LOCALLY CONTRACTIVE MAPS ON PERFECT
POLISH ULTRAMETRIC SPACES

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Abstract. In this paper we present a result concerning locally contractive maps defined on subsets of perfect Polish ultrametric spaces (i.e. separable complete ultrametric spaces). Specifically, we show that a perfect compact ultrametric space cannot be contained in its locally contractive image, a corollary relating this result to minimal dynamical systems, and pose a conjecture for the general Polish ultrametric case.

1. Preliminaries

A *topological space* X is *completely metrizable* if X admits a compatible *complete* metric d, and *separable* if X contains a countable *dense* subset. A *Polish space* is a separable completely metrizable topological space. Likewise, a metric space that is both separable and complete we will call a *Polish metric space*. Alexander Kechris provides a thorough treatment of Polish spaces in [6, Section 3, p. 13], and we follow the notation contained in this reference for the majority of the paper.

This paper focuses on a particular subclass of Polish spaces, namely *perfect* Polish *ultrametric* spaces. A topological space is perfect if and only if it contains no *isolated* points.

DEFINITION 1.1. A metric space (X, d) is an *ultrametric* space if and only if d has the property that for all $x, y \in X$, $d(x, y) \leq \max\{d(x, z), d(y, z)\}$. In this case we call d an ultrametric.

Briefly, the *diameter* of a set A, will be denoted $\operatorname{diam}(A)$, where $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. The following are some fundamental facts about ultrametric spaces which we will state without proof, see [9, p. 58], [6, p. 35].

PROPOSITION 1.2. *For an ultrametric space (X, d) and ball $B(x, r) \subseteq X$ with center x and radius r, the following holds for all $x, y, z \in X$ and nonempty $Y \subseteq X$,*

(i) $d(x, z) \neq d(y, z)$, *implies* $d(x, y) = \max\{d(x, z), d(y, z)\}$,

(ii) *All balls are clopen sets,*

(iii) *If two balls meet, then one is contained in the other,*

(iv) *If* $y \in B(x, r)$*, then* $B(x, r) = B(y, r)$,

(v) *If* $a \in Y$*, then* $\mathrm{diam}(Y) = \sup\{d(a, b) : b \in Y\}$.

Recall that a topological space X is *zero-dimensional* if and only if X is *Hausdorff* and has a basis consisting of *clopen* sets. Therefore, any Polish space that admits a compatible ultrametric is zero-dimensional, see [6, 7.1].

Some examples of zero-dimensional spaces that admit a ultrametric compatible with their usual topology are the p-adic numbers $\mathbb{Q}_p$, the Cantor Space 2^ω, the Baire Space ω^ω.

A *Lipschitz* function possesses a stronger form of continuity, which is defined below.

DEFINITION 1.3. For metric space (X, d), a function $f \colon X \to X$ is *Lipschitz* if and only if there is some constant $\alpha \geq 0$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. If $0 < \alpha < 1$, then f is a *contractive* map.

There exist other definitions for contractive functions in the mathematical literature. For the interested reader, a thorough comparison of various definitions for contractive maps was initiated by B.E. Rhoades, see [8]. But, for the purposes of this paper we have chosen a commonly used definition. In particular, this definition of a contractive map is fundamental to the *Banach Fixed Point Theorem*, a theorem of Analysis, also known as the *Banach Contraction Principle* [3].

A *neighborhood (nbhd)* of x, denoted N_x, is an open set containing the point x of a topological space X.

DEFINITION 1.4. Let (X, d) be a metric space. Then a map $f \colon X \to X$ is a *locally contractive* map, if and only if for each $x \in X$, there exists a $N_x \subseteq X$ such that for some $\alpha_x \in (0, 1)$, $d(f(u), f(v)) \leq \alpha_x d(u, v)$ for all $u, v \in N_x$.

Notice, the radial nature of the diameter of a ball in an ultrametric space allows us to consider a weaker version of a locally contractive map for the remainder of this paper, see Proposition 1.2 item (v).

DEFINITION 1.5. Let (X, d) be a metric space. Then a map $f \colon X \to X$ is a *local radial contraction*, if and only if for each $x \in X$, there exists a $N_x \subseteq X$ such that for some $\alpha_x \in (0, 1)$, $d(f(x), f(u)) \leq \alpha_x d(x, u)$ for all $u \in N_x$.

For clarity and convenience, we will introduce the definition of an *f-contractive* nbhd.

DEFINITION 1.6. Let (X, d) be a metric space, $f \colon X \to X$ a local radial contraction, and let $N_x \subseteq X$ be a nbhd of x for some $x \in X$. Then N_x is an

f-*contractive* nbhd of x, if and only if for some $\alpha_x \in (0,1)$ we have for all $u \in N_x$, $d(f(x), f(u)) \leq \alpha_x d(x, u)$.

Our next lemma demonstrates the effect a local radial contraction f has on compact f-contractive nbhds.

LEMMA 1.7. *Let (X, d) be a perfect compact Polish ultrametric space, let the map $f\colon X \to X$ be a local radial contraction, and $N_x \subseteq X$ be a compact f-contractive nbhd for some $x \in X$. Then, for some $\alpha \in (0,1)$, $\operatorname{diam}(f[N_x]) \leq \alpha \operatorname{diam}(N_x)$.*

Proof. Let X be a perfect compact Polish ultrametric space, and let the map $f\colon X \to X$ be a local radial contraction. Also, let $N_x \subseteq X$ be a compact f-contractive nbhd for some $x \in X$. By Definition 1.6, we know there exists an $\alpha \in (0,1)$ such that for all $u \in N_x$, $d(f(x), f(u)) \leq \alpha d(x, u)$. Since $f[N_x]$ is compact, by the continuity of f, it follows from Proposition 1.2 that $\operatorname{diam}(N_x) = d(x, x_1)$ for some $x_1 \in N_x$ such that $x_1 \neq x$, by perfectness, and $\operatorname{diam}(f[N_x]) = d(f(x), f(x_2))$ for some $x_2 \in N_x$. So, since N_x is a f-contractive nbhd, we have that $d(f(x)), f(x_2)) \leq \alpha d(x, x_2) \leq \alpha d(x, x_1)$. Therefore,

$$\operatorname{diam}(f[N_x]) \leq \alpha \operatorname{diam}(N_x). \quad \blacksquare$$

We will now state a useful fact without proof (see [9, p. 48–49]).

THEOREM 1.8. [W. H. Schikhof] *Let (X, d) be an ultrametric space. If $U \subseteq X$ is a nonempty open set, then U can be partitioned into balls.*

We now package Theorem 1.8 into a form more convenient for use in the proofs to follow, and note that compact metric spaces are necessarily Polish.

LEMMA 1.9. *Let (X, d) be a perfect compact Polish ultrametric space, and let $f\colon X \to X$ be a local radial contraction. Then, there exists a finite partition of X into f-contractive balls.*

Proof. (Sketch) Let (X, d) be a perfect compact ultrametric space, and let $f\colon X \to X$ be a local radial contraction. Let $\mathcal{N}_f$ be a collection of f-contractive nbhds corresponding to each $x \in X$. This collection forms an open cover of X, and by compactness can be assumed to be finite. By Theorem 1.8 we can create a partition of each nbhd in $\mathcal{N}_f$ into balls, and again by compactness each of these partitions can be assumed to be finite. Thus, by ultrametric property 1.2(iii) it is easy to see that we can form $\mathcal{B}$ a finite partition of X into f-contractive balls. $\blacksquare$

2. Main results

The proof of our main theorem relies upon the concept of a *generalized tree* or *R-tree* introduced by Gao and Shao [4]. This generalizes the concept of a *descriptive set-theoretic tree* on ω (see [6, Section 2, Chapter 1]), by allowing the tree's *level* or *depth* to be indexed by any *countable* set R with limit point 0 instead of just by ω.

This is done through the introduction of a *distance set* for a metric space (X, d), that is the set $R = \{d(x, y) : x \neq y \in X\}$.

Distance sets of separable ultrametric spaces are countable, see J.D. Clemens [2] and C. Shao [10]. The set $\mathbb{R}_+$ denotes the set of non-negative real numbers.

DEFINITION 2.1. [4] For countable $R \subseteq \mathbb{R}_+$, (X, d) is an R-*ultrametric space* if $\{d(x, y) : x \neq y \in X\} \subseteq R$.

For the remainder of this section, unless otherwise specified, fix $R \subseteq \mathbb{R}_+$ to be countable with $0 \in R'$, where R' denotes the set of all limit points of R.

DEFINITION 2.2. [4] Let $\omega^{<R}$ denote the set of all functions.

$$u \colon [a, +\infty) \cap R \to \omega$$

where $a \in R$, such that the set

$$\mathrm{supp}(u) = \{b \in R \cap [a, +\infty) : u(b) \neq 0\}$$

is finite. We call $(\omega^{<R}, \subseteq)$ the *full R-tree*.

For some $b \in \mathrm{dom}(u)$, we denote $u \restriction b$ the function $u \restriction ([b, +\infty) \cap R) \in \omega^{<R}$. For $u, v \in \omega^{<R}$, u is an *initial segment* of v, denoted $u \subseteq v$, if there exists $b \in \mathrm{dom}(v)$ such that $u = v \restriction b$. For every $u \in \omega^{<R}$ the *level* or *depth* of u is defined by $l(u) = \inf \mathrm{dom}(u) = \min \mathrm{dom}(u)$. For $T \subseteq \omega^{<R}$, if $v \in T$ and $u \subseteq v$ implies $u \in T$, then we call T an R-*tree*, and T is *pruned* if for all $u \in T$, there exists a *proper extension* v of u. A *branch* of an R-tree T is a function $f \in \omega^R$ such that for all $a \in R$, $f \restriction a = f \restriction ([a, +\infty) \cap R) \in T$, and $u \subseteq f$ if $u = f \restriction a$. We will let $[T]$ denote the set of all branches, also known as the *end space* or *body* of T, [4].

DEFINITION 2.3. [4] Let T be an R-tree. Define a metric on $[T]$ by

$$d(f, g) = \begin{cases} 0, & f = g \\ \max\{a \in R : f(a) \neq g(a)\}, & \text{otherwise} \end{cases}$$

THEOREM 2.4. [4] $([T], d)$ *is a Polish R-ultrametric space.*

THEOREM 2.5. [4] *For any Polish R-ultrametric space (X, ρ) there is a R-tree T such that (X, ρ) is isometric to $([T], d)$.*

DEFINITION 2.6. [4] For any $u \in \omega^{<R}$ define

$$N_u = \{f \in [\omega^{<R}] : u \subseteq f\}$$

We see above that any branch of an R-tree T restricted to any element of of R is an element of T. Together with the fact that for any perfect compact R-ultrametric space, the set containing all $u \in T$ where $l(u) = r$ for some $r \in R$ is finite (see [4, p. 500]), we show the existence of at least one *uniform diameter* finite partition of $[T]$ into balls, where uniform diameter is meant in the sense that each

ball is of equal diameter. Briefly, for any R-tree T and $u, v \in T$ if $u \subseteq v$ or $v \subseteq u$ we will call u and v *compatible*, and if this is not the case, *incompatible*, which we will denote by $u \perp v$. Also, it is easy to show that if $u \perp v$, then $N_u \cap N_v = \emptyset$.

The following result by J.D. Clemens is helpful in our next lemma and theorem.

THEOREM 2.7. [2] *A set $A \subseteq [0, \infty)$ is a set of distances of some perfect, compact, ultrametric space if and only if A can be enumerated as a countable decreasing sequence $\langle d_i : i \geq 0 \rangle$ with $\lim_{i \to \infty} d_i = 0$.*

Now, we define a property of a diameter for use in Lemma 2.9.

DEFINITION 2.8. Let $\mathcal{B}$ be a cover of a metric space X. For $r \in R$, where R is the distance set to X, r is an *infimum diameter* of R if there exists $B_0 \in \mathcal{B}$ such that for all $B \in \mathcal{B}$, $\mathrm{diam}(B_0) = r \leq \mathrm{diam}(B)$.

LEMMA 2.9. *Let $([T], d)$ be a perfect compact R-ultrametric space, where T is an R-tree. If $R \subseteq \mathbb{R}_+$ is countable and with $R' \ni 0$, then $[T]$ has a uniform diameter partition for each $r \in R$ such that r is an infimum diameter of R.*

Proof. Since R is countable and $R' \ni 0$ there exists an R-tree T such that $([T], d)$ is an infinite compact ultrametric space. Therefore, we can refine any cover of $[T]$ to a finite partition $\{N_{u_i} : i < n\}$ where each $u_i \in T$ for some $n < \omega$. Without loss of generality, we can assume this partition is ordered in a non-decreasing fashion with respect to level, and thus label $l_0 = l(u_0)$.

Notice, since the set $\{u \in T : l(u) = l_0\}$ is finite it follows that the set $V = \{u, v \in T : l(u) = l(v) = l_0 \ \& \ u \perp v\}$ is finite. The collection $\mathcal{C} = \{N_u : u \in V\}$ is clearly finite, and $\mathcal{C}$ is a cover of $[T]$, since for all $f \in [T]$, there exists $f \upharpoonright u \in T$ where $u \in V$, implying $f \in N_u \in \mathcal{C}$, by the definition of a branch of T. Then it follows that $\mathcal{C}$ is a finite partition of $[T]$, since for all $u, v \in V$ $u \perp v$ implies $N_u \cap N_v = \emptyset$ (see remarks following Definition 2.8.

Now, since R can be enumerated into a strictly decreasing sequence $\langle r_i : i \geq 0 \rangle$, by Theorem 2.7, then it follows that for all $u \in V$,

$$
\begin{aligned}
l(u) &= l_0 \\
&= r_j \text{ for some } j \in \omega \\
&> r_{j+1} \\
&= \mathrm{diam}(N_u) \text{ following from the definition of a branch of } T.
\end{aligned}
$$

Therefore it follows, that for all $N, M \in \mathcal{C}$, $\mathrm{diam}(N) = \mathrm{diam}(M)$, and l_0 is a infimum diameter by Definition 2.8. ∎

Now, we are ready to present the proof of our main theorem.

THEOREM 2.10. *Let (X, ρ) be a nonempty perfect Polish ultrametric space. If X is compact, then any local radial contraction $\varphi \colon X \to X$ is necessarily not surjective.*

Proof. Let (X, ρ) be a perfect compact Polish ultrametric space. Let $R \subseteq \mathbb{R}_+$ be such that $R = \{\rho(x, y) : x \neq y \in X\}$. Since this is the distance set of a compact perfect ultrametric space, we see by 2.7 that R can be enumerated as a strictly decreasing sequence $\langle r_j : j \geq 0 \rangle$ and $\lim_{j \to \infty} r_j = 0$.

Since R is countable and $R' \ni 0$, then by Theorem 2.5 since X is a Polish R-ultrametric space there exists a unique pruned R-tree T such that (X, ρ) is isometric to $([T], d)$. So, let $\sigma \colon X \to [T]$ be an isometry, and define $\tau = \sigma \circ \varphi \circ \sigma^{-1} \colon [T] \to [T]$. It is easy to see that τ is local radial contraction. Thus, it suffices to show that τ is not surjective.

By Lemma 1.9 for some $n \in \omega$, we have a finite partition $\{N_{u_i} : u_i \in T$ and $i < n\}$ of τ-contractive balls in $[T]$. We may assume that $l(u_i) \leq l(u_j)$ whenever $i \leq j$. Label $l_0 = l(u_0)$ (i.e. the lowest level nbhd). Now for each $i < n$, let $\{v_i^k : k < s_i\}$ be the set of all incompatible extensions of u_i with $l(v_i^k) = l_0$, which we know is possible by Lemma 2.9.

So, $\operatorname{diam} N_{v_i^k} = r_t$ for some $t < \omega$ and $\operatorname{diam} \tau[N_{v_i^k}] \leq r_{t+1} < r_t$ for all $i < n$ and $k < s_i$.

Notice for all $i < n$, $\tau[N_{v_i^k}]$ is contained in some ball B_i^k where $\operatorname{diam} B_i^k \leq r_{t+1}$, and hence by Proposition 1.2(iii)

$$\tau[N_{v_i^k}] \subseteq B_i^k \subsetneq N_{v_{i'}^{k'}}$$

for some $i' < n$ and $k' < s_{i'}$.

Since $\{N_{v_i^k} : i < n \ \& \ k < s_i\}$ is a finite partition of $[T]$, it is easy to see that $\tau[[T]] \subsetneq [T]$. ∎

Below are some corollaries related to the above result. The *generic* element of a set has a particular property if and only if the set of all elements with this property forms a *comeager* subset (i.e. a subset which is the complement of a countable union of *nowhere dense* subsets). The *hyperspace* of compact sets of a topological space X, denoted $K(X)$, is equipped with the *Vietoris Topology* generated by the sets,

$$\{K \in X : K \subseteq U\}, \{K \in X : K \cap U \neq \emptyset\},$$

where U is open in X. It is well known that for a perfect Polish space X the hyperspace $K(X)$ is a *Baire* space (see [6, p. 41]), which allows us to take advantage of implications of the *Baire Category Theorem*. So, from [6, p. 42] we see that the set $K_p(X) = \{K \in K(X) : K$ is perfect $\}$ is dense G_δ, since X is perfect Polish. Thus, since $K(X)$ is a Baire space, $K_p(X)$ is comeager, and therefore we see that the generic compact subset of a perfect Polish space is perfect. So, recalling the above theorem, since all self-maps which local radial contractions defined on a perfect compact ultrametric space are not surjective, then the generic compact set of a perfect Polish ultrametric space has this property, which is stated as a corollary below.

COROLLARY 2.11. *Let (X, d) be a perfect Polish ultrametric space. Then for generic compact $K \subseteq X$ and local radial contraction $\varphi \colon K \to K$, $K \nsubseteq \varphi[K]$.*

Now leading toward a original motivation of the study of this phenomena concerning locally contractive maps, we state the following corollary related to *topological dynamical systems*, which is a pair $\langle X, \varphi \rangle$, where X is a closed topological space, and $\varphi \colon X \to X$ is a continuous map. A set $E \subseteq X$ is *φ-invariant*, if $\varphi(E) \subseteq E$. A topological dynamical system $\langle X, \varphi \rangle$ is *minimal* if and only if X contains no nonempty closed proper φ-invariant sets, and in this case φ is called a *minimal map*. Observe, that if φ is a minimal map and X is compact Hausdorff, then φ is surjective (see S. Kolyada and L. Snoha [7]), and from this, the subsequent corollary follows.

COROLLARY 2.12. *If (X, d) is a perfect compact ultrametric space and $\langle X, \varphi \rangle$ is a topological dynamical system with $\varphi \colon X \to X$ local radial contraction, then $\langle X, \varphi \rangle$ is not minimal.*

In Corollary 2.12, the assumption that X is ultrametric is essential. K. Ciesielski and J. Jasinski recently proved that there exists an uncountable perfect compact subset X of $\mathbb{R}$ and a surjective local radial contraction $f \colon X \to X$ such that $\langle X, f \rangle$ is a minimal dynamical system, see [1].

Briefly, considering the general Polish ultrametric case as a whole, it should be noted that any finite space (ultrametric or not) is necessarily discrete and thus there exists a trivial locally contractive map (i.e. the identity map). Thus, it suffices for us to consider infinite spaces only. Also, related to the ability to employ an R-tree in this characterization, any Polish R-ultrametric space such that R' does not contain 0 is necessarily a countable discrete space (see [10, p. 17 and Remark 3.5, p. 88]), and thus the identity map again can be used. So, together with the fact that for any Polish ultrametric space R is countable, we can assume that there exists an R-tree with end space isometric to the Polish spaces we are considering, and motivated by this, the following conjecture is posed.

CONJECTURE 2.13. *Let (X, ρ) be an infinite Polish ultrametric space. Then X is compact if and only if any locally contractive map $\varphi \colon X \to X$ is necessarily not surjective.*

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