. M. Lyapunov, V. I. Arnold, A. A. Andronov, H. Hopf and many other mathematicians.

In mechanical systems, as a rule, the steady motions (equilibrium states or relative equilibrium states) depend on parameters. The values of parameters, at which the change of the number of equilibriums occurs, are called the bifurcation ones. The curves or surfaces showing the sets of equilibriums in the state space of parameters, are called the bifurcation curves or bifurcation surfaces. When the parameter passes over the bifurcation value, the change of stability properties of equilibriums occurs as a rule. The bifurcations of equilibriums can result with arising periodical and more complicated motions.

Stationary equation SH (Swift–Hohenberg [15,19,21]) formally coincides with the nonlinear equation of deflections a beam on an elastic foundation, which is investigated by Y. A. Mitropolsky, B. I. Moseenkov [17], J. M. T. Thompson, G. W. Hunt [22,23], B. S. Bardin, S. D. Furth [2] et al. The linear equation of the beam bending on an elastic foundation, is a well-known model suggested by Fuss–Winkler–Zimmermann

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(FWZ) and serving, up to the present time, the basis for majority of engineering calculations of stability of the building structures [24]. Nonlinear generalizations of FWZ model started to be considered more or less recently [2, 23].

Bifurcation analysis of periodic solutions of nonlinear equations, close to the FWZ type, carried out by B. M. Darinskii and Y. I. Sapronov [4–6], the main attention in their papers was paid to the question of the appearance of post-critical periodic structures in crystals. Deflections of nonhomogeneous beams on elastic foundation were studied by the author of this article in [12, 13].

The subject of research in this article is infinite-dimensional dynamical systems defined by the differential equations:

$$\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} + \kappa \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0\,, \tag{1}$$

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + \kappa \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0\,, \tag{2}$$

that are modeling, in different interpretations and with different initial-boundary conditions, various phenomena in nature and technology. Those equations have common stationary equation:

$$\frac{\partial^4 w}{\partial x^4} + \kappa \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0\,. \tag{3}$$

Equations similar to (1)–(3), were applied in investigation of the problems of periodic structures in physical media and of the problem of deflections of beams on elastic foundation by J. Swift, P.S. Hohenberg [21], H.E. Kulagin, L.M. Lerman, T.G. Shmakova [15], J.A. Mitropolsky, B.I. Moseenkov [17], J.M.T. Thompson, G.W. Hunt [22, 23], B.S. Bardeen, S.D. Furth [2], and others. Equation (1) got the name *Swift-Hohenberg equation* (SH equation). It is modeling the appearance of space one-dimensional structures in hydrodynamics, in chemical media and nonlinear optics (see, e.g., the survey [15]). Equations similar to (2)–(3), besides the theory of elastic supercritical states of elastic beams, found applications in the theory of ferroelectric phase transitions [9].

An analysis of equations (1)–(3) can be conditionally divided into three parts, the first of which analyses the stationary supercritical state (depressions, post-critical phase, cluster reconstructions etc.). The author carried out the analysis of stationary states on the basis of a modification of the Lyapunov-Schmidt variation method with the use of some methods from the theory of singularities of smooth functions [6,12,13]. This analysis is done under the Dirichlet, Neumann, periodical and other boundary conditions. We pay special attention to the case of weak inhomogeneity of the medium, in which there is a functional parameter at the highest derivative that has the physical meaning of inhomogeneity of the material: $\frac{d^2}{dx^2}(q\frac{d^2w}{dx^2})$. The second and the third parts of analysis of the equations are connected with the study of dynamic descent trajectories to the minimum of the energy functional (SH equation) and of oscillations of the infinite-dimensional dynamical systems (FWZ equation).

The main result of the bifurcation analysis of stationary states here is a construction of the complete qualitative picture of the behavior of solutions, their quantity and character, dependence on parameters, and construction of the first asymptotics of the solutions. Proposed bifurcation analysis is based on the local reduction of equation (3) to the Lyapunov-Schmidt function, inheriting all structural properties of the solutions of original problem [6].

In the third section, we study the dynamic of the SH equation by constructing the descent trajectory to stationary points with the minimum energy and other graphical images of the states along the descent trajectories. This approach made a good showing in studying structural rearrangements of physical media based on model equations of "reaction-diffusion" and Cahn-Hilliard [11] types. As the result of this approach, the graphical images of solutions along descent paths (from an arbitrary state to stationary points of lowest energy) are obtained.

2. Analysis of deflection of the elastic beam on elastic foundation

2.1 Supercritical deflections of the homogeneous beam on elastic foundation

Nonlinear scaled model of oscillatory motion of the homogeneous beam on elastic foundation is given by equation (2) (see [23]), where w is deflection of the beam (the offset of points along the midline of elastic beam located along x axis). A similar equation arises in the theory of crystals [9], where w is an order parameter.

The first step in the study of this problem is determination of the equilibrium (stationary) conditions, defined by the equation

$$\frac{d^4w}{dx^4} + \kappa\frac{d^2w}{dx^2} + \alpha w + w^3 = 0, \tag{4}$$

which we consider below with standard boundary conditions

$$w(0) = w(1) = w''(0) = w''(1) = 0. \tag{5}$$

Initial boundary value problem with boundary condition (5) admits 2-dimensional degeneration.

Equation (4) is the Euler equation for the extremals of the functional (action)

$$V = \int_0^1 \left(\frac{1}{2}\left(\left(\frac{d^2w}{dx^2}\right)^2 - \kappa\left(\frac{dw}{dx}\right)^2 + \alpha w^2\right) + \frac{w^4}{4}\right) dx. \tag{6}$$

The two-dimensional degeneration of zero extremal occurs at

$$\kappa = \kappa_1 := (p^2 + q^2)\pi^2, \qquad \alpha = \alpha_1 := p^2q^2\pi^4, \qquad p, q \in \mathbb{N},$$

with standard modes of bifurcation (the basis and kernel of the second differential)

$$e_1 = \sqrt{2}\sin(p\,\pi x), \qquad e_2 = \sqrt{2}\sin(q\,\pi x).$$

Below it is assumed that $p = 1,\ q = 2$ and, respectively, $\kappa_1 = 5\pi^2,\ \alpha_1 = 4\pi^4$ (these values are the smallest ones of those, in which there is a 2-dimensional degeneration; in the other cases the analysis is similar).

The Lapunov-Schmidt reduction [6] to key function (of two variables)

$$W(\xi,\delta)=\inf_{w:\ \substack{\langle w,e_1\rangle=\xi_1,\\ \langle w,e_2\rangle=\xi_2}} V(w,\alpha_1+\delta_1,\kappa_1+\delta_2), \tag{7}$$

$\xi=(\xi_1,\xi_2)$, $\delta=(\delta_1,\delta_2)$, preserves the symmetry of the action functional. Since functional (6) is invariant under the involution of J_1, J_2

$$J_2(p)(x):=p\,(1-x),\qquad J_1:=-J_2,$$

for the function (7) we also have

$$W(-\xi_1,\xi_2,\delta_1,\delta_2)=W(\xi_1,-\xi_2,\delta_1,\delta_2)=W(\xi_1,\xi_2,\delta_1,\delta_2)$$

(symmetry of the rectangle). This yields (see [6]) the asymptotic representation

$$W(\xi,\delta)=U(\xi,\delta)+o(|\xi|^4)+O(|\xi|^4)O(\delta),$$

where $U(\xi,\delta)=V(\xi_1e_1+\xi_2e_2,\delta)$ is a linear Ritz approximation of functional V with respect to the modes e_1,e_2. Therefore, for the key function the following asymptotic representation takes place

$$\frac{\lambda_1}{2}\xi_1^2+\frac{\lambda_2}{2}\xi_2^2+\frac{1}{4}\left(A\xi_1^4+2B\xi_1^2\xi_2^2+C\xi_2^4\right)+o(|\xi|^4)+O(|\xi|^4)O(\delta), \tag{8}$$

where

$$\lambda_1=\delta_1-\pi^2\delta_2,\qquad \lambda_2=\delta_1-4\pi^2\delta_2,$$

$$A=\int_0^1 e_1^4\,dx=\frac{3}{2},\quad B=\int_0^1 e_1^2e_2^2\,dx=3,\quad C=\int_0^1 e_2^4\,dx=\frac{3}{2}.$$

On contracting the above formula by factor 3/2, we obtain function (8) with normalized principal part

$$\widetilde{W}_0(\xi,\delta)=\widetilde{U}(\xi,\delta)+o(|\xi|^4)+O(|\xi|^4)O(\delta),$$

where

$$\widetilde{U}(\xi,\delta)=\frac{\tilde\lambda_1}{2}\xi_1^2+\frac{\tilde\lambda_2}{2}\xi_2^2+\frac{1}{4}\left(\xi_1^4+4\xi_1^2\xi_2^2+\xi_2^4\right).$$

The "geometric subject" for bifurcation of critical points and the first asymptotics for branches of bifurcating points (in supercritical increments of control parameters) for the function $\widetilde{W}_0(\xi,\delta)$ are completely determined by its principal part of $\widetilde{U}(\xi,\delta)$, which is a perturbed two-dimensional cusp (with the coefficient of double ratio $a=4$), even with respect to each variable [6].

Let $\theta_1=\tilde\lambda_1-2\tilde\lambda_2$, $\theta_2=\tilde\lambda_2-2\tilde\lambda_1$. Since the Hessian of function $\tilde U$ can be represented in the form

$$\begin{pmatrix}\tilde\lambda_1+3\xi_1^2+2\xi_2^2 & 4\xi_1\xi_2\\ 4\xi_1\xi_2 & \tilde\lambda_2+2\xi_1^2+3\xi_2^2\end{pmatrix},$$

it is easy to check that for $\theta_1>0$ and $\theta_2>0$ there are four 2-mode critical points of index 1. All 1-mode points here are local minima, and zero-mode ones are critical point of index 2. The 1-mode critical point arise when the parameters $\tilde\lambda_1$ and $\tilde\lambda_2$ come to the domain of negative values.

The caustic (bifurcation diagram of functions [1]) $\Sigma_{\tilde U}$ of the function $\tilde U$ separates the plane of control parameters into six zones

$$\begin{aligned}\omega_0 &= \tilde{\lambda}_1 > 0,\ \tilde{\lambda}_2 > 0\ ,\\ \omega_1 &= \tilde{\lambda}_1 < 0,\ \tilde{\lambda}_2 > 0\ ,\\ \omega_2 &= \tilde{\lambda}_1 > 0,\ \tilde{\lambda}_2 < 0\ ,\\ \omega_3 &= \tilde{\lambda}_1 < 0,\ \tilde{\lambda}_2 < 0,\ \theta_1 < 0,\ \theta_2 > 0\ ,\\ \omega_4 &= \tilde{\lambda}_1 < 0,\ \tilde{\lambda}_2 < 0,\ \theta_1 > 0,\ \theta_2 < 0\ ,\\ \omega_5 &= \tilde{\lambda}_1 < 0,\ \tilde{\lambda}_2 < 0,\ \theta_1 > 0,\ \theta_2 > 0\ .\end{aligned}$$

Each zone has its own variety (bif-variety) of bifurcating critical points:

- parameters of the zone of ω_0 correspond to the case of the only critical point (minimum point at zero);
- ω_1, ω_2 corresponds to a pair of symmetrically arranged (relative to zero) 1-mode minimum points and the saddle at the origin;
- ω_3, ω_4 corresponds to the pair of symmetrically located 1-mode minimum points, the pair of 1-mode saddles and the local maximum point at zero;
- and, at last, ω_5 corresponds to four symmetrically located 1-mode minimum points, four 2-mode saddles and the point of local maximum at zero.

On making circuit of the plane of control parameters counterclockwise around zero, starting with the area ω_0, we obtain the corresponding metamorphosis of level lines and the distribution of critical points, represented on the following diagram:

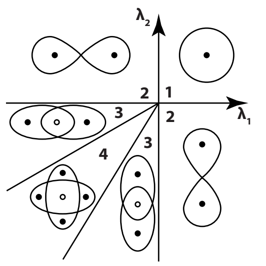

Figure 1: Bifurcation diagram

In this case, the varieties of bifurcating critical points (bif-variety) correspond to the following integer vectors: $(1,0,0)$, $(2,1,0)$, $(2,2,1)$, $(4,4,1)$ and only they arise.

2.2 The case of slightly inhomogeneous beams

Equilibrium configuration of slightly inhomogeneous beams are described by the equation

$$\frac{d^2}{dx^2}\left(q\frac{d^2w}{dx^2}\right) + \kappa\frac{d^2w}{dx^2} + \alpha w + w^3 = 0, \qquad q(x) = 1 + \varepsilon\gamma(x), \tag{9}$$

where ε is a small parameter and γ is a smooth function. Equation (9), considered on the interval $[0,1]$ of real axis with boundary conditions (5), determines the extremals of functional

$$V = \int_0^1 \left(\frac{1}{2} \left(q \left(\frac{d^2 w}{dx^2} \right)^2 - \kappa \left(\frac{dw}{dx} \right)^2 + \alpha w^2 \right) + \frac{w^4}{4} \right) dx. \tag{10}$$

The presence of "weight" factor q does not allow us to apply research scheme of [4]. Indeed, in this scheme, the condition of constancy of the bifurcation modes is not satisfied, while on this condition the computational algorithm of [4] is based. However, the bifurcation analysis of this boundary value problem can also be realized by the Lyapunov– Schmidt reduction to the key functions of a more general form

$$\widetilde{W}(\xi,\delta) = \inf_{w:\ \substack{\langle w,\widetilde{e}_1\rangle = \xi_1, \\ \langle w,\widetilde{e}_2\rangle = \xi_2}} V(w,\alpha_1+\delta_1,\kappa_1+\delta_2), \tag{11}$$

where $\widetilde{e}_k$ are "perturbed" bifurcation modes

$$\widetilde{e}_k = e_k + \varepsilon h_k + o(\varepsilon), \qquad e_k = \sqrt{2}\sin(k\pi x),$$

that form the basis of 2-dimensional root subspace of the Hesse operator $\mathcal{H} = \mathcal{A} + \varepsilon\mathcal{B}$ at the zero of functional (10), where

$$\mathcal{A}u := \frac{d^4 w}{dx^4} + \kappa\frac{d^2 w}{dx^2} + \alpha I, \quad \mathcal{B}w := \frac{d^2}{dx^2}\left(\gamma\frac{d^2 w}{dx^2}\right)$$

(elements of $\widetilde{e}_k$ are not, generally speaking, eigenfunctions of the operator $\mathcal{H}$).

The main technical difficulty in the construction of the principal part of the key function (11) is to calculate h_k. They can be determined by using the formula of the orthogonal projection on the root subspace of a perturbed symmetric operator [16].

So, instead of the eigenfunctions, we consider the elements $\widetilde{e}_j(\lambda)$, $j = 1,\ 2$, (below they are called the root ones), for which

$$\frac{\partial f}{\partial x}(0,\lambda)\widetilde{e}_j(\lambda) = \sum_k \alpha_{jk}(\lambda)\,\widetilde{e}_k(\lambda).$$

An important concomitant of the proposed approach here is that the functions, included in these relations, i.e. $\alpha_{jk}(\lambda)$, and $\widetilde{e}_j(\lambda)$, smoothly depend on λ. As the required basic elements one can take $\widetilde{e}_k(\lambda) = \mathbf{P}(\lambda)(e_k)$, where

$$\mathbf{P}(\lambda) = \frac{1}{2\pi i}\oint_\ell \mathcal{R}(\lambda,z)dz$$

is an orthogonal projection onto a two-dimensional root space, ℓ is a circle of sufficiently small radius centered at the origin (in the complex plane) and $\mathcal{R}(\lambda,z)$ is resolvent $\mathcal{R}(\lambda,z) = (\mathcal{A} + \varepsilon\mathcal{B} - zI)^{-1}$. Thus,

$$\widetilde{e}_k = e_k + \varepsilon h_k + o(\varepsilon), \tag{12}$$

where

$$h_k = \mathcal{M}e_k, \tag{13}$$

$$\mathcal{M} = \frac{1}{2\pi i}\oint_\ell (\mathcal{A} - zI)^{-1}\mathcal{B}(\mathcal{A} - zI)^{-1}\,dz. \tag{14}$$

So, the following assertion is true:

THEOREM 2.1. *Perturbed root vectors* $\widetilde{e}_k$, $k = 1, 2$, *can be represented in form* (12), *where* h_k *are determined by relations* (13)–(14).

2.3 Calculation of integral coefficients

From the preceding discussion it follows that for obtaining the root vectors $\widetilde{e}_k$ it is necessary to calculate the integral

$$\mathcal{M}\ e_k \ = \ \frac{1}{2\pi i}\oint_{|z|=1}(\mathcal{A}-zI)^{-1}\mathcal{B}(\mathcal{A}-zI)^{-1}e_k\,dz, \tag{15}$$

taking into account boundary conditions (5) (where ℓ loop $|z| = 1$).

The eigenvalue $z = 0$ has multiplicity two, it corresponds to the eigenelements

$$e_k = \sqrt{2}\sin(k\pi x),\ k = 1, 2.$$

Note that if e_k is an eigenvector corresponding to eigenvalue z_k : $\mathcal{A}e_k = z_k e_k$, then

$$(\mathcal{A}-zI)^{-1}(\mathcal{A}-z_kI)e_k = 0,$$

and we obtain

$$(\mathcal{A}-zI)^{-1}e_k = \frac{e_k}{z_k - z}.$$

Thus, formula (15) can be rewritten as

$$\mathcal{M}e_k \ = \frac{1}{2\pi i}\oint_{|z|=1}\frac{(\mathcal{A}-zI)^{-1}\mathcal{B}e_k}{z_k - z}\,dz. \tag{16}$$

It is obvious that the operator $\mathcal{A}$ with given domain and boundary conditions is symmetrical, and its eigenfunctions form a complete system in $L^2[0,1]$ [18].

Consider the function $g = \mathcal{B}e_k,\ g(x) \in L^2[0,1]$, and expand it in the series with respect to eigenfunctions of $\mathcal{A}$:

$$g(x) = \sum_{n=1}^{\infty} c_n e_n(x), \tag{17}$$

where

$$c_n = \int_0^1 g(s)\, e_n(s)ds = \int_0^1 e_n(s)(\gamma(s)e_k''(s))''\,ds. \tag{18}$$

On integrating by parts, we transform integral (18) to the form:

$$c_n = -(n\pi)^2\int_0^1 e_n(s)\gamma(s)e_k''(s)ds = (nk\pi^2)^2\int_0^1 \gamma(s)e_n(s)e_k(s)ds.$$

Then using formula (17) in integral (16), where $\kappa = 5\pi^2,\ \alpha = 4\pi^4$, we obtain the following expansions

$$\begin{aligned}
\mathcal{M}e_k(x) = & \ \frac{1}{2\pi i}\oint_{|z|=1} \frac{(\mathcal{A}-zI)^{-1}}{z}\sum_{n=1}^{\infty} c_n e_n(x)\,dz \\
& = \sum_{n=1}^{\infty} c_n \ \frac{1}{2\pi i}\oint_{|z|=1} \frac{e_n(x)dz}{z(z_n-z)} \\
& = \sum_{n=3}^{\infty} c_n \ e_n(x) \ \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{z_n}\left(\frac{1}{z}+\frac{1}{z_n-z}\right)\,dz = \sum_{n=3}^{\infty}\frac{c_n e_n(x)}{z_n}.
\end{aligned}$$

Here we take into account that $z_1 = z_2 = 0$ and that

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z^2} = 0.$$

Thus, on the basis of (15), we obtain for all perturbed eigenfunctions that

$$\left.\begin{aligned}
Me_1 &= \sum_{n=3}^{\infty} \tfrac{2\sqrt{2}\,n^2}{(n^2-4)(n^2-1)} \left(\int_0^1 \gamma(s)\sin(n\pi s)\sin(\pi s)\,ds\right)\sin(n\pi x), \\
Me_2 &= \sum_{n=3}^{\infty} \tfrac{8\sqrt{2}\,n^2}{(n^2-4)(n^2-1)} \left(\int_0^1 \gamma(s)\sin(n\pi s)\sin(2\pi s)\,ds\right)\sin(n\pi x).
\end{aligned}\right\} \tag{19}$$

Now, since the asymptotic behavior of the first root vectors $\tilde{e}_1$, $\tilde{e}_2$ is known, we can construct the principal part of the key functions (up to the linear change of coordinates)

$$\begin{aligned}
W_q(\xi,\nu) = & \ \frac{1}{2}\left(\nu_1\,\xi_1^2+\nu_2\,\xi_2^2+2\,\nu_3\,\xi_1\xi_2\right)+\frac{1}{4}\left(A\xi_1^4+2B\xi_1^2\xi_2^2+C\xi_2^4\right) \\
& + O(|\xi|^4)+O(|\xi|^4)O(\nu)+o(\nu),
\end{aligned}$$

where

$$\nu_1 = \int_0^1 \left(q\left(\frac{d^2\tilde{e}_1}{dx^2}\right)^2-(\kappa_1+\delta_1)\left(\frac{d\tilde{e}_1}{dx}\right)^2+(\alpha_1+\delta_2)\tilde{e}_1^2\right)\,dx, \tag{20}$$

$$\nu_2 = \int_0^1 \left(q\left(\frac{d^2\tilde{e}_2}{dx^2}\right)^2-(\kappa_1+\delta_1)\left(\frac{d\tilde{e}_2}{dx}\right)^2+(\alpha_1+\delta_2)\tilde{e}_2^2\right)\,dx, \tag{21}$$

$$\nu_3 = \varepsilon\int_0^1 \left(\gamma\frac{d^2\tilde{e}_1}{dx^2}\cdot\frac{d^2\tilde{e}_2}{dx^2}\right)\,dx, \tag{22}$$

$$A = C = \int_0^1 e_1^4\,dx = \frac{3}{2} \quad B = \int_0^1 e_1^2 e_2^2\,dx = 3\,.$$

After contraction by the factor $3/2$ we obtain the representation of the key function that we are looking for. Thus, the following statement takes place.

Theorem 2.2. *For key function* (11) *the representation*

$$\begin{aligned}
\widetilde{W}_q(\xi,\nu) = & \ \frac{1}{2}\left(\widetilde{\nu}_1\,\xi_1^2+\widetilde{\nu}_2\,\xi_2^2+2\,\widetilde{\nu}_3\,\xi_1\xi_2\right)+\frac{1}{4}\left(\xi_1^4+4\xi_1^2\xi_2^2+\xi_2^4\right) \\
& + O(|\xi|^4)+O(|\xi|^4)O(\widetilde{\nu})+o(\widetilde{\nu})
\end{aligned}$$

is true, where $\widetilde{\nu}_j$ *is defined by relations* (13), (14), (19)–(22).

As compared with the function W_0, defined by equation (8), here the additional term $\widetilde{\nu}_3\,\xi_1\xi_2$ appears that destroys the symmetry of rectangle.

2.4 The structure of the caustic. Stable equilibrium

Caustic Σ (discriminant set of initial equation) is a two-dimensional surface in the three-dimensional space of parameters $\tilde{\nu}_i$, $i = 1, 2, 3$. Here the critical points of the key functions are determined by the system of equations

$$\frac{\partial \tilde{U}}{\partial \xi_1} = \tilde{\nu}_1\xi_1 + \xi_1^3 + 2\xi_1\xi_2^2 + \tilde{\nu}_3\xi_2 = 0,$$

$$\frac{\partial \tilde{U}}{\partial \xi_2} = \tilde{\nu}_2\xi_2 + 2\xi_1^2\xi_2 + \xi_2^3 + \tilde{\nu}_3\xi_1 = 0,$$

$$\det H = \begin{vmatrix} \tilde{\nu}_1 + 3\xi_1^2 + 2\xi_2^2 & 4\xi_1\xi_2 + \tilde{\nu}_3 \\ 4\xi_1\xi_2 + \tilde{\nu}_3 & \tilde{\nu}_2 + 2\xi_1^2 + 3\xi_2^2 \end{vmatrix} = 0,$$

where H is the Hessian. Solving this system with respect to $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3$ and going to the polar coordinates $\xi_1 = \rho\,\sin\varphi$, $\xi_2 = \rho\,\cos\varphi$, we obtain

$$\tilde{\nu}_1 = -\rho^2\frac{(3 + \cos^3(2\varphi))}{2 + \sin^2(2\varphi)} = 0,$$

$$\tilde{\nu}_2 = -\rho^2\frac{(3 - \cos^3(2\varphi))}{2 + \sin^2(2\varphi)} = 0,$$

$$\tilde{\nu}_3 = -3\rho^2\frac{\sin^3(2\varphi)}{4 + 2\sin^2(2\varphi)} = 0.$$

On the basis of this system, we obtain the image of Σ, the level lines of the key function (with "typical" values of the parameters $\widetilde{\nu}_1$, $\widetilde{\nu}_2$, $\widetilde{\nu}_3$ and $\gamma = 0{,}01 \cdot \sin(2\pi x)$), as well as the separatrix level surfaces of the key function and the forms of deflection corresponding to the points of minimum.

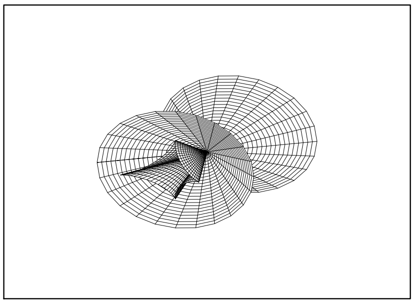

Figure 2: Caustic surface

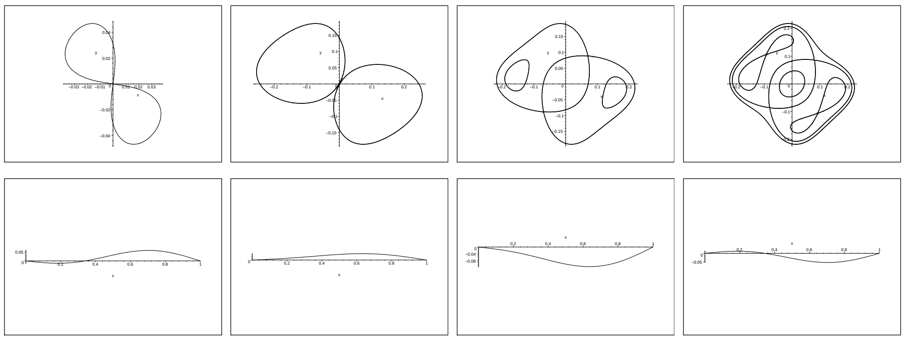

Figure 3: Contour curves and solutions

3. Tracing the descent along the trajectories of the Swift-Hohenberg equation to the points of minimum energy

Let us turn back to the Swift–Hohenberg equation (1). The change of κ, and α causes loss of stability of the initial phase and, as a result (as the response of the system), its transition to a new state (new structural properties). Such a transition can be accompanied by, for example, spinodal stratification (the distraction) expressed in changing the local concentrations of components, in appearing first the granular structure and then clusters and domains of the new phase. During the recent years, restructuring the physical environment is often explained on the basis of the nonlinear diffusion SH equation (Cahn-Hilliard [3], [20])

$$\dot{u} = \delta \operatorname{grad} V(u) := \mathcal{D}\, \delta(u^3 - u - \gamma\delta(u)), \tag{23}$$

where $u = u(x)$ is the relative concentration of the component substance, $x \in U \subset \mathbb{R}^m$, $1 \leq m \leq 3$, $\mathcal{D}$ is the diffusion coefficient,

$$V_{CH}(u) := \mathcal{D} \int_U \left(\frac{(u^2-1)^2}{4} + \frac{\gamma}{2} |\nabla u|^2 \right) dx$$

is the energy integral and U is the domain occupied by the medium under consideration.

The SH nonlinear equation is close to equation (23), and can also simulate the structural transformations. It can be represented as a gradient dynamical system (infinite number of degrees of freedom)

$$\dot{w} = -\operatorname{grad} V(w) := -\left(\frac{\partial^4 w}{\partial x^4} + \kappa \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 \right), \tag{24}$$

$w = w(x,t)\,,\ x \in [0,1]\,,\ t \geq 0,$

$$V(w) := \int_0^{2\pi} \left(\frac{(w'')^2}{2} - \kappa \frac{(w')^2}{2} + \alpha \frac{w^2}{2} + \frac{w^4}{4} \right) dx$$

is the energy integral. We further assume that the concentration satisfies the boundary condition

$$w(0,t) = w(1,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(1,t) = 0\,. \tag{25}$$

In the study of local bifurcations of extremals, the Ritz functional approximation is used

$$W_R(\xi) := V(c + \xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_n e_n),$$

constructed from initial eigenfunctions (modes) e_j of the operator $\frac{d^4}{d^4 x}$. In the nonlocal problems, one can also use the Ritz approximations, but in order to achieve the required accuracy of the solution it is necessary to use large number of modes that leads to the large dimension of approximating system. To reduce the dimension of the approximating system it is possible to apply the nonlinear Ritz approximation, say, in the form of nonlocal extended key function, i.e. to pass in fact to the finite-dimensional problem $\dot{\xi} = \operatorname{grad} W(\xi)$, $\xi \in \mathbb{R}^n$, where $W(\xi) := \inf_{\langle w, e_j \rangle = \xi_j} V(w)$ is the Lyapunov–Schmidt key function.

Below, we use the procedure of the shortest direct descent to the minimum point of V (without going to the approximating key function). The first step of this procedure is to select the shift amount along the gradient, starting from (generating) point, in order to reduce the value of the energy functional.

As the final states of the desired trajectories of dynamic system (24) we use the orbits of minimum points that branched off (with increasing parameter κ and α) from subcritical zero equilibrium.

The main step in the construction of “line of the shortest” descent to the minimum is the solution (relative to s) of

$$\langle \operatorname{grad}\widetilde{V}(a_0 + s h_0), h \rangle = 0. \tag{26}$$

Here $h_0 = -\operatorname{grad}\widetilde{V}(a_0)$, $g = -\operatorname{grad}\widetilde{V}$, a_0 is the initial (generating) point. For example, it is possible to take (to be specific)

$$a_0 = \sin(7\pi x_1) + \varepsilon \sum_{k=1}^{6} \sin(k\pi x)\,,$$

ε is a certain specified small value. Using for $g(a + sh_0)$ the Taylor expansion, we obtain the relation

$$g(a_0 + sh_0) = g(a_0) + s\frac{\partial g}{\partial x}(a_0) h_0 + o(s),$$

where $\frac{\partial g}{\partial x}(a_0)$ is the Frechet derivative of the gradient mapping g, and

$$\langle g(a_0 + sh_0), h_0 \rangle = \langle g(a_0), h_0 \rangle + s\langle (\partial g/\partial x)(a_0) h_0, h_0 \rangle + o(|s|).$$

Hence, starting from equation (26), we can set (with some accuracy)

$$s = s_0 := -\frac{\langle g(a_0), h_0 \rangle}{\langle (\partial g/\partial x)(a_0) h_0, h_0 \rangle} = -\frac{|g(a_0)\ |^2}{\langle (\partial g/\partial x)(a_0) h_0, h_0 \rangle}\,.$$

The search for the value of s_0 can be done more accurately by finding the minimum point of the fourth-degree polynomial

$$p(s) = s^4 + p_3 s^3 + p_2 s^2 + p_1 s \ \sim\ \widetilde{V}(a_0 + s\, h_0)$$

(considered to within a constant factor and constant summand).

The next step of the shortest descent is the repetition of the main step for new generating point $a_1 := a_0 + s_0 h_0$, etc. Below, on Figure 4, we show the graphs, obtained for $\kappa = 5\pi^2 + 0.5$, $\alpha = 4\pi^4 + 0.2$, of the corresponding intermediate and final solution functions after a certain preliminary choice of the Fourier coefficients of the expansion for a randomly chosen initial function $\sin(7\pi x) + 0.05\sin(\pi x) + 0.06\sin(2\pi x) + 0.05\sin(3\pi x) + 0.01\sin(4\pi x) - 0.7\sin(5\pi x) + 0.9\sin(6\pi x)$.

On the base of this algorithm we obtain the curves that show the dynamics of concentrations along the trajectories of descent for problem (24), (25).

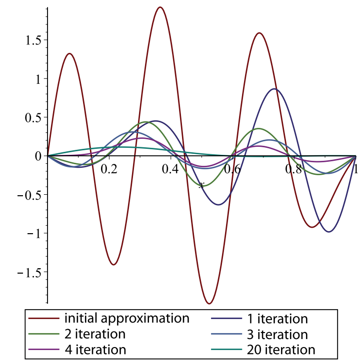

Figure 4: The initial value of the function and the first four and the twentieth iteration

In addition, a numerical experiment is carried out, the result of which is in the following figure that shows different minima for different values of the parameter $\kappa = 5\pi^2 - 15$, $\kappa = 5\pi^2$, $\kappa = 5\pi^2 + 15$ for one primary function: $\sin(5\pi x) + 0.05\sin(\pi x) + 0.06\sin(2\pi x) + 0.05\sin(3\pi x) + 0.01\sin(4\pi x)$.

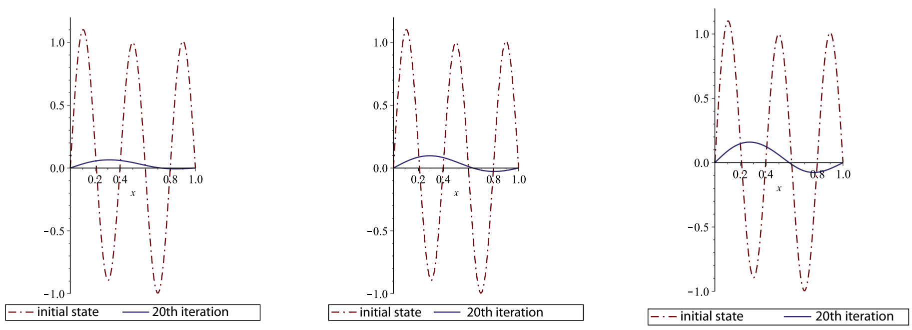

Figure 5: 1) $\kappa = 5\pi^2 - 15$; 2) $\kappa = 5\pi^2$; 3) $\kappa = 5\pi^2 + 15$ and the initial value of the function 10th iteration

Under constant values of the parameters $\kappa = 5\pi^2$ and $\alpha = 4\pi^4$ for different primary functions the minima are constructed. The graph shows that for various characters of the initial approximations the minima are different. For example, for the functions:

1) $f_1 = \sin(5\pi x) - 0.05\sin(\pi x) + 0.6\sin(2\pi x) + 0.5\sin(3\pi x) + 0.01\sin(4\pi x)$;
2) $f_2 = 0.1\sin(5\pi x) + 0.9\sin(\pi x) + 0.6\sin(2\pi x) + 0.5\sin(3\pi x) + 0.01\sin(4\pi x)$;
3) $f_3 = -f_2$

the graphs have the form, respectively:

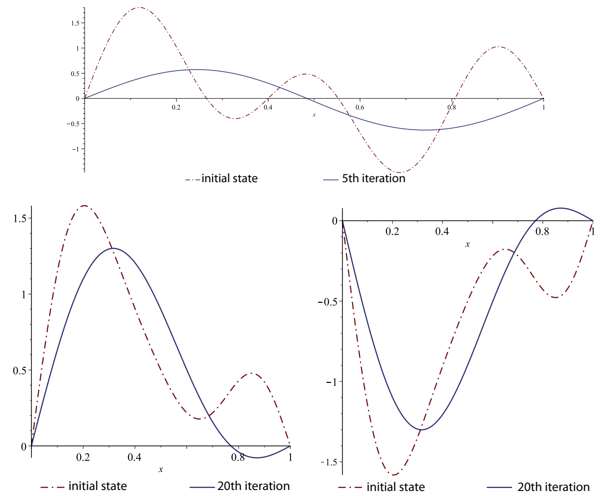

Figure 6: 1) Function f_1; 2) function f_2; 3) 3) function f_3

Conclusion

The methods presented in this paper, in future will allow one to investigate more exactly the character of dependence of a descent trajectory to the points of local minima on the values of parameters and on a starting point of the function.

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